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# Power Exponential Distributions

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**Abstract.** By applying Theorem 2.6.4 (Fang and Zhang, 1990, p.66) the dispersion matrix of a multivariate power exponential (MPE) distribution is derived. It is shown that the MPE and the gamma distributions are related and thus the MPE and chi-square distributions are related. By extending Fang and Xu's Theorem (1987) from the normal distribution to the Univariate Power Exponential (UPE) distribution an explicit expression is derived for calculating the probability of an UPE random variable over an interval. A representation of the characteristic function (c.f.) for an UPE distribution is given. Based on the MPE distribution the probability density functions of the generalized non-central chi-square, the generalized non-central t, and the generalized non-central F distributions are derived.

**Key Words** : *multivariate power exponential distribution, gamma distribution, generalized non-central chi-square, generalized non-central t, generalized non-central F, characteristic function*

## 1. INTRODUCTION

The power exponential distribution can be used to model both light and heavy tailed, symmetric and unimodal continuous data sets. Gomez et al.(1998) generalized the Univariate Power Exponential (UPE) distribution, which was established by Subbotin (1923), to the Multivariate Power Exponential (MPE) distribution. Gomez et al have a FORTRAN program which can be used to simulate the MPE distributions, but this program is not available in a public domain. Both Johnson (1979) and Gomez et al. (1998) mentioned a relationship between the UPE distribution and a Gamma distribution. Gomez et al. (1998) studied the properties of MPE intensively, including the stochastic representation, the moments, the characteristic function for $n > 1$ and the marginal and conditional distributions and asymmetry and kurtosis coefficients. The dispersion matrix of a multivariate power exponential (MPE) distribution is derived. Using Theorem 2.6.4 (Fang and Zhang, 1990, p.66) a dispersion matrix of a multivariate power exponential

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(MPE) distribution is derived and it is shown that the MPE and the gamma distributions are related and thus the MPE and chi-square distributions are related. We extend Fang and Xu's Theorem (1987) to the Univariate Power Exponential (UPE) distribution.. An explicit expression is derived for calculating the probability of an UPE random variable over an interval. A representation of the characteristic function (c.f.) for an UPE distribution is given. Based on the MPE distribution the probability density functions of the generalized non-central chi-square, the generalized non-central t, and the generalized non-central F distributions are derived.

## 2. POWER EXPONENTIAL DISTRIBUTIONS

Lindsey (1999) defined the UPE distribution as follows:

$$f(y;\mu,\sigma,\beta)=\left[\sigma\Gamma\left(1+\frac{1}{2\beta}\right)2^{\left(1+\frac{1}{2\beta}\right)}\right]^{-1}\exp\left(-\frac{1}{2}\left|\frac{y-\mu}{\sigma}\right|^{2\beta}\right),-\infty<\mu<\infty,\sigma>0,0<\beta\leq\infty. \quad (1)$$

If Y is distributed as a UPE with parameters $\mu$, $\sigma$, and $\beta$ we write Y ~ UPE ($\mu$, $\sigma$, $\beta$) and we write Y~UPE ($\beta$) if $\mu = 0$, $\sigma = 1$. Lindsey (1999) defined the MPE distribution as follows:

$$f(Y;\mu,\Sigma,\beta)=\frac{n\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n}{2}}\sqrt{|\Sigma|}\Gamma\left(1+\frac{n}{2\beta}\right)2^{\left(1+\frac{n}{2\beta}\right)}}\exp\left(-\frac{1}{2}\left[(y-\mu)'\Sigma^{-1}(y-\mu)\right]^{\beta}\right), \quad (2)$$

$$-\infty<\mu<\infty,\Sigma>0,0<\beta\leq\infty.$$

If $\mathbf{y}$ is distributed as an MPE distribution with parameters $\mu$, $\Sigma$, and $\beta$ we write $\mathbf{y}$ ~ UPE($\mu$, $\Sigma$, $\beta$) and we write $\mathbf{y}$ ~ MPE( $\beta$) if $\mu = 0$ , $\Sigma = \mathbf{I}_n$ . The parameter $\beta$ in (1) and (2) is called the shape parameter.

## 3. THE MOMENTS OF POWER EXPONENTIAL DISTRIBUTIONS

From now on throughout this paper we adopt some notations from Seber's (1984). We use the upper case letter $E$ to denote the expectation operation for a univariate random variable, the script upper case letter $E$ to denote the expectation operation for a random variable vector or matrix; the upper case letter $V$ to denote the variance operation for the univariate random variable, the script upper case letter $D$ to denote the variance-covariance operation for a random variable vector, the three lower case letters *cov* to denote the covariance operation between two random univariate variables, the upper case script letter $C$ to denote the covariance operation between two random variable vectors or between one univariate random variable and a random vector.

**Theorem 3.1.** Suppose $\mathbf{y}$ is distributed as the MPE as defined in (2). Then there exists a nonsingular matrix **L** such that $\Sigma = \mathbf{L}\mathbf{L}'$. Let $\mathbf{x} = \mathbf{L}^{-1}(\mathbf{y} - \boldsymbol{\mu})$. Then the p.d.f. of **x** is given by

$$\frac{n\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n}{2}}\Gamma\left(1+\frac{n}{2\beta}\right)2^{\left(1+\frac{n}{2\beta}\right)}}\exp\left(-\frac{1}{2}\left[X'X\right]^{\beta}\right). \tag{3}$$

The expectation vector and variance-covariance matrix of **x** are given, respectively, by

$$E(X)=0, \qquad D(X)=\frac{2^{\frac{1}{\beta}}\Gamma\left(\frac{n+2}{2\beta}\right)}{n\Gamma\left(\frac{n}{2\beta}\right)}I_n. \tag{4}$$

**Proof.** The Stochastic Representation of **x** is $\mathbf{x} \approx R\mathbf{u}^{(n)} \sim Sn(\phi)$. Let $g(r)$ be the density function of R. Then

$$g(r)=\frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}r^{n-1}f(r^2). \tag{5}$$

Substituting the function f of (2) into the above formula we have

$$g(r)=\left(\frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}\right)\frac{n\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n}{2}}\Gamma\left(1+\frac{n}{2\beta}\right)2^{\left(1+\frac{n}{2\beta}\right)}}r^{n-1}\exp\left(-\frac{1}{2}r^{2\beta}\right)$$

$$=\frac{n}{\Gamma\left(1+\frac{n}{2\beta}\right)2^{\left(\frac{n}{2\beta}\right)}}r^{n-1}\exp\left(-\frac{1}{2}r^{2\beta}\right).$$

The expectation of $R^2$ is given by

$$E(R^2)=\frac{n}{\Gamma\left(1+\frac{n}{2\beta}\right)2^{\left(\frac{n}{2\beta}\right)}}\int_0^{\infty}r^{n+1}\exp\left(-\frac{1}{2}r^{2\beta}\right)dr.$$

Let $t = \frac{1}{2} r^{2\beta}$.

Then $r = \left(2t^{\frac{1}{2\beta}}\right)$, $dr = \left(\frac{1}{2\beta}\right) 2^{\frac{1}{2\beta}} t^{\left(\frac{1}{2\beta}-1\right)} dt$

and $r^{n+1} = 2^{\frac{n+1}{2\beta}} t^{\left(\frac{n+1}{2\beta}\right)}$.

Therefore we have

$$E(R^2) = \frac{n}{\Gamma\left(1+\frac{n}{2\beta}\right) 2^{\left(\frac{n}{2\beta}\right)}} \int_0^\infty \left(2^{\frac{n+1}{2\beta}} t^{\frac{n+1}{2\beta}}\right)\left(\frac{1}{2\beta}\right) 2^{\frac{1}{2\beta}} t^{\frac{1}{2\beta}-1} \exp(-t) dt$$

$$= \frac{n 2^{\left(\frac{1}{\beta}\right)}}{2\beta\Gamma\left(1+\frac{n}{2\beta}\right)} \int_0^\infty t^{\frac{n+2}{2\beta}-1} \exp(-t) dt$$

$$= \frac{2^{\left(\frac{1}{\beta}\right)} \Gamma\left(\frac{n+2}{2\beta}\right)}{\Gamma\left(\frac{n}{2\beta}\right)}.$$

**Theorem 3.2** (Lindsey, 1999): Suppose **y** is distributed as the MPE as defined in (2). The expectation and variance-covariance matrix of **y** are given, respectively, by

$$E(y) = \mu, \qquad D(y) = \frac{2^{\frac{1}{\beta}} \Gamma\left(\frac{n+2}{2\beta}\right)}{n\Gamma\left(\frac{n}{2\beta}\right)} \Sigma. \tag{6}$$

**Proof.** By Theorem 2.6.4 (Fang and Zhang, 1990, p.66) we have E(**y**) = $\boldsymbol{\mu}$ and

$$D(y) = \frac{E(R^2)}{n} \Sigma = \frac{2^{\frac{1}{\beta}} \Gamma\left(\frac{n+2}{2\beta}\right)}{n\Gamma\left(\frac{n}{2\beta}\right)} \Sigma,$$

since we found that in the last theorem:

$$E(R^2) = \frac{2^{\left(\frac{1}{\beta}\right)}\Gamma\left(\frac{n+2}{2\beta}\right)}{\Gamma\left(\frac{n}{2\beta}\right)}.$$

This completes the proof of the theorem 3.2.

Gomez et al.(1998) presented the characteristics function of the MPE distribution for n > 1. In the next theorem we present the characteristics function of the UPE distribution as a moment series.

**Theorem 3.3.** If Y ~ UPE (0,1, $\beta$) then the characteristics function (c.f.) of Y is as follows

$$\phi(t) = \sum_{r=0}^{\infty}(-1)^r 2^{\frac{r}{\beta}} \frac{\Gamma\left(\frac{2r+1}{2\beta}\right)t^{2r}}{\Gamma\left(\frac{1}{2\beta}\right)(2r)!}, \quad -\infty < t < \infty$$

**Proof.** By (24.84) (Johnson et al., 1995, p.196) the $r$ th central moment of UPE ($\mu$, $\sigma$, $\beta$) is given by

$$\mu_r = \begin{cases} 0, & if \quad r \quad is \quad odd, \\ \sigma^r 2^{\frac{r}{\beta}} \frac{\Gamma\left(\frac{r+1}{2\beta}\right)}{\Gamma\left(\frac{1}{2\beta}\right)}, & if \quad r \quad is \quad even \end{cases} \tag{7}$$

Substituting $\sigma = 1$ into the above formula we can get the $r$ th moment of Y and by the using the expression (1.39) (Fang and Xu, 1987, p.38) we can get the c.f. of Y.

We let z = $\sigma$y + $\mu$ and use the formula $\phi_z(t) = e^{i\mu t}\phi_y(\sigma t)$ to get the c.f. of the UPE ($\mu$, $\sigma$, $\beta$) distribution as follows:

$$\phi(t) = e^{i\mu t}\sum_{r=0}^{\infty}(-1)^r 2^{\frac{r}{\beta}} \frac{\Gamma\left(\frac{2r+1}{2\beta}\right)(\sigma t)^{2r}}{\Gamma\left(\frac{1}{2\beta}\right)(2r)!}, \quad -\infty < t < \infty.$$

If we let $\beta = 1$ in the above formula we can prove that

$$e^{i\mu t}\sum_{r=0}^{\infty}(-1)^r 2^r \frac{\Gamma\left(\frac{2r+1}{2}\right)(\sigma t)^{2r}}{\Gamma\left(\frac{1}{2}\right)(2r)!} = e^{i\mu t-\frac{1}{2}\sigma^2 t^2},$$

which is the c.f. of the normal distribution $N(\mu,\sigma^2)$.

## 4. THE RELATIONSHIP BETWEEN POWER EXPONENTIAL DISTRIBUTIONS AND OTHER DISTRIBUTIONS

Putting n = 1 in Theorem 3.2 we obtain the expectation and variance of UPE distribution. That is if Y is defined as in (1) E (Y) = μ and

$$D(Y) = \frac{2^{\frac{1}{\beta}}\Gamma\left(\frac{3}{2\beta}\right)}{\Gamma\left(\frac{1}{2\beta}\right)}\sigma^2.$$

Furthermore putting n = 1, $\beta = 1$ in Theorem 3.2 we have E(Y) = μ and D(Y) = $\sigma^2$, the UPE distribution now becomes a univariate normal distribution. If we let $\beta = 1/2$, n = 1 we have E(Y) = μ and D(Y) $=8\sigma^2$, and the UPE distribution now becomes the Laplace distribution. If we let $\beta = 1/2$, n = 1, $\mu = 0$, $\sigma = 1$ we have the double exponential distribution. If n >1 and $\beta = 1/2$, then the MPE distribution becomes a multivariate Laplace distribution. If $\beta = \infty$ then the MPE distribution becomes a multivariate uniform distribution. If n > 1 and $\beta = 1$ then the MPE distribution becomes a multivariate normal distribution. If $0 < \beta < 1$ we have heavier tails than the normal distributions. If $\beta > 1$ we have lighter tails than normal distributions.

If the random variable U is distributed as a Gamma, the p.d.f. of U is written as $f(\mu) = \frac{\lambda^r}{\Gamma(r)}u^{r-1}e^{-\lambda x}$, $\mu \geq 0$, where r > 0, λ > 0. Simply we write this as u ~ Gamma ($\mu$, λ, r) or u ~ Gamma (λ, r).

**Theorem 4.1.** Suppose Y is distributed as the UPE ($\beta$) distribution as defined in (1), and U is distributed as a Gamma. Then we have the following relationship

$$\left(\frac{1}{2}\right)|U|^{2\beta} \approx U \sim Gamma\left(1,\frac{1}{2\beta}\right). \tag{8}$$

**Proof.** First suppose

$$U \sim Gamma\left(1,\frac{1}{2\beta}\right).$$

Let $u = \left(\frac{1}{2}\right)|y|^{2\beta}$

then $du = \beta y^{2\beta-1} dy$ and

$$f(y) = \frac{1}{\Gamma\left(\frac{1}{2\beta}\right)} \left(\beta y^{2\beta-1}\right) \left(\frac{1}{2} y^{2\beta}\right)^{\frac{1}{2\beta}-1} \exp\left(-\frac{1}{2} y^{2\beta}\right)$$

$$= \frac{\beta}{\Gamma\left(\frac{1}{2\beta}\right)(2)^{\frac{1}{2\beta}-1}} \exp\left(-\frac{1}{2} y^{2\beta}\right). \qquad (9)$$

Conversely, if y > 0 and Y ~ UPE($\beta$) then

$$U = \left(\frac{1}{2}\right)|y|^{2\beta},$$

$$dy = \frac{1}{2\beta}(2)^{\frac{1}{2\beta}}(u)^{\frac{1}{2\beta}-1} du \text{ and}$$

$$f(u) = \frac{\frac{1}{2\beta}(2)^{\frac{1}{2\beta}}}{\Gamma\left(1+\frac{1}{2\beta}\right)(2)^{\frac{1}{2\beta}}} \left(u^{\frac{1}{2\beta}-1}\right) \exp(-u)$$

$$= \frac{1}{\Gamma\left(\frac{1}{2\beta}\right)} \left(u^{\frac{1}{2\beta}-1}\right) \exp(-u).$$

Similarly, if y $\le 0$ we can prove the above conclusion.
In Theorem 4.2 we extended Fang and Xu's theorem (1987, p.239) from the normal to the UPE distribution and found an explicit expression for using a Gamma distribution to calculate the probability of the UPE distribution over an interval.

**Theorem 4.2.** Suppose Y ~ UPE($\beta$), $U \sim \Gamma\left(1, \frac{1}{2\beta}\right)$ then we have

$$P\{Y \le y\} = \frac{1}{2}\left[P\{U \le \frac{1}{2} y^{2\beta}\} + 1\right], \quad y \ge 0 \text{ and}$$

$$P\{Y \le y\} = \frac{1}{2} - \frac{1}{2}\left[P\{U \le \frac{1}{2}(-y)^{2\beta}\}\right], \quad y < 0.$$

**Proof.** If y ≥ 0 then $P\{-y \le Y \le y\} = P\{U \le \frac{1}{2} y^{2\beta}\}$.

But
$$\begin{aligned} P\{Y \le y\} &= P\{-y \le Y \le y\} + P\{Y \le -y\} \\ &= P\{-y \le Y \le y\} + \frac{1}{2}[1 - P\{-y \le Y \le y\}] \\ &= \frac{1}{2} + \frac{1}{2} P\{-y \le Y \le y\} \\ &= \frac{1}{2}\left[ P\{U \le \frac{1}{2}(y)^{2\beta}\} + 1 \right]. \end{aligned}$$

Similarly, when y < 0,
$$\begin{aligned} P\{Y \le y\} &= 1 - P\{Y \le -y\} \\ &= 1 - \frac{1}{2}\left[ P\{U \le \frac{1}{2}(-y)^{2\beta}\} + 1 \right] \\ &= \frac{1}{2} - \frac{1}{2}\left[ P\{U \le \frac{1}{2}(-y)^{2\beta}\} \right]. \end{aligned}$$

In the following Theorem 4.3 we derive a relationship between the MPE distribution and a Gamma distribution.

**Theorem 4.3.** Suppose x > 0 and y is distributed as the MPE distribution as defined in (2) and u is distributed as a Gamma. We then have the following relationship

$$\overbrace{\int\int \cdots \int}^{n}_{(y-\mu)'\Sigma^{-1}(y-\mu)\le x} f(y)dy = Gamma\left(\frac{1}{2}x^{\beta}, 1, \frac{n}{2\beta}\right), \tag{10}$$

where Gamma(.) is the same as defined as in the Theorem 4.1.

**Proof.** Let $x = \Sigma^{-\frac{1}{2}}(y-\mu)$, where $\sum$ is nonsingular and $\Sigma^{-\frac{1}{2}}\Sigma^{-\frac{1}{2}} = \Sigma^{-1}$. Then we have

$$(y-\mu)' = x'\Sigma^{\frac{1}{2}},$$

$$(y-\mu)'\Sigma^{-1}(y-\mu) = \left( x'\Sigma^{\frac{1}{2}} \right)\Sigma^{-1}\left( \Sigma^{\frac{1}{2}}x \right) = x'x = \sum_{i=1}^{n} x_i^2.$$

The Jacobian of this transformation is $|J|^{+} = \left|\Sigma^{\frac{1}{2}}\right|^{+}$.

Therefore

$$\overbrace{\int\int \cdots \int}^{n}_{(y-\mu)'\Sigma^{-1}(y-\mu)\le x} f(y)dy$$

$$= \frac{n\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n}{2}}\Gamma\left(1+\frac{n}{2\beta}\right)(2)^{1+\frac{n}{2\beta}}} \int_{\sum_{i=1}^{n} x_i^2 \le x} \exp\left(-\frac{1}{2}\left(\sum_{i=1}^{n} x_i^2\right)^{\beta}\right) dx_1 dx_2 \ldots dx_n$$

$$= \pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)^{-1} \int_0^x \frac{n\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n}{2}}\Gamma\left(1+\frac{n}{2\beta}\right)2^{1+\frac{n}{2\beta}}} \exp\left(-\frac{1}{2}y^{\beta}\right) y^{\frac{n}{2}-1} dy$$

$$= \frac{n2^{-1-\frac{n}{2\beta}}}{\Gamma\left(1+\frac{n}{2\beta}\right)} \int_0^x \exp\left(-\frac{1}{2}y^{\beta}\right) y^{\frac{n}{2}-1} dy$$

$$= \frac{n}{\Gamma\left(1+\frac{n}{2\beta}\right)2^{1+\frac{n}{2\beta}}} \left(\frac{1}{\beta}\right) 2^{\frac{n}{2\beta}} \int_0^{\frac{1}{2}x^{\beta}} w^{\frac{n}{2\beta}-1} \exp(-w) dw$$

(by applying a change of variable, w = $\frac{1}{2}y^{\beta}$)

$$= \frac{1}{\Gamma\left(\frac{n}{2\beta}\right)} \int_0^{\frac{1}{2}x^{\beta}} w^{\frac{n}{2\beta}-1} \exp(-w) dw$$

$$= Gamma\left(\frac{1}{2}x^{\beta}, 1, \frac{n}{2\beta}\right).$$

In the following Theorem 4.4 we derive a relationship between the UPE distribution and a $\chi^2$ distribution.

**Theorem 4.4.** Suppose Y is distributed as the UPE(0,1, $\beta$) as defined in (1) and T is distributed as a $\chi^2$. If $1/\beta$ is a positive integer, then T $\approx |Y|^{2\beta}$ is distributed as a $\chi^2$ with degree of freedom $1/\beta$.

**Proof.** We discussed the definition of UPE in the section 2. Substituting $\mu = 0$ and $\sigma = 1$ into (1) we have the density function of UPE(0, 1, $\beta$) as follows:

$$f(y;0,1,\beta) = \frac{1}{\Gamma\left(1+\frac{1}{2\beta}\right)2^{1+\frac{1}{2\beta}}} \exp\left(-\frac{1}{2}|y|^{2\beta}\right).$$

Applying a change of variable t = $|y|^{2\beta}$ to the above density function we have the density function of T as follows:

$$f(t) = \frac{1}{\Gamma\left(\frac{1}{2\beta}\right) 2^{\frac{1}{2\beta}}} t^{\frac{1}{2\beta}-1} \exp\left(-\frac{1}{2}t\right), \quad t \geq 0,$$

which is the density function of a $\chi^2$ with the degree of freedom $1/\beta$.
The above relationship can be used to calculate the probability or the moments of some UPE distributions through a $\chi^2$ distribution. For example, if Y ~ UPE (0, 1, 0, 1), then T $\approx$ $|Y|^{2\beta} \sim \chi^2_{10}$, and E(T) = 10, V(T) = 20, and E(Y) = E($T^5$) = $\frac{2^5\Gamma(15)}{\Gamma(10)}$, since Y $= T^{\frac{1}{2\beta}} = T^5$ by Theorem 4.4.

## 5. MARGINAL AND CONDITIONAL DISTRIBUTION OF POWER EXPONENTIAL DISTRIBUTIONS

**Theorem 5.1.** Suppose (X, Y) is distributed as 2-dimensional MPE distribution. That is

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim MPE_2(\mu, \Sigma, \beta),$$

where

$$\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}, \quad rk(\Sigma) = 2.$$

The joint p.d.f. of (X, Y) is given as follows:

$$f(x,y) = \frac{2}{\pi\sqrt{\sigma_1^2\sigma_2^2(1-\rho^2)}\Gamma\left(1+\frac{1}{\beta}\right)2^{1+\frac{1}{\beta}}} \exp\left(-\frac{1}{2}\left[(x,y)\Sigma^{-1}(x,y)'\right]^\beta\right)$$

$$= \frac{2}{\pi\sqrt{\sigma_1^2\sigma_2^2(1-\rho^2)}\Gamma\left(1+\frac{1}{\beta}\right)2^{1+\frac{1}{\beta}}} \exp\left(-\frac{1}{2}\left[\frac{1}{1-\rho^2}\left(\frac{x^2}{\sigma_1^2} - \frac{2\rho xy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2}\right)\right]^\beta\right), \qquad (11)$$

where $\rho$ is the correlation coefficient between X and Y. We have the following three conclusions.

(1) The marginal distribution of X is given by

$$f(x)=\frac{1}{\sigma_1\Gamma\left(1+\frac{1}{2\beta}\right)2^{1+\frac{1}{2\beta}}}\exp\left(-\frac{1}{2}\left(\frac{x^2}{\sigma_1^2}\right)^\beta\right). \tag{12}$$

(2) The marginal distribution of Y is given by

$$f(y)=\frac{1}{\sigma_2\Gamma\left(1+\frac{1}{2\beta}\right)2^{1+\frac{1}{2\beta}}}\exp\left(-\frac{1}{2}\left(\frac{y^2}{\sigma_2^2}\right)^\beta\right). \tag{13}$$

(3) The conditional distribution of X given Y = b is given by

$$f(x\mid b)=\frac{\Gamma\left(1+\frac{1}{2\beta}\right)2^{1-\frac{1}{2\beta}}}{\pi\sqrt{\sigma_1^2(1-\rho^2)}\Gamma\left(1+\frac{1}{\beta}\right)}\exp\left(-\frac{1}{2}\left[(x,b)\begin{pmatrix}\sigma_1^2 & \sigma_{12}\\ \sigma_{12} & \sigma_2^2\end{pmatrix}^{-1}(x,b)'\right]^\beta+\frac{1}{2}\left(\frac{b^2}{\sigma_2^2}\right)^\beta\right) \tag{14}$$

**Proof.** By Theorem 2.6.3 (Fang& Zhang, 1990, p.66) we can see that (12) and (13) follow from: $\Sigma_{11} = \sigma_1^2$, $\Sigma_{22} = \sigma_2^2$, $\mu^{(1)} = \mu^{(2)} = 0$, n = 1. Next we prove (14). The conditional distribution of X given Y = b is f(x | b) = f(x, b) / $f_Y$(b). Then (14) follows from (11) and (13).

**Theorem 5.2.** Suppose (X, Y, T) is distributed as 3-dimensional MPE distribution. That is

$$\begin{pmatrix}X\\ Y\\ T\end{pmatrix}\sim MPE_3(\mu,\Sigma,\beta),$$

where

$$\mu=\begin{pmatrix}0\\ 0\\ \mu_t\end{pmatrix},\quad \Sigma=\begin{pmatrix}\sigma_1^2 & \sigma_{12} & \sigma_{13}\\ \sigma_{12} & \sigma_2^2 & \sigma_{23}\\ \sigma_{13} & \sigma_{23} & \sigma_3^2\end{pmatrix},\quad rk(\Sigma)=3.$$

The joint p.d.f. of (X,Y,T) is given as follows

$$f(x,y,t)=\frac{3\Gamma\left(\frac{3}{2}\right)}{\pi^{\frac{3}{2}}\sqrt{|\Sigma|}\Gamma\left(1+\frac{3}{2\beta}\right)2^{1+\frac{3}{2\beta}}}\exp\left(-\frac{1}{2}\left[z\Sigma^{-1}z'\right]^\beta\right), \tag{15}$$

where z = (x, y, t - $\mu_t$). We then have the conditional distribution of T given (X, Y) = (a, b) as follows

$$f(t\mid a,b)=\frac{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)\sqrt{\sigma_1^2\sigma_2^2\left(1-\rho^2\right)}\Gamma\left(1+\frac{1}{\beta}\right)}{\sqrt{\pi}\sqrt{|\Sigma|}\Gamma\left(1+\frac{3}{2\beta}\right)2^{\frac{1}{2\beta}}}\exp\left(-\frac{1}{2}\left[z\Sigma^{-1}z'\right]^{\beta}+\frac{1}{2}\left[(a,b)\begin{pmatrix}\sigma_1^2 & \sigma_{12}\\ \sigma_{12} & \sigma_2^2\end{pmatrix}^{-1}\begin{pmatrix}a\\ b\end{pmatrix}\right]^{\beta}\right), \quad (16)$$

where z = (a, b, t - μ) and $\rho$ is the correlation coefficient between X and Y.

**Proof.** The conditional distribution of T given X = a, Y = b is f (t | a, b) = f(a, b, t) / $f_{X,Y}$ (a,b). It is an immediate result that (16) follows from (11) and (12) or (13).

## 6. SOME NON-CENTRAL DISTRIBUTIONS

In the classic multivariate analysis we derive the non-central chi-square, non-central t and non-central F distributions under the normal distribution assumptions. We can generalize these three non-central distributions under ECD assumptions. We utilize Theorem 2.9.2, Theorem 2.9.4 and Theorem 2.9.5 (Fang and Zhang, 1990, p.82-87) to derive the following three theorems.

Suppose x ~ MPE ($\mu$, $\mathbf{I}_n$, $\beta$) we define u = $\mathbf{x}'\mathbf{x}$ as the generalized $\chi^2$ distribution and write u~ G $\chi^2_n$ ($\delta^2$, f), where $\delta^2 = \mu'\mu$ is the non-centrality parameter and function f is the same as defined in definition (2). There exists an orthogonal matrix Γ such that

$\Gamma\mu = \left(\|\mu\|,0,0,\ldots,0\right)' = \left(\delta,0,0,\ldots,0\right)'$. We denote $\Gamma\mu$ by $\nu$.. After applying the orthogonal transformation MPE ($\mu$, $\mathbf{I}_n$, $\beta$) becomes MPE ($\nu$, $\mathbf{I}_n$, $\beta$).

**Theorem 6.1.** The density of the generalized chi-square U = $\mathbf{x}'\mathbf{x}$ = G $\chi^2_n(\delta^2, f)$ is

$$\frac{\Gamma\left(\frac{n}{2}\right)\beta\mu^{\frac{n}{2}-1}}{\pi^{\frac{1}{2}}\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n}{2\beta}\right)2^{\frac{n}{2\beta}}}\int_0^{\pi}\exp\left(-\frac{1}{2}\left[\mu+\delta^2-2\sqrt{\delta^2\mu}\cos\theta\right]^{\beta}\right)\sin^{n-2}\theta d\theta. \quad (17)$$

**Proof.** The above theorem can be proved by using Theorem 2.9.2 (Fang and Zhang, 1990, p.82) and the definition of (1) of this paper.

Suppose $\mathbf{x} = (x_1, \mathbf{x}^{(2)\prime})'$ ~MPE(μ, $\mathbf{I}_{n+1}$, $\beta$) , where $\mathbf{x}^{(2)}$ is n x 1 and $\mu = (\delta, 0, \ldots, 0)'$ , the generalized non-central t distribution is defined as

$$T=\frac{\sqrt{n}x_1}{\sqrt{X^{(2)\prime}X^{(2)}}}. \quad (18)$$

and we write T ~ G $t_n$ (δ, f) where f(($\mathbf{x} - \mu$)$'$ ($\mathbf{x} - \mu$)) is the density of **x**.

**Theorem 6.2.** The generalized non-central t distribution is defined as above whose density function can be written as

$$\frac{n^{\frac{n}{2}}(n+1)\Gamma\left(\frac{n+1}{2}\right)\left(n+t^2\right)^{-\frac{n+1}{2}}}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)\Gamma\left(1+\frac{n+1}{2\beta}\right)2^{\frac{n+1}{2\beta}}}\int_0^\infty \exp\left(-\frac{1}{2}\left(y^2-2\delta_1 y+\delta^2\right)^\beta\right)y^n dy,$$

$$-\infty < t < \infty, \quad where \quad \delta_1 = \frac{t\delta}{\sqrt{n+t^2}}.$$

**Proof.** We can prove the above theorem by using Theorem 2.9.4 (Fang and Zhang, 1990, p.85) and the definition of (2) of this paper.

If $\mathbf{x} = (\mathbf{x}^{(1)'}, \mathbf{x}^{(2)'})' \sim$ MPE ($\mu$, $\mathbf{I}_{m+n}$, $\beta$), where $\mathbf{x}^{(1)}$ is m x 1, $\mathbf{x}^{(2)}$ is n x 1 and $\mu = (\mathrm{v}', \mathbf{0}')'$, v is m x 1, we define the generalized non-central F distribution as follows:

$$F = \frac{nX^{(1)'}X^{(1)}}{mX^{(2)'}X^{(2)}}.$$

and we write F~ G $F_{m,n}$ ($\delta^2$, f), $f((\mathbf{x}-\mu)'(\mathbf{x}-\mu))$ is the density of $\mathbf{x}$, $\delta^2 = \mu'\mu = \mathrm{v}'\mathrm{v}$.

**Theorem 6.3.** The generalized non-central F distribution is defined as above whose density function can be written as

$$\frac{(m+n)\left(\frac{m}{n}\right)\Gamma\left(\frac{m+n}{2}\right)\left(\frac{m}{n}F\right)^{\frac{m-2}{2}}\left(1+\frac{m}{n}F\right)^{-\frac{m+n}{2}}}{\Gamma\left(1+\frac{m+n}{2\beta}\right)\Gamma\left(\frac{m-1}{2}\right)\Gamma\left(\frac{n}{2}\right)2^{\frac{m+n}{2\beta}}}\times$$

$$\int_0^\pi\int_0^\infty \exp\left(-\frac{1}{2}\left(y^2-2\delta_1 y\cos\theta+\delta^2\right)^\beta\right)y^{m+n-1}\sin^{m-2}\theta d\theta dy, F>0,$$

$$where \quad \delta_1 = \left(\sqrt{\frac{mF}{n+mF}}\right)\delta.$$

**Proof.** We can prove the above theorem by using Theorem 2.9.5 (Fang and Zhang, 1990, p.86) and the definition of (2) of this paper.

## 7. CONCLUSION

Several extensions and developments were made in this paper. We derived the dispersion matrix of an MPE distribution.. Our method is much simpler than the method of Gomez et al. (1998). Second, we showed a relationship between any MPE distribution and a Gamma distribution. Third, we obtained a link between the UPE distribution and a $\chi^2$ distribution. Fourth, we extend Fan and Xu's (1987) from the normal to the UPE distribution and found an explicit expression using a Gamma distribution to calculate the probability of the UPE distribution over an interval. Fifth, we found a representation of the characteristic function for any UPE distribution. Finally, we derived the p.d.f. of the generalized non-central chi-square, the generalized non-central t, and the generalized non-central F distribution based on the MPE distributions.

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