일반화된 모델에 대한 최적 교체정책에 관한 연구

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On Optimal Replacement Policy for a Generalized Model

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Absract

In this paper, the properties on the optimal replacement policies for the general failure model are developed. In the general failure model, two types of system failures may occur : one is Type I failure (minor failure) which can be removed by a minimal repair and the other, Type II failure (catastrophic failure) which can be removed only by complete repair. It is assumed that, when the unit fails, Type I failure occurs with probability $1-p$ and Type II failure occurs with probability p, $0\leq p\leq 1$. Under the model, the system is minimally repaired for each Type I failure, and it is repaired completely at the time of the Type II failure or at its age T, whichever occurs first. We further assume that the repair times are non-negligible. It is assumed that the minimal repair times in a renewal cycle consist of a strictly increasing geometric process. Under this model, we study the properties on the optimal replacement policy minimizing the long-run average cost per unit time.

1. Introduction

There are many works in reliability that deal with the repair or replacement of components and systems. Two of the most popular repair types that have been considered in the literature are minimal and complete repairs. In complete repair, a failed system is repaired 'as good as new'. This means that the lifetime of the repaired system is independent of that of the original system and it has the same distribution function. On the other hand, a minimal repair enables the system to continue its task or work, but does not affect the failure rate of the system; i.e., the failure rate of a system after minimal repair is equal to the failure rate of the system immediately before the failure. Many repair models assume that a repaired system will yield a functioning state which is as good as new. In this case, if the repair times are ignored, the successive lifetimes generate a renewal process. However, in many practical situations, because of deterioration or friction in the system, the systems are deteriorating, where the lifetimes after each repair tend to be shorter and shorter whereas, in view of the ageing and cumulative wear, the repair times increase.

In this paper, we study the properties on the optimal replacement policy for a deteriorating system which has two types of system failures and corresponding repair times. We assume that the repair times are stochastically increasing. Under a suitable cost structure, we derive the long-run average cost rate and develop the properties on the optimal replacement policy that minimizes the long-run average cost rate.

2. Model and Preliminaries

We consider the general failure model. In the general failure model, when the unit fails, Type I failure occurs with probability $1-p$ and Type II failure occurs with probability p, $0 \leq p \leq 1$. It is assumed that Type I failure is a minor one thus can be removed by a minimal repair, whereas Type II failure is a catastrophic one thus can be removed only by a complete repair. Such a model has been considered in the literature. See, for example, Beichelt and Fischer (1980), Nakagawa (1981), Sheu and Griffith (1996) and Sheu (1998). Under the general failure model described above, during system operation, the system is repaired completely at its age T or at the time of the first Type II failure, whichever occurs first. For each Type I failure which occurs during operation, only minimal repair is done. In the following we call this replacement policy 'Policy T'. Furthermore, we assume that the repair times are not negligible. It is assumed that the minimal repair times in a renewal cycle constitute a strictly increasing geometric process and the complete repair times are i.i.d. positive random variables. The geometric process is introduced in the following definitions.

Definition 1.(Yeh, 1988) Given a sequence of random variables $X_1, X_2, \cdots$, if for some $a>0$, $\{a^{n-1}X_n, n=1,2,\cdots\}$ forms a renewal process, then $\{X_n, n=1,2,\cdots\}$ is a geometric process. a is called the parameter of the geometric process.

Definition 2.(Yeh, 1988) A geometric process is called a *non-increasing* geometric process if $a \geq 1$ and a non-decreasing geometric process if $a \leq 1$.

Note that, when $a = 1$, the geometric process corresponds to a renewal process.

Now we state the detailed assumptions as below.

- **Assumption** 1. At the beginning, a new system is used and we apply the 'Policy T' .
- **Assumption** 2. Let Y_j be the jth minimal repair time of the system after the jth Type I failure in a particular renewal period, $j = 1, 2, \cdots$. Then the sequence $\{Y_j, j = 1, 2, \cdots\}$ forms a geometric process with parameter $a < 1$ and the mean of the random variable Y_1 be given by $E(Y_1) = \nu_1$. Furthermore, let ν_2 be the mean of the complete repair time.
- **Assumption** 3. Let the cost rate(cost per unit repair time) of a minimal repair be C_1, and the cost rate of a complete repair, C_2. Furthermore, let the reward rate whenever the system is working be r.
- **Assumption** 4. Throughout this paper we assume that $0 < C_1 < C_2$. Then this means that the cost rate of a complete repair is higher than that of a minimal repair.

Observe that when $p = 0$ the model under our consideration reduces to that considered in Cha et al. (2000), and when $p = 1$ it reduces to the ordinary simple age-replacement policy model. Hence throughout this paper, we assume that $0 < p < 1$.

Now, we describe the model and derive some preliminary results, which will be useful in obtaining the long-run average cost rate. Temporarily, for convenience, we do not consider the repair times in our model. We define some notations to be used as follows.

- X : the lifetime of the system
- $F(x)$: the distribution function of X
- $f(x)$: the density function of X
- $r(x)$: the failure rate function of X, which is defined by $r(x) = f(x) / \overline{F}(x)$ where $\overline{F}(x) = 1 - F(x)$ is the reliability function of X
- Y : the time to the first Type II failure of the system when the repair times are ignored
- $G(x)$: the distribution function of Y
- $\overline{G}(x)$: $1 - G(x)$
- Z_t : $\min\{Y, t\}$, where t is a non-negative constant

Then, by the results presented in Beichelt (1993), $\overline{G}(x)$ is given by

$$\begin{aligned}\overline{G}(x) &= \exp\left[-\int_0^x pr(u)du\right]\\ &= \exp[-p\Lambda(x)],\ \ \forall x\geq 0,\end{aligned}$$

where $\Lambda(x)\equiv\int_0^x r(u)du,$ and the expectation of Z_t is given by

$$\begin{aligned}E(Z_t) &= E(Z_t\,|\,Y\leq t)P(Y\leq t)\\ &\quad + E(Z_t\,|\,Y>t)P(Y>t) \qquad (1)\\ &= \int_0^t udG(u) + t\overline{G}(t)\\ &= \int_0^t \overline{G}(u)du.\end{aligned}$$

Let N_t be the random number of minimal repairs during $(0, Z_t)$ when the repair times are ignored and $P(N_t=k\,|\,Y=t)$ be the conditional probability of $N_t=k$ given that $Y=t$. Then, from the results of Beichelt (1993), we can see that

$$\begin{aligned}P(N_t=k|Y=t) &= \frac{1}{k!}\left(\int_0^t(1-p)r(u)du\right)^k\\ &\times\exp\left(-\int_0^t(1-p)r(u)du\right),\\ &k=0,1,2,\cdots.\end{aligned}$$

(2)

Hence, on the condition that $Y=t$, the conditional distribution of the random number N_t is the Poisson distribution with mean

$$E(N_t|\,Y=t)=\int_0^t(1-p)r(u)du.$$

We now consider the repair times and derive the expected value of the sum of total minimal repair times in a renewal cycle. Note that

$$\begin{aligned}&E[\sum_{i=1}^{N_t} Y_i I(N_t\geq 1)|N_t=k,\,Y=t]\\ &= \frac{1-(1/a)^k}{1-1/a}\cdot\nu_1,\ \ k=0,1,2,\cdots,\end{aligned} \qquad (3)$$

where

$$I(N_t\geq 1)\equiv\begin{cases}1, \text{ if } N_t\geq 1,\\ 0, otherwise.\end{cases}$$

Then to derive the expected value of the sum of total minimal repair times in a renewal cycle, the following two separate cases are considered.

(Case I) When $a\neq 1-p$

In this case, from (2) and (3), we have

$$\begin{aligned}&E[\sum_{i=1}^{N_t} Y_i I(N_t\geq 1)|\,Y=t]\\ &= \sum_{k=0}^{\infty}\frac{1-(1/a)^k}{1-1/a}\frac{1}{k!}\\ &\quad\times\left(\int_0^t(1-p)r(u)\,du\right)^k\\ &\quad\times\exp\left(-\int_0^t(1-p)r(u)\,du\right)\cdot\nu_1\\ &= \frac{a}{1-a}\Big[\exp[(1/a-1)\\ &\quad\times\int_0^t(1-p)r(u)\,du]-1\Big]\cdot\nu_1.\end{aligned}$$

Therefore, the conditional expectations are given by

$$\begin{aligned}&E[\sum_{i=1}^{N_t} Y_i I(N_t\geq 1)|\,Y<t]\\ &= \int_0^t\frac{a}{1-a}\Big[\exp[(1/a-1)\\ &\quad\times\int_0^s(1-p)r(u)\,du]-1\Big]\cdot\nu_1\\ &\quad\times pr(s)\exp[-p\int_0^s r(u)du]/G(t)ds\end{aligned}$$

$$= \nu_1 \int_0^t a/(1-a) \cdot pr(s) \exp[((1/a)(1-p)-1)\int_0^s r(u)du]ds/G(t) - \nu_1(a/(1-a))$$

$$= \nu_1 \frac{p(a/(1-a))}{(1/a)(1-p)-1} \times [\exp[((1/a)(1-p)-1)\Lambda(t)]-1]/G(t) - \nu_1(a/(1-a)),$$

(4)

and

$$E[\sum_{i=1}^{N_t} Y_i I(N_t \geq 1) | Y \geq t] = \nu_1(a/(1-a))\exp[((1/a)-1)\times(1-p)\Lambda(t)] - \nu_1(a/(1-a)). \quad (5)$$

Then, from (4) and (5), the unconditional expectation is given by

$$E[\sum_{i=1}^{N_t} Y_i I(N_t \geq 1)] = E[\sum_{i=1}^{N_t} Y_i I(N_t \geq 1) | Y < t] \cdot G(t) + E[\sum_{i=1}^{N_t} Y_i I(N_t \geq 1) | Y \geq t] \cdot \overline{G}(t)$$

$$= \frac{(1-p)}{(1/a)(1-p)-1} \times [\exp[((1/a)(1-p)-1)\Lambda(t)]-1]\nu_1.$$

(6)

(Case II) When $a = 1-p$

In this case, by similar fashion,

$$E[\sum_{i=1}^{N_t} Y_i I(N_t \geq 1) | Y = t] = \frac{1-p}{p}\left[\exp[p\Lambda(t)]-1\right]\nu_1.$$

Thus

$$E[\sum_{i=1}^{N_t} Y_i I(N_t \geq 1) | Y < t] = \nu_1(1-p)\Lambda(t)/G(t) - \nu_1((1-p)/p),$$

(7)

and

$$E[\sum_{i=1}^{N_t} Y_i I(N_t \geq 1) | Y \geq t] = \nu_1((1-p)/p)\exp[p\Lambda(t)] - \nu_1((1-p)/p). \quad (8)$$

Then, from (7) and (8), we obtain

$$E[\sum_{i=1}^{N_t} Y_i I(N_t \geq 1)] = \nu_1(1-p)\Lambda(t). \quad (9)$$

3. Main Results

In this section, the optimal replacement policy for the long run expected cost rate is considered. From the renewal theory, we can see that the long-run average cost per unit time is given by

$C(T) \equiv$ [*the expected cost incurred in a renewal cycle*] / [*the expected length of a renewal cycle*].

(10)

Hence the long-run average cost per unit time can be obtained by calculating the expected cost incurred in a renewal cycle and the expected length of a renewal cycle. Using the preliminary results in section 2, the long-run average cost per unit time is obtained in the following Lemma 1.

Lemma 1.

Under the Policy T described in section 2, the long-run average cost per unit time $C(T)$ is given by

$$C(T) = C_1 + \frac{1}{\eta(T)} \times (-(C_1 + r)\int_0^T \overline{G}(t)dt + (C_2 - C_1)\nu_2),$$

where (i) when $a \neq 1-p$

$$\eta(T) \equiv \int_0^T \overline{G}(u)du + \frac{(1-p)}{(1/a)(1-p)-1} \times [\exp[((1/a)(1-p)-1)\Lambda(T)]-1]\nu_1 + \nu_2,$$

and (ii) when $a = 1-p$

$$\eta(T) \equiv \int_0^T \overline{G}(u)du + (1-p)\Lambda(T)\nu_1 + \nu_2.$$

proof.

Let us define $\eta(T)$ as the expected length of a renewal cycle. Then, when $a \neq 1-p$, from (1) and (6),

$$\eta(T) = \int_0^T \overline{G}(u)du + \frac{(1-p)}{(1/a)(1-p)-1} \times [\exp[((1/a)(1-p)-1)\Lambda(T)]-1]\nu_1 + \nu_2,$$

and, when $a = 1-p$, from (1) and (9), it is given by

$$\eta(T) = \int_0^T \overline{G}(u)du + (1-p)\Lambda(T)\nu_1 + \nu_2.$$

Furthermore, when $a \neq 1-p$, the expected cost incurred in a renewal cycle is given by

$$\frac{(1-p)}{(1/a)(1-p)-1}\Big[\exp[((1/a)(1-p)-1) \times \Lambda(T)]-1\Big]\nu_1 C_1 + \nu_2 C_2 - r\int_0^T \overline{G}(u)du,$$

and it is determined by

$$(1-p)\Lambda(T)\nu_1 C_1 + \nu_2 C_2 - r\int_0^T \overline{G}(u)du,$$

when $a = 1-p$. Therefore, from (10), we obtain the desired results. ■

Now the properties of the optimal replacement policy T^* which minimizes $C(T)$ can be obtained as follows.

Theorem 1.

Suppose that the failure rate function $r(t)$ is increasing(non-decreasing) failure rate function(IFR) and

$$E(Y) = \int_0^\infty \overline{G}(t)dt > [(C_2 - C_1)\nu_2]/(C_1 + r).$$

(Case I) When $a \neq 1-p$

If $a < (1-p)$ then there exists the unique optimal T^*, and it is the unique solution of the following equation :

$$\left[(C_1+r)\int_0^T \overline{G}(t)dt - (C_2 - C_1)\nu_2\right] \times (1-p)r(T) \times \exp[((1/a)(1-p)-1)\Lambda(T)]\nu_1$$
$$= (C_1 + r)/[(1/a)(1-p)-1]\cdot(1-p) \times [\exp[((1/a)(1-p)-1)\Lambda(T)]-1] \times \exp[-p\Lambda(T)]\nu_1 + (C_2 + r) \times \exp[-p\Lambda(T)]\nu_2.$$

Furthermore, if we set

$$T_1 \equiv \inf\{T: \int_0^T \overline{G}(t)dt \geq [(C_2 - C_1)\nu_2]/(C_1 + r)\}. \tag{11}$$

then the non-trivial lower bound for the optimal T^* is given by T_1, that is, $T^* \geq T_1$.

(Case II) When $a = 1-p$

There exists the unique optimal T^*, and it is the unique solution of the following equation :

$$\left[(C_1+r)\int_0^T \overline{G}(t)dt - (C_2 - C_1)\nu_2\right] \times (1-p)r(T)\nu_1$$
$$= (C_1 + r)(1-p)\Lambda(T)\exp[-p\Lambda(T)]\nu_1 + (C_2 + r)\cdot\exp[-p\Lambda(T)]\nu_2.$$

Furthermore, the non-trivial lower bound for the optimal T^* is given by T_1, that is, $T^* \geq T_1$, where T_1 is defined in (11).

proof.

(Case I) When $a \neq 1-p$

Observe that

$$C'(T) = \frac{1}{\eta(T)^2} \cdot \Psi(T),$$

where

$$\begin{aligned}\Psi(T) &= -(C_1+r)/[(1/a)(1-p)-1] \\ &\times(1-p) \\ &\times[\exp[((1/a)(1-p)-1)\Lambda(T)]-1] \\ &\times\exp[-p\Lambda(T)]\nu_1 \\ &\quad -(C_2+r)\cdot\exp[-p\Lambda(T)]\nu_2 \\ &+\left[(C_1+r)\int_0^T \overline{G}(t)dt-(C_2-C_1)\nu_2\right] \\ &\times(1-p)r(T) \\ &\times\exp[((1/a)(1-p)-1)\Lambda(T)]\nu_1.\end{aligned}$$

Then the following two separate subcases are considered.

Subcase Ia. When $\frac{(1-p)}{(1+p)} < a < (1-p)$.

Since

$$\frac{1}{a}(1-p)-1>0, \quad \frac{1}{a}(1-p)-1-p<0,$$

and $(C_1+r)\int_0^\infty \overline{G}(t)dt > (C_2-C_1)\nu_2$, we can see that

$$\Psi(\infty) \equiv \lim_{T\to\infty} \Psi(T) = \infty, \quad (12)$$

and, by the definition of T_1, we also have

$$\Psi(T) < 0, \text{ for all } 0 \leq T \leq T_1. \quad (13)$$

Furthermore we can obtain

$$\begin{aligned}\Psi'(T) &= (C_1+r)/[(1/a)(1-p)-1] \\ &\times(1-p)\cdot pr(T) \\ &\times[\exp[((1/a)(1-p)-1)\Lambda(T)]-1] \\ &\times\exp[-p\Lambda(T)]\nu_1 \\ &+(C_2+r)pr(T)\exp[-p\Lambda(T)]\nu_2 \\ &+\left[(C_1+r)\int_0^T \overline{G}(t)dt-(C_2-C_1)\nu_2\right] \\ &\times(1-p)r'(T) \\ &\times\exp[((1/a)(1-p)-1)\Lambda(T)]\nu_1 \\ &+\left[(C_1+r)\int_0^T \overline{G}(t)dt-(C_2-C_1)\nu_2\right] \\ &\times(1-p)r(T)[(1/a)(1-p)-1]r(T) \\ &\quad\times\exp[((1/a)(1-p)-1)\Lambda(T)]\nu_1,\end{aligned}$$

and thus we can see that

$$\Psi'(T) > 0 \text{ for } T > T_1, \quad (14)$$

which implies that $\Psi(T)$ is strictly increasing when $T \geq T_1$. Then, from (12), (13) and (14), we can see that there exists a unique T^* which satisfies

$$\Psi(T^*) = 0, \quad (15)$$

and that $\Psi(T) < 0$ for all $T < T^*$ and $\Psi(T) > 0$ for all $T > T^*$. Therefore, the optimal replacement policy is unique and is given by the unique solution of the equation (15). Moreover, from (13), we can conclude that $T^* > T_1$.

Subcase Ib. When $a \leq \frac{(1-p)}{(1+p)}$.

In this case,

$$\frac{1}{a}(1-p)-1 > \frac{1}{a}(1-p)-1-p \geq 0,$$

and $(C_1+r)\int_0^\infty \overline{G}(t)dt > (C_2-C_1)\nu_2$.

Therefore, in this case, it still holds that $\Psi(\infty) \equiv \lim_{T\to\infty} \Psi(T) = \infty$. Then, by applying the same arguments described in Subcase Ia, we can obtain the same result.

(Case II) When $a=1-p$

In this case, observe that

$$C'(T)=\frac{1}{\eta(T)^2}\cdot\Psi(T),$$

where

$$\begin{aligned}\Psi(T)&=-(C_1+r)(1-p)\Lambda(T)\\ &\times\exp[-p\Lambda(T)]\nu_1-(C_2+r)\cdot\\ &\times\exp[-p\Lambda(T)]\nu_2\\ &+\left[(C_1+r)\int_0^T\overline{G}(t)dt-(C_2-C_1)\nu_2\right]\\ &\times(1-p)r(T)\nu_1,\end{aligned}$$

and note that $\Psi(\infty)\equiv\lim_{T\to\infty}\Psi(T)>0$. Then, for the proof of the Case II, we can apply the same method described in the proof of Case I. Hence we omit the details for brevity. ∎

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