

REGULARITY OF WEAK SOLUTIONS OF THE COMPRESSIBLE NAVIER-STOKES EQUATIONS

HI JUN CHOE AND BUM JA JIN

ABSTRACT. In this paper, we assume a density with integrability on the space $L^\infty(0, T; L^{q_0})$ for some q_0 and $T > 0$. Under the assumption on the density, we obtain a regularity result for the weak solutions to the compressible Navier-Stokes equations. That is, the supremum of the density is finite and the infimum of the density is positive in the domain $\mathbf{T}^3 \times (0, T)$. Moreover, Moser type iteration scheme is developed for L^∞ norm estimate for the velocity.

1. Introduction

It has been well known that the compressible Navier-Stokes equations have the unique classical solution for arbitrary large initial data and external force, which is local solution in time (see Itaya [7] and Tani [12]). On the other hand, the global time existence of the strong solution with small initial data and small force has been proved by Matsumura and Nishida [10], Zajackowski [13] and etc. The existence problem independent of the size of the initial data and external force has been many mathematicians' concern. As far as weak solutions are concerned, in 1991, Lions [9] claimed the global time existence of the solutions of the isentropic compressible Navier-Stokes equations for arbitrary size of data when the pressures depend on the large power of the density. More precisely, he considered the isentropic compressible Navier-Stokes

Received November 28, 2002.

2000 Mathematics Subject Classification: Primary 76N10, 35D10.

Key words and phrases: compressible, regularity, generalized solution.

The first author is supported by KOSEF and KRF. The second author is supported by the Brain Korea 21 project.

equations for the periodic domain $\mathbf{T}^N = \mathbf{R}^N / \mathbf{Z}^N$:

$$(1.1) \quad \begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0 & \text{in } (0, T) \times \mathbf{T}^N, \\ \rho \mathbf{u}_t + \rho \mathbf{u} \cdot \nabla \mathbf{u} - \mu \Delta \mathbf{u} - (\lambda + \mu) \nabla \operatorname{div} \mathbf{u} + \nabla p(\rho) \\ = \rho \mathbf{f} & \text{in } (0, T) \times \mathbf{T}^N, \end{cases}$$

where the pressure p is an increasing function satisfying that

$$(1.2) \quad \begin{cases} \int_0^1 p(s)/s^2 ds < +\infty, \\ \liminf_{s \rightarrow +\infty} p(s)/s^\gamma > 0. \end{cases}$$

His results for periodic domains are the followings: we let γ be a constant such that $\gamma > \frac{N}{2}$ if $N \geq 4$, $\gamma > \frac{9}{5}$ if $N = 3$ and $\gamma > \frac{3}{2}$ if $N = 2$. Also we assume

$$\rho_0 \in L^1(\mathbf{T}^N) \cap L^\gamma(\mathbf{T}^N), \quad \rho_0 \geq 0 \text{ and } \rho_0 |\mathbf{u}_0|^2 \in L^1(\mathbf{T}^N).$$

Then, there exists a weak solution $(\rho, \mathbf{u}) \in L^\infty(0, \infty; L^\gamma(\mathbf{T}^N)) \times L^2(0, \infty; H^1(\mathbf{T}^N))^3$ satisfying in addition: $\rho \in C([0, \infty); L^r(\mathbf{T}^N))$ if $1 \leq r < \gamma$, $\rho |\mathbf{u}|^2 \in L^\infty(0, \infty; L^1(\mathbf{T}^N))$, $\rho \in L^q_{loc}([0, \infty); L^q(\mathbf{T}^N))$ for $1 \leq q \leq \gamma - 1 + \frac{2\gamma}{N}$. Moreover, when $\mathbf{f} \equiv 0$, for almost all $t \geq 0$, we have the energy inequality

$$(1.3) \quad \begin{aligned} & \int_{\mathbf{T}^N} \left(\frac{1}{2} \rho |\mathbf{u}|^2 + \frac{1}{\gamma - 1} p(\rho) \right) (t, x) dx \\ & + \int_0^t ds \int_{\mathbf{T}^N} (\mu |\nabla \mathbf{u}|^2 + (\lambda + \mu) (\operatorname{div} \mathbf{u})^2) (s, x) dx ds \\ & \leq \int_{\mathbf{T}^N} \left(\frac{1}{2} \rho_0 |\mathbf{u}_0|^2 + \frac{1}{\gamma - 1} p(\rho_0) \right) (x) dx. \end{aligned}$$

Later on, Feireisl, Novotny and Petzeltova [1] improved the above existence result for the case $\gamma > 3/2$ in the three dimensional domain. See also the results of Feireisl and Petzeltova [2], [3], [4] for the related problems.

Now, it has been issued if the weak solution is in fact strong solution or not. Indeed, for a short time and regular enough data, the unique existence of strong solution has been well known. And the weak solution should be consistent with strong solution for the short time. However, for less regular data, still it has been open question if a strong solution could be developed for any time. In 1997, Desjardins [5] showed the existence of globally existing weak solution whose regularity can be extended for some positive time. More precisely, he considered the isentropic compressible Navier-Stokes equations (1), where the pressure p

satisfies the assumptions (1.2) and the following assumptions

$$(1.4) \quad \begin{cases} \inf_{s>0} \frac{p(s)^2}{s \int_0^s p(\tau)^2/\tau^2 d\tau} > \frac{1}{2}, \\ \limsup_{t \rightarrow 0^+} p(t)/\sqrt{t} < +\infty. \end{cases}$$

We observe that the assumption (1.2) and (1.4) hold for the flow with $p(\rho) = R\rho^\gamma$ for $R > 0$ and $\gamma > 1$. The following results are his works: suppose the initial data (ρ_0, \mathbf{u}_0) and external force \mathbf{f} satisfy $\rho_0 \in L^\infty(\mathbf{T}^N)$, $\rho_0 \geq 0$, $\mathbf{u}_0 \in H^1(\mathbf{T}^N)^3$, and $\mathbf{f} \in L^1(0, T; L^{\frac{\gamma}{\gamma-1}}(\mathbf{T}^N)) \cap L^2((0, T) \times \mathbf{T}^N)$ for all $T > 0$. Then the followings hold:

- (i) there exists $T_0 \in (0, \infty]$ and a weak solution (ρ, \mathbf{u}) to the Navier-Stokes equations in $(0, T_0)$ such that for all $T < T_0$

$$\rho \in L^\infty((0, T) \times \mathbf{T}^N) \cap C([0, T]; L^q(\mathbf{T}^N)) \quad \text{for all } q \in [1, \infty),$$

and

$$\begin{cases} \sqrt{\rho} \partial_t \mathbf{u} \in L^2((0, T) \times \mathbf{T}^N)^3 \\ P\mathbf{u} \in L^2((0, T); H^2(\mathbf{T}^N))^3 \\ G = (\lambda + 2\mu) \operatorname{div} \mathbf{u} - p(\rho) \in L^2(0, T; H^1(\mathbf{T}^N)), \\ \nabla \mathbf{u} \in L^\infty(0, T; L^2(\mathbf{T}^N))^3, \end{cases}$$

where P denotes the Leray projection on the divergence-free vector fields.

- (ii) in the case $N = 2$, the above regularity properties hold as long as $\sup_{t \in [0, T]} |\rho(t, \cdot)|_{L^\infty(\mathbf{T}^N)} < \infty$.

Throughout this paper, we assume the initial data ρ_0 has positive lower bound and finite upper bound, that is,

$$(1.5) \quad 0 < m \leq \rho_0(x) < M < +\infty$$

for some positive real number m and M and we consider a weak solution which has L^{q_0} integrability for some positive time interval. We develop a blow up criterion of some regularity, which is similar to Desjardins's (see [5]) in the case of three dimensional space and the zero bulk viscosity. The following is precise statement of our results.

THEOREM 1.1. *Let $N = 3$ and $\gamma > 1$. Suppose that (ρ, \mathbf{u}) is a weak solutions to the isentropic compressible Navier-Stokes equations with zero bulk viscosity with the initial data satisfying (1.5) and the following integrability conditions*

$$\rho_0 \in L^\infty(\mathbf{T}^3) \text{ and } \mathbf{u}_0 \in H^1(\mathbf{T}^3) \cap L^\infty(\mathbf{T}^3).$$

If $\rho \in L^\infty(0, T; L^{q_0}(\mathbf{T}^3))$ for some large q_0 and for a positive time T , then

$$(1.6) \quad 0 < \inf_{(x,t) \in \mathbf{T}^3 \times [0,T]} \rho(x, t) \leq \sup_{(x,t) \in \mathbf{T}^3 \times [0,T]} \rho(x, t) < \infty,$$

$$(1.7) \quad \sup_{(t,x) \in [0,T] \times \mathbf{T}^3} |\mathbf{u}(t, x)| < +\infty.$$

Moreover, \mathbf{u} and ρ satisfy the regularity

$$(1.8) \quad \rho \in L^\infty((0, T) \times \mathbf{T}^3) \cap C([0, T]; L^q(\mathbf{T}^3))$$

for all $q \in [1, \infty)$ and

$$(1.9) \quad \begin{cases} \sqrt{\rho} \partial_t \mathbf{u} \in L^2((0, T) \times \mathbf{T}^3)^3, \\ P\mathbf{u} \in L^2(0, T; H^2(\mathbf{T}^3))^3, \\ G = \mu \operatorname{div} \mathbf{u} - p(\rho) \in L^2(0, T; H^1(\mathbf{T}^3)), \\ \nabla \mathbf{u} \in L^\infty(0, T; L^2(\mathbf{T}^3))^3, \end{cases}$$

where P denotes the Leray projection on the divergence-free vector fields. Here q_0 depends only on γ .

2. A priori bounds for \mathbf{u}

DEFINITION 2.1. Suppose that $\rho \in L^\infty(0, T; L^1(\mathbf{T}^3)) \cap L^\infty(0, T; L^\gamma(\mathbf{T}^3))$ and $\mathbf{u} \in L^2(0, T; H^1(\mathbf{T}^3)^3)$ satisfy the equations (1) in the sense of distributions in $\mathbf{T}^3 \times [0, T]$ with initial data ρ_0 and \mathbf{u}_0 , that is,

$$\begin{aligned} & \int_0^T \int_{\mathbf{T}^3} dt dx \{ \mu \nabla \mathbf{u} \cdot \nabla \phi + (\mu + \lambda) \operatorname{div} \mathbf{u} \operatorname{div} \phi - \rho \mathbf{u} \cdot \frac{\partial \phi}{\partial t} \} \\ &= \int_0^T \langle \rho f, \phi \rangle_{H^{-1} \times H_0^1} dt + \int_{\mathbf{T}^3} \rho_0 \mathbf{u}_0 \cdot \phi(x, 0) dx, \end{aligned}$$

for all $\phi \in C_0^\infty(\mathbf{T}^3 \times [0, T])^3$ and

$$\int_0^T \int_{\mathbf{T}^3} dt dx \{ \rho \frac{\partial \psi}{\partial t} + \rho \mathbf{u} \cdot \nabla \psi \} = \int_0^T \rho_0 \psi(x, 0) dx,$$

for all $\psi \in C_0^\infty(\mathbf{T}^3 \times [0, T])$. Here \langle, \rangle denotes a duality pairing between $H^{-1}(\mathbf{T}^3)$ and $H_0^1(\mathbf{T}^3)$. Then we call (ρ, \mathbf{u}) to be a weak solution of the isentropic compressible Navier-Stokes equations (1) in $\mathbf{T}^3 \times [0, T]$ with initial data ρ_0 and \mathbf{u}_0 .

Throughout this paper, all the a priori estimates in this paper are based on the assumption that ρ and \mathbf{u} are C^∞ for the time interval. For C^∞ given data, it can be justified by the global time existence of weak solution and short time existence of strong solution. Furthermore, it has been well known that the length of time interval of existence of smooth solution depends on the size of the given data and the smoothness of solution depends on the smoothness of given data. For $\rho_0 \in L^\infty$, $\mathbf{u}_0 \in H^1 \cap L^\infty$, we can consider C^∞ approximation $\rho_{0,\epsilon}$, $\mathbf{u}_{0,\epsilon}$ of ρ_0 , \mathbf{u}_0 and the solutions corresponding to $\rho_{0,\epsilon}$, $\mathbf{u}_{0,\epsilon}$. And then derive uniform a priori estimates independent of ϵ .

Let (ρ, \mathbf{u}) be a solution of the isentropic compressible Navier-Stokes equations with zero bulk viscosity

$$(2.1) \quad \begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \text{ for } (t, x) \in (0, T) \times \mathbf{T}^3, \\ \rho \mathbf{u}_t + \rho \mathbf{u} \cdot \nabla \mathbf{u} - \mu \Delta \mathbf{u} + \nabla p(\rho) = \rho f, \end{cases}$$

where we let $p = R\rho^\gamma$, $R > 0$ and $\gamma > 1$, without loss of generality. We let $Q_T = [0, T) \times \mathbf{T}^3$, $m_T = \inf_{(t,x) \in Q_T} \rho(x, t)$ and $M_T = \sup_{(t,x) \in Q_T} \rho(x, t)$.

Standard energy estimates show that

$$\begin{aligned} & \int_{\mathbf{T}^3} (\rho + g(\rho))(t, x) dx + \int_{\mathbf{T}^3} \frac{1}{2}(\rho \mathbf{u}^2)(t, x) dx \\ & + \int_0^t \int_{\mathbf{T}^3} \mu |\nabla \mathbf{u}(s, x)|^2 dx ds \\ & \leq \int_{\mathbf{T}^3} (\rho_0 + g(\rho_0))(x) dx + \int_{\mathbf{T}^3} \frac{1}{2}(\rho_0 \mathbf{u}_0^2)(x) dx + \int_{\mathbf{T}^3} (\rho f \mathbf{u})(t, x) dx, \end{aligned}$$

where $g(s) = s \int_0^s \frac{p(w)}{w^2} dw$. For the simplicity of our computations, we let $f = 0$. Let α be a nonnegative real number. If we multiply $|\mathbf{u}|^\alpha \mathbf{u}$ to (2.1) and integrate over \mathbf{T}^3 , then we obtain the inequality

$$\begin{aligned} (2.2) \quad & \frac{1}{\alpha + 2} \frac{d}{dt} \int_{\mathbf{T}^3} (\rho |\mathbf{u}|^{\alpha+2})(t, x) dx + \int_{\mathbf{T}^3} (\mu |\nabla \mathbf{u}|^2 |\mathbf{u}|^\alpha)(t, x) dx \\ & + \frac{\alpha}{2} \int_{\mathbf{T}^3} (\mu |\mathbf{u}|^{\alpha-2} |\nabla |\mathbf{u}|^2|^2)(t, x) dx \\ & = \int_{\mathbf{T}^3} (R\rho^\gamma |\mathbf{u}|^\alpha \operatorname{div} \mathbf{u})(t, x) dx + \alpha \int_{\mathbf{T}^3} (R\rho^\gamma |\mathbf{u}|^{\alpha-2} \nabla |\mathbf{u}|^2 \cdot \mathbf{u})(t, x) dx \\ & = I + II. \end{aligned}$$

We remark that C^∞ approximation of $(\rho_{0,\epsilon}, \mathbf{u}_{0,\epsilon})$ justifies the estimates of test function $|\mathbf{u}|^\alpha \mathbf{u}$.

Estimating I and II separately, the following lemma is obtained.

LEMMA 2.1. *Suppose that $\rho \in L^\infty(0, T; L^{\frac{(2\gamma-1)\alpha+4}{2}}(\mathbf{T}^3))$. Then, for all $t \leq T$ the following inequality holds.*

$$\begin{aligned} & \int_{\mathbf{T}^3} (\rho|\mathbf{u}|^{\alpha+2})(x, t)dx \\ & \leq 2^{\frac{\alpha}{2}} \int_{\mathbf{T}^3} (\rho_0|\mathbf{u}_0|^{\alpha+2})(x)dx \\ & \quad + (2R)^{\alpha+2}(\alpha + 1)^{\frac{\alpha+2}{2}} \sup_{0 \leq t \leq T} \|\rho\|_{L^{\frac{(2\gamma-1)\alpha+4}{2}}}^{\frac{(2\gamma-1)\alpha+4\gamma}{2}} t^{\frac{\alpha+2}{2}}. \end{aligned}$$

Proof. From Hölder inequality and Young’s inequality, we have that

$$\begin{aligned} |I| & \leq R\|\rho\|_{L^{\frac{(2\gamma-1)\alpha+4}{2}}}^{\frac{(2\gamma-1)\alpha+4\gamma}{2(\alpha+2)}} \left(\int_{\mathbf{T}^3} (|\mathbf{u}|^\alpha|\nabla\mathbf{u}|^2)(t, x)dx \right)^{\frac{1}{2}} \\ & \quad \times \left(\int_{\mathbf{T}^3} (\rho|\mathbf{u}|^{\alpha+2})(t, x)dx \right)^{\frac{\alpha}{2(\alpha+2)}} \\ & \leq \frac{1}{2} \int_{\mathbf{T}^3} (|\mathbf{u}|^\alpha|\nabla\mathbf{u}|^2)(t, x)dx + 2R^2 \sup_{0 \leq t \leq T} \|\rho\|_{L^{\frac{(2\gamma-1)\alpha+4}{2}}}^{\frac{(2\gamma-1)\alpha+4\gamma}{(\alpha+2)}} \\ & \quad \times \left(\int_{\mathbf{T}^3} (\rho|\mathbf{u}|^{\alpha+2})(t, x)dx \right)^{\frac{\alpha}{(\alpha+2)}}, \end{aligned}$$

and

$$\begin{aligned} |II| & \leq \alpha R\|\rho\|_{L^{\frac{(2\gamma-1)\alpha+4}{2}}}^{\frac{(2\gamma-1)\alpha+4\gamma}{2(\alpha+2)}} \\ & \quad \times \left(\int_{\mathbf{T}^3} (|\mathbf{u}|^{\alpha-2}|\nabla|\mathbf{u}|^2|^2)(t, x)dx \right)^{\frac{1}{2}} \\ & \quad \times \left(\int_{\mathbf{T}^3} (\rho|\mathbf{u}|^{\alpha+2})(t, x)dx \right)^{\frac{\alpha}{2(\alpha+2)}} \\ & \leq \frac{\alpha}{4} \int_{\mathbf{T}^3} (|\mathbf{u}|^{\alpha-2}|\nabla|\mathbf{u}|^2|^2)(t, x)dx + \alpha R^2 \sup_{0 \leq t \leq T} \|\rho\|_{L^{\frac{(2\gamma-1)\alpha+4}{2}}}^{\frac{(2\gamma-1)\alpha+4\gamma}{(\alpha+2)}} \\ & \quad \times \left(\int_{\mathbf{T}^3} (\rho|\mathbf{u}|^{\alpha+2})(t, x)dx \right)^{\frac{\alpha}{\alpha+2}}. \end{aligned}$$

Hence from (2.2) we get the following inequality

$$\frac{1}{\alpha + 2} \frac{d}{dt} \int_{\mathbf{T}^3} \rho|\mathbf{u}|^{\alpha+2}(t, x)dx + \frac{1}{2} \int_{\mathbf{T}^3} |\nabla\mathbf{u}|^2|\mathbf{u}|^\alpha(t, x)dx$$

$$\begin{aligned}
 & + \frac{\alpha}{4} \int_{\mathbf{T}^3} (|\mathbf{u}|^{\alpha-2} |\nabla |\mathbf{u}|^2|^2)(t, x) dx \\
 \leq & 2R^2 \sup_{0 \leq t \leq T} \|\rho\|_{L^{\frac{(2\gamma-1)\alpha+4\gamma}{(\alpha+2)}}} \left(\int_{\mathbf{T}^3} (\rho |\mathbf{u}|^{\alpha+2})(t, x) dx \right)^{\frac{\alpha}{\alpha+2}} \\
 & + \alpha R^2 \sup_{0 \leq t \leq T} \|\rho\|_{L^{\frac{(2\gamma-1)\alpha+4\gamma}{(\alpha+2)}}} \left(\int_{\mathbf{T}^3} (\rho |\mathbf{u}|^{\alpha+2})(t, x) dx \right)^{\frac{\alpha}{\alpha+2}}.
 \end{aligned}$$

Since α is nonnegative, we get

$$\begin{aligned}
 & \frac{1}{\alpha+2} \frac{d}{dt} \int_{\mathbf{T}^3} (\rho |\mathbf{u}|^{\alpha+2})(t, x) dx \\
 \leq & 2R^2 \sup_{0 \leq t \leq T} \|\rho\|_{L^{\frac{(2\gamma-1)\alpha+4\gamma}{(\alpha+2)}}} \left(\int_{\mathbf{T}^3} (\rho |\mathbf{u}|^{\alpha+2})(t, x) dx \right)^{\frac{\alpha}{\alpha+2}} \\
 & + \alpha R^2 \sup_{0 \leq t \leq T} \|\rho\|_{L^{\frac{(2\gamma-1)\alpha+4\gamma}{(\alpha+2)}}} \left(\int_{\mathbf{T}^3} (\rho |\mathbf{u}|^{\alpha+2})(t, x) dx \right)^{\frac{\alpha}{\alpha+2}} \\
 = & (\alpha+2)R^2 \sup_{0 \leq t \leq T} \|\rho\|_{L^{\frac{(2\gamma-1)\alpha+4\gamma}{(\alpha+2)}}} \left(\int_{\mathbf{T}^3} (\rho |\mathbf{u}|^{\alpha+2})(t, x) dx \right)^{\frac{\alpha}{\alpha+2}}.
 \end{aligned}$$

If we set $Y(t) = \int_{\mathbf{T}^3} (\rho |\mathbf{u}|^{\alpha+2})(t, x) dx$, $Y(t)$ satisfies the following inequality

$$\frac{1}{\alpha+2} \frac{d}{dt} Y(t) \leq 2(\alpha+2)R^2 \sup_{0 \leq t \leq T} \|\rho\|_{L^{\frac{(2\gamma-1)\alpha+4\gamma}{(\alpha+2)}}} [Y(t)]^{\frac{\alpha}{\alpha+2}},$$

with initial condition

$$Y(0) = \int_{\mathbf{T}^3} (\rho_0 |\mathbf{u}_0|^{\alpha+2})(x) dx.$$

Since α is nonnegative, we have

$$\frac{d}{dt} \left(Y(t)^{\frac{2}{\alpha+2}} \right) \leq 2(\alpha+2)R^2 \sup_{0 \leq t \leq T} \|\rho\|_{L^{\frac{(2\gamma-1)\alpha+4\gamma}{(\alpha+2)}}} [Y(t)]^{\frac{\alpha}{\alpha+2}},$$

and hence if we integrate the above inequality over $(0, t)$, then we obtain the inequality

$$Y(t)^{\frac{2}{\alpha+2}} \leq Y(0)^{\frac{2}{\alpha+2}} + 2(\alpha+2)R^2 \sup_{0 \leq t \leq T} \|\rho\|_{L^{\frac{(2\gamma-1)\alpha+4\gamma}{(\alpha+2)}}} t.$$

This reduces to the result of our lemma. □

3. A priori bounds for $\nabla P\mathbf{u}$ and ∇G

Let P be the Leray projection operator from $L^2(\mathbf{T}^3)$ to divergence free vector fields. In this section, we shall derive the H^2 norm estimates of $P\mathbf{u}$ and G . At first, we take the Leray projection operator to the equation (2.1) and we get the equation

$$P(\rho\mathbf{u}_t + \rho\mathbf{u} \cdot \nabla\mathbf{u}) - \mu\Delta P\mathbf{u} = 0.$$

Hence from the above equation, $\Delta P\mathbf{u}$ can be written as

$$(3.1) \quad -\mu\Delta P\mathbf{u} = -P(\rho\mathbf{u}_t + \rho\mathbf{u} \cdot \nabla\mathbf{u}).$$

Secondly, we take divergence operator to the equation (2.1). Then we obtain the equation

$$\operatorname{div}(\rho\mathbf{u}_t + \rho\mathbf{u} \cdot \nabla\mathbf{u}) - \Delta(\mu\operatorname{div}\mathbf{u} - p(\rho)) = 0.$$

Set $G = \mu\operatorname{div}\mathbf{u} - R\rho^\gamma$. Thus from the above equation, G can be written as

$$(3.2) \quad G = \Delta^{-1}\operatorname{div}(\rho\mathbf{u}_t + \rho\mathbf{u} \cdot \nabla\mathbf{u}).$$

From (3.1) and (3.2), we get the Stokes equations

$$\begin{aligned} -\mu\Delta P\mathbf{u} + \nabla G &= F, \\ \operatorname{div} P\mathbf{u} &= 0, \end{aligned}$$

where $F = F^1 + F^2$ and F^1 and F^2 are defined by

$$F^1 = -P(\rho\mathbf{u}_t + \rho\mathbf{u} \cdot \nabla\mathbf{u}),$$

and

$$F^2 = \nabla\Delta^{-1}(\operatorname{div}(\rho\mathbf{u}_t + \rho\mathbf{u} \cdot \nabla\mathbf{u})).$$

From the L^r theory for Stokes operator (see Theorem 6.1 of Chapter IV in Galdi [6]), we have the inequality

$$(3.3) \quad \|\nabla^2 P\mathbf{u}\|_{L^r(\mathbf{T}^3)} + \|\nabla G\|_{L^r(\mathbf{T}^3)} \leq \|F\|_{L^r(\mathbf{T}^3)} \text{ for } 1 < r < \infty.$$

Since both Leray projection operator and Riesz transform, that is, P and $\partial_i\partial_j\Delta^{-1}$, are strong L^r bounded operator for $1 < r < +\infty$,

$$\|F\|_{L^r(\mathbf{T}^3)} \leq C(\|\rho\mathbf{u}_t\|_{L^r(\mathbf{T}^3)} + \|\rho\mathbf{u} \cdot \nabla\mathbf{u}\|_{L^r(\mathbf{T}^3)}),$$

where C depends only on the dimension N . Let $1 < r < 2$. We note that

$$\|\rho\mathbf{u}_t\|_{L^r} \leq \|\rho\|_{L^{\frac{r}{2-r}}}^{\frac{1}{2}} \left(\int \rho|\mathbf{u}_t|^2 \right)^{\frac{1}{2}}$$

and

$$\|\rho\mathbf{u} \cdot \nabla\mathbf{u}\|_{L^r} \leq \|\rho\|_{L^{\frac{5r-2}{4r}}}^{\frac{5r-2}{4r}} \left(\int_{\mathbf{T}^3} \rho|\mathbf{u}|^{\frac{4r}{2-r}} \right)^{\frac{2-r}{4r}} \|\nabla\mathbf{u}\|_{L^2(\mathbf{T}^3)}.$$

Thus, if we integrate (3.3) over $(0, T)$, we have the following lemma.

LEMMA 3.1. We let $\frac{6}{5} < r < 2$. Suppose that

$$\rho \in L^\infty(0, T; L^{p_0}(\mathbf{T}^3)),$$

where $p_0(r) = \max\{\frac{5r-2}{2-r}, \frac{(2\gamma-1)\alpha+4\gamma}{2}\}$ for $\alpha + 2 = \frac{4r}{2-r}$. Then

$$\begin{aligned} & \int_0^T \|\nabla^2 P\mathbf{u}\|_{L^r(\mathbf{T}^3)}^2 + \|\nabla G\|_{L^r(\mathbf{T}^3)}^2 dt \\ \leq & C \sup_{0 < t < T} \|\rho\|_{L^{\frac{r}{2-r}}}^2 \int_{Q_T} \rho |\mathbf{u}_t|^2 dx dt \\ & + C \sup_{0 < t < T} \left(\|\rho\|_{L^{\frac{5r-2}{2-r}}}^{\frac{5r-2}{2-r}} \left(\int_{\mathbf{T}^3} \rho |\mathbf{u}|^{\frac{4r}{2-r}} dx \right)^{\frac{2-r}{2r}} \right) \int_{Q_T} |\nabla \mathbf{u}|^2 dx dt. \end{aligned}$$

Let $t \leq T$. We multiply \mathbf{u}_t to the momentum equation (2.1) and integrate over $(0, t) \times \mathbf{T}^3$, then we have the inequality

$$\begin{aligned} & \int_0^t \int_{\mathbf{T}^3} \rho |\mathbf{u}_t|^2(s, x) dx ds + \frac{\mu}{2} \sup_{0 \leq s \leq t} \int_{\mathbf{T}^3} |\nabla \mathbf{u}(s, x)|^2 dx \\ & + \int_0^t \int_{\mathbf{T}^3} R \nabla \rho^\gamma \cdot \mathbf{u}_t dx ds \\ (3.4) \quad \leq & \frac{\mu}{2} \int_{\mathbf{T}^3} |\nabla \mathbf{u}_0(x)| dx + \int_0^t \int_{\mathbf{T}^3} |\rho(\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \mathbf{u}_t|(s, x) dx ds. \end{aligned}$$

Let

$$I = \int_0^t \int_{\mathbf{T}^3} \nabla(R\rho^\gamma) \cdot \mathbf{u}_t dx ds.$$

From the conservation of mass, we find that

$$(R\rho^\gamma)_t = R\gamma\rho^{\gamma-1}\rho_t = -R\gamma\rho^{\gamma-1}\operatorname{div}(\rho\mathbf{u}) = -\mathbf{u} \cdot \nabla(R\rho^\gamma) - R\gamma\rho^\gamma \operatorname{div} \mathbf{u}.$$

Thus I can be written as

$$\begin{aligned} & \int_{\mathbf{T}^3} R \nabla \rho^\gamma \cdot \mathbf{u}_t dx \\ = & -\frac{d}{dt} \int_{\mathbf{T}^3} R\rho^\gamma \operatorname{div} \mathbf{u} dx + \int_{\mathbf{T}^3} (R\rho^\gamma)_t \operatorname{div} \mathbf{u} dx \end{aligned}$$

$$\begin{aligned}
 &= -\frac{d}{dt} \int_{\mathbf{T}^3} R\rho^\gamma \operatorname{div} \mathbf{u} dx - \int_{\mathbf{T}^3} \operatorname{div} (R\rho^\gamma \mathbf{u}) \operatorname{div} \mathbf{u} dx \\
 &\quad - \int_{\mathbf{T}^3} R(\gamma - 1)\rho^\gamma (\operatorname{div} \mathbf{u})^2 dx \\
 &= -\frac{d}{dt} \int_{\mathbf{T}^3} R\rho^\gamma \operatorname{div} \mathbf{u} dx + \frac{1}{\mu} \int_{\mathbf{T}^3} R\rho^\gamma \mathbf{u} \cdot \nabla(G + R\rho^\gamma) dx \\
 &\quad - \int_{\mathbf{T}^3} \frac{1}{\mu^2} R(\gamma - 1)\rho^\gamma (G^2 - R^2\rho^{2\gamma} + 2\mu R\rho^\gamma \operatorname{div} \mathbf{u}) dx \\
 &= -\frac{d}{dt} \int_{\mathbf{T}^3} R\rho^\gamma \operatorname{div} \mathbf{u} dx \\
 &\quad + \frac{1}{\mu} \int_{\mathbf{T}^3} R\rho^\gamma \mathbf{u} \cdot \nabla G dx + \frac{1}{\mu^2} \int_{\mathbf{T}^3} R^3(\gamma - 1)\rho^{3\gamma} dx \\
 &\quad - \frac{1}{\mu^2} \int_{\mathbf{T}^3} R(\gamma - 1)\rho^\gamma G^2 dx - \frac{(4\gamma - 3)}{2\mu} \int_{\mathbf{T}^3} R^2\rho^{2\gamma} \operatorname{div} \mathbf{u} dx.
 \end{aligned}$$

If we integrate the above inequality over $(0, t)$, then we have

$$\begin{aligned}
 (3.5) \quad &\int_0^t I(s) ds \\
 &= - \int_{\mathbf{T}^3} (R\rho^\gamma \operatorname{div} \mathbf{u})(t, x) dx + \int_{\mathbf{T}^3} R\rho_0^\gamma \operatorname{div} \mathbf{u}_0 dx \\
 &\quad + \frac{1}{\mu^2} \int_0^t \int_{\mathbf{T}^3} R^3(\gamma - 1)\rho^{3\gamma} dx ds \\
 &\quad - \frac{1}{\mu^2} \int_0^t \int_{\mathbf{T}^3} R(\gamma - 1)\rho^\gamma G^2 dx dt - \frac{(4\gamma - 3)}{2\mu} \int_0^t \int_{\mathbf{T}^3} R^2\rho^{2\gamma} \operatorname{div} \mathbf{u} dx dt.
 \end{aligned}$$

From (3.5), (3.4) can be written again as

$$\begin{aligned}
 (3.6) \quad &\int_0^T \int_{\mathbf{T}^3} \rho |\mathbf{u}_t|^2 dx ds + \frac{\mu}{2} \sup_{0 \leq t \leq T} \int_{\mathbf{T}^3} |\nabla \mathbf{u}(t, x)|^2 dx \\
 &\quad + \frac{1}{\mu^2} \int_0^T \int_{\mathbf{T}^3} R^3(\gamma - 1)\rho^{3\gamma} dx ds \\
 &\leq \frac{\mu}{2} \int_{\mathbf{T}^3} |\nabla \mathbf{u}_0(x)|^2 dx + \int_{\mathbf{T}^3} |R\rho_0^\gamma \operatorname{div} \mathbf{u}_0|(x) dx \\
 &\quad + \sup_{0 \leq t \leq T} \int_{\mathbf{T}^3} |R\rho^\gamma \operatorname{div} \mathbf{u}|(t, x) dx
 \end{aligned}$$

$$\begin{aligned}
 & + \int_0^T \int_{\mathbf{T}^3} |R\rho^\gamma \mathbf{u} \cdot \nabla G| dxdt + \frac{1}{\mu^2} \int_0^T \int_{\mathbf{T}^3} |R(\gamma - 1)\rho^\gamma G^2| dxdt \\
 & + \int_0^T \int_{\mathbf{T}^3} |\rho(\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \mathbf{u}_t| dxdt + \frac{(4\gamma - 3)}{2\mu} \int_0^t \int_{\mathbf{T}^3} R^2 |\rho^{2\gamma} \operatorname{div} \mathbf{u}| dxdt.
 \end{aligned}$$

Estimating the right hand side of the previous inequality from the elliptic structure of (3.1) and (3.2), we get the following lemma.

LEMMA 3.2. Let $p_1(r) = \max\{4\gamma, \frac{(2\gamma-1)r+1}{r-1}, \frac{7\gamma-3}{5\gamma-6}, \frac{(2\gamma-1)\alpha+4}{2}, \frac{(2\gamma-1)\beta+4\gamma}{2}\}$. Here $\alpha + 2 = 2r$, and $\beta + 2 = \frac{6r}{3-r}$. We assume $\rho \in L^\infty([0, T]; L^{p_1}(\mathbf{T}^3))$. Then, for any given $\epsilon > 0$,

$$\begin{aligned}
 & \int_0^T \int_{\mathbf{T}^3} \rho |\mathbf{u}_t|^2 dxds + \frac{\mu}{2} \sup_{0 \leq t \leq T} \int_{\mathbf{T}^3} |\nabla \mathbf{u}(t, x)|^2 dx \\
 & + \frac{1}{\mu^2} \int_0^T \int_{\mathbf{T}^3} R^3 (\gamma - 1) \rho^{3\gamma} dxdt \\
 & \leq 3\epsilon \int_0^T (\|\nabla^2 P\mathbf{u}\|_{L^r(\mathbf{T}^3)}^2 + \|\nabla G\|_{L^r(\mathbf{T}^3)}^2) ds + C.
 \end{aligned}$$

Here C depends only on $\mu, R, \|\mathbf{u}_0\|_{H^1(\mathbf{T}^3)}, \|\rho_0\|_{L^{2\gamma}(\mathbf{T}^3)}, \sup_{0 < t < T} \|\rho\|_{L^{p_1}}$.

Proof. Let

$$\begin{aligned}
 i & = \sup_{0 \leq t \leq T} \int_{\mathbf{T}^3} |R\rho^\gamma \operatorname{div} \mathbf{u}|(t, x) dx, \\
 ii & = \int_0^T \int_{\mathbf{T}^3} |R\rho^\gamma \mathbf{u} \cdot \nabla G| dxdt, \\
 iii & = \frac{1}{\mu^2} \int_0^T \int_{\mathbf{T}^3} |R(\gamma - 1)\rho^\gamma G^2| dxdt, \\
 iv & = \int_0^T \int_{\mathbf{T}^3} |\rho(\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \mathbf{u}_t| dxdt
 \end{aligned}$$

and

$$v = \frac{R^2(4\gamma - 3)}{2\mu} \int_0^T \int_{\mathbf{T}^3} \rho^{2\gamma} |\operatorname{div} \mathbf{u}| dxdt.$$

We apply the following inequalities to estimates $i - iv$:

$$\begin{aligned}
 \|\nabla \mathbf{u}\|_{L^q} & \leq \|\nabla P\mathbf{u}\|_{L^q} + \|\operatorname{div} \mathbf{u}\|_{L^q} \text{ for } 1 < q < \infty, \\
 \|\mathbf{u}\|_{L^{\frac{3q}{3-q}}} & \leq \|\nabla \mathbf{u}\|_{L^q} \text{ for } 1 < q < 3.
 \end{aligned}$$

We also observe that $\|\operatorname{div} \mathbf{u}\|_{L^q} \leq \frac{1}{\mu} (\|G\|_{L^q} + \|\rho\|_{L^{\gamma q}}^\gamma)$.

i, *ii*, *iii* and *iv* can be estimated by

$$\begin{aligned}
 |i| &\leq \frac{1}{\mu} R^2 \sup_{0 \leq t \leq T} \|\rho\|_{L^{2\gamma}}^2 + \frac{\mu}{4} \sup_{0 \leq t \leq T} \int_{\mathbf{T}^3} |\nabla \mathbf{u}|^2(t, x) dx, \\
 |ii| &\leq \varepsilon \int_0^T \|\nabla G\|_{L^r(\mathbf{T}^3)}^2 dt \\
 &\quad + \frac{1}{4\varepsilon} R^2 \sup_{0 < t < T} \left(\left(\int \rho |\mathbf{u}|^{\frac{2r}{r-1}} \right)^{\frac{r-1}{r}} \|\rho\|_{L^{\frac{(2\gamma-1)r+1}{r-1}}}^{\frac{(2\gamma-1)r+1}{r-1}} \right) T, \\
 |iii| &\leq \varepsilon \int_0^T \|\nabla G\|_{L^2(\mathbf{T}^3)}^2 dt \\
 &\quad + \frac{1}{4\varepsilon} R^2 (\gamma - 1)^2 \sup_{0 < t < T} \|\rho\|_{L^{\frac{6\gamma r}{5r-6}}}^{2\gamma} \int_{Q_T} |G|^2 dx dt, \\
 |iv| &\leq \frac{1}{2} \int_0^T \int_{\mathbf{T}^3} \rho \mathbf{u}_t^2 dx dt + \varepsilon \int_0^T \|\nabla \mathbf{u}\|_{L^{\frac{3r}{3-r}}}^2 dt \\
 &\quad + \frac{1}{16\varepsilon} \sup_{0 < t < T} \left(\left(\int \rho |\mathbf{u}|^{\frac{6r}{3-r}} \right)^{\frac{2(3-r)}{3r}} \|\rho\|_{L^{\frac{7r-3}{5r-6}}}^{\frac{2(7r-3)}{3r}} \right) T \\
 &\leq \frac{1}{2} \int_0^T \int_{\mathbf{T}^3} \rho \mathbf{u}_t^2 dx dt + \varepsilon \int_0^T \|\nabla^2 P \mathbf{u}\|_{L^r}^2 + \|\nabla G\|_{L^r}^2 + \|\rho\|_{L^{\frac{3\gamma r}{3-r}}}^{2\gamma} dt \\
 &\quad + \frac{1}{16\varepsilon} \sup_{0 < t < T} \left(\left(\int \rho |\mathbf{u}|^{\frac{6r}{3-r}} \right)^{\frac{2(3-r)}{3r}} \|\rho\|_{L^{\frac{7r-3}{5r-6}}}^{\frac{2(7r-3)}{3r}} \right) T
 \end{aligned}$$

and

$$|v| \leq \frac{R^2(4\gamma - 3)}{2\mu} \left(\int_0^T \int_{\mathbf{T}^3} |\operatorname{div} \mathbf{u}|^2 dx dt \right)^{1/2} \left(\int_0^T \int_{\mathbf{T}^3} \rho^{4\gamma} dx dt \right)^{1/2}.$$

Hence, our lemma holds. □

Combining Lemma 3.1 and Lemma 3.2, we have a closed form of the inequality for $\nabla \mathbf{u}$ and \mathbf{u}_t . If we take ε small enough, then we get the following theorem.

THEOREM 3.3. *Let $p_2(r) = \max\{p_0(r), p_1(r)\}$. Suppose that ρ_0 and \mathbf{u}_0 are the same as in Lemma 3.2. We assume that $\rho \in L^\infty(0, T; L^{p_2}(\mathbf{T}^3))$. Then*

$$\begin{aligned}
 \int_{Q_T} \rho \mathbf{u}_t^2 dx dt + \sup_{0 < t < T} \int_{\mathbf{T}^3} |\nabla \mathbf{u}(t, x)|^2 dx &\leq C, \\
 \int_0^T \|\nabla^2 P \mathbf{u}\|_{L^r(\mathbf{T}^3)}^2 + \|\nabla G\|_{L^r(\mathbf{T}^3)}^2 dt &\leq C,
 \end{aligned}$$

where C depends only on $\mu, R, \|\mathbf{u}_0\|_{H^1(\mathbf{T}^3)}, \|\rho_0\|_{L^{2\gamma}(\mathbf{T}^3)}, \sup_{0 < t < T} \|\rho\|_{L^{p_2}}$.

4. A priori estimates for density

We define the Lagrangian flow X of \mathbf{u} by

$$\begin{cases} \partial_t X(t, s, x) = \mathbf{u}(t, X(t, s, x)), \\ X|_{t=s} = x. \end{cases}$$

From the transport equation representing mass conservation law in the isentropic compressible Navier-Stokes equations (1), we derive the following relations:

$$\ln \rho(X(t, 0, x), t) = \rho_0(x) - \int_0^t \operatorname{div} \mathbf{u}(s, X(s, 0, x)) ds.$$

We replace $\operatorname{div} \mathbf{u}$ by $\frac{1}{\mu}(G + R\rho^\gamma)$, then we get equality

$$\begin{aligned} & \ln \rho(X(t, 0, x), t) \\ = & \ln \rho_0(x) - \frac{1}{\mu} \int_0^t R\rho^\gamma(X(s, 0, x), s) ds \\ & - \frac{1}{\mu} \left(\Delta^{-1} \operatorname{div} (\rho \mathbf{u})(X(t, 0, x), t) - \Delta^{-1} \operatorname{div} (\rho_0 \mathbf{u}_0)(x) \right) \\ & - \frac{1}{\mu} \int_0^t \left(\Delta^{-1} \operatorname{div}^2 (\rho \mathbf{u} \otimes \mathbf{u}) - \mathbf{u} \cdot \nabla \Delta^{-1} \operatorname{div} (\rho \mathbf{u}) \right) (X(s, 0, x), s) ds. \end{aligned}$$

We set $A := \Delta^{-1} \operatorname{div}^2 (\rho \mathbf{u} \otimes \mathbf{u}) - \mathbf{u} \cdot \nabla \Delta^{-1} \operatorname{div} (\rho \mathbf{u})$ and $B := \Delta^{-1} \operatorname{div} (\rho \mathbf{u})$. Here div^2 is an operator defined by $\operatorname{div}^2 M = \partial_{ij} M_{ij}$ for a 3×3 matrix $M = (M_{ij})$. We observe that

$$\Delta A = \operatorname{div} \left(\rho (\mathbf{u} \cdot \nabla) \mathbf{u} + \operatorname{div} \mathbf{u} \nabla B - (\nabla \mathbf{u} \cdot \nabla) B + (\nabla B \cdot \nabla) \mathbf{u} \right).$$

From Calderon-Zygmund theorem, we get the inequality

$$\|\nabla A\|_{L^p} \leq c \|(\rho|\mathbf{u}| + |\nabla B|)|\nabla \mathbf{u}|\|_{L^p(\mathbf{T}^3)} \text{ for all } 1 < p < \infty.$$

We consider following Sobolev inequality

$$\|v\|_{L^\infty(\mathbf{T}^3)} \leq c \|\nabla v\|_{L^4(\mathbf{T}^3)} \text{ for all } v \in W^{1,4}(\mathbf{T}^3).$$

Then we get the following inequality

$$(4.1) \quad \ln \rho(x, t) \leq \ln \|\rho_0\|_{L^\infty(\mathbf{T}^3)} + M(t),$$

$$(4.2) \quad \ln \rho(x, t) \geq \ln(\inf_{\mathbf{T}^3} \rho_0) - M(t) - \int_0^t R \|\rho\|_{L^\infty(\mathbf{T}^3)}^\gamma ds,$$

where

$$M(t) = \frac{1}{\mu} \left(\|\rho \mathbf{u}(\cdot, t)\|_{L^4(\mathbf{T}^3)} + \|\rho_0 \mathbf{u}_0\|_{L^4(\mathbf{T}^3)} \right) + \int_0^t \|\nabla A\|_{L^4(\mathbf{T}^3)} ds.$$

(4.1) implies that

$$\rho(x, t) \leq \|\rho_0\|_{L^\infty(\mathbf{T}^3)} \exp(M(t)) \text{ for all } x \in \mathbf{T}^3 \text{ and } t > 0.$$

Therefore

$$\ln \rho(x, t) \geq \ln(\inf_{\mathbf{T}^3} \rho_0) - M(t) - \int_0^t R \|\rho_0\|_{L^\infty(\mathbf{T}^3)}^\gamma \exp(\gamma M(s)) ds.$$

This implies that

$$\rho(x, t) \geq (\inf_{\mathbf{T}^3} \rho_0) \exp \left\{ -M(t) - \int_0^t R \|\rho_0\|_{L^\infty(\mathbf{T}^3)}^\gamma \exp(\gamma M(s)) ds \right\}.$$

Hence, $\sup_{0 < t < T} M(t) < \infty$ implies that $\sup_{Q_T} \rho(x, t) < \infty$ and $\inf_{Q_T} \rho(x, t) > 0$.

THEOREM 4.1. *Suppose that $\rho_0 \in L^\infty(\mathbf{T}^3)$ and $\mathbf{u}_0 \in H^1(\mathbf{T}^3)$. We assume that $\rho \in L^\infty(0, T; L^{q_0}(\mathbf{T}^3))$ for some large q_0 . Then*

$$\sup_{Q_T} \rho(x, t) < \infty$$

and

$$\inf_{Q_T} \rho(x, t) > 0.$$

Proof. We have only to show that $\sup_{0 < t < T} M(t) < \infty$. We apply Hölder inequality and Young inequality to each term of $M(t)$:

$$\|\rho \mathbf{u}\|_{L^4(\mathbf{T}^3)} \leq c \|\rho\|_{L^{7/3}(\mathbf{T}^3)}^{7/8} \left(\int_{\mathbf{T}^3} \rho |\mathbf{u}|^8 dx \right)^{1/8}$$

and

$$\begin{aligned} \|\nabla A\|_{L^4(\mathbf{T}^3)} &\leq c(\|\rho|\mathbf{u}\|_{L^{20}(\mathbf{T}^3)} + \|\nabla B\|_{L^{20}(\mathbf{T}^3)}) \|\nabla \mathbf{u}\|_{L^5(\mathbf{T}^3)} \\ &\leq c(\|\rho|\mathbf{u}\|_{L^{20}(\mathbf{T}^3)}^2 + \|\nabla B\|_{L^{20}(\mathbf{T}^3)}^2 + \|\nabla \mathbf{u}\|_{L^5(\mathbf{T}^3)}^2) \\ &= I^2 + II^2 + III^2. \end{aligned}$$

We apply Hölder inequality to I and Calderon-Zygmung inequality and Hölder inequality to II :

$$\begin{aligned} I &\leq c \|\rho\|_{L^{39/40}(\mathbf{T}^3)}^{39/40} \left(\int_{\mathbf{T}^3} \rho |\mathbf{u}|^{40} dx \right)^{1/40}, \\ II &\leq c \|\rho \mathbf{u}\|_{L^{20}(\mathbf{T}^3)} \\ &\leq c \|\rho\|_{L^{39/40}(\mathbf{T}^3)}^{39/40} \left(\int_{\mathbf{T}^3} \rho |\mathbf{u}|^{40} dx \right)^{1/40}. \end{aligned}$$

We note that

$$\|\nabla \mathbf{u}\|_{L^5(\mathbf{T}^3)} \leq c(\|\nabla P \mathbf{u}\|_{L^5(\mathbf{T}^3)} + \|\operatorname{div} \mathbf{u}\|_{L^5(\mathbf{T}^3)})$$

and $\operatorname{div} \mathbf{u} = \frac{1}{\mu}(G + R\rho^\gamma)$. Therefore,

$$\begin{aligned} III &\leq c\left(\|\nabla P\mathbf{u}\|_{L^5(\mathbf{T}^3)} + \frac{1}{\mu}(\|G\|_{L^5(\mathbf{T}^3)} + R\|\rho\|_{L^{5\gamma}(\mathbf{T}^3)}^\gamma)\right) \\ &\leq c\left(\|\nabla^2 P\mathbf{u}\|_{L^{\frac{15}{8}}(\mathbf{T}^3)} + \frac{1}{\mu}(\|\nabla G\|_{L^{\frac{15}{8}}(\mathbf{T}^3)} + R\|\rho\|_{L^{5\gamma}(\mathbf{T}^3)}^\gamma)\right). \end{aligned}$$

By replacing r by $\frac{15}{8}$ in Theorem 3.3, we prove our theorem. More precisely, we can take q_0 by

$$q_0 = \max\left\{p_2\left(\frac{15}{8}\right), 39, 5\gamma, \frac{(2\gamma - 1)\alpha + 4\gamma}{2}\right\},$$

where $\alpha + 2 = 40$. □

5. L^∞ estimate of \mathbf{u}

A more precise inspection of (2.2) suggests the parabolic Moser iteration provides the $L^\infty((0, T) \times \mathbf{T}^N)$ bound of \mathbf{u} . With the Gronwall inequality, we obtain local L^∞ estimate of \mathbf{u} .

LEMMA 5.1. *Suppose $0 < m_T$ and $M_T < +\infty$. Then there is a positive constant $C(m_T, M_T, T)$ such that*

$$\sup_{(t,x) \in Q_T} |\mathbf{u}(x, t)| \leq C(m_T, M_T, T) + C(m_T, M_T, T) \|\mathbf{u}\|_{L^{\frac{25}{9}}(Q_T)}.$$

Proof. We integrate (2.2) from 0 to T , then we obtain the following inequality

$$\begin{aligned} &\frac{1}{\alpha} \sup_{0 \leq t \leq T} \int_{\mathbf{T}^3} (\rho|\mathbf{u}|^{\alpha+2})(t, x)dx + \int_0^T \int_{\mathbf{T}^3} (|\nabla \mathbf{u}|^2 |\mathbf{u}|^\alpha)(t, x)dxdt \\ &+ \frac{\alpha}{2} \int_0^T \int_{\mathbf{T}^3} (|\mathbf{u}|^{\alpha-2} |\nabla |\mathbf{u}|^2|^2)(t, x)dxdt \\ &\leq \frac{1}{\alpha} \int_{\mathbf{T}^3} (\rho_0 |\mathbf{u}_0|^{\alpha+2})(x)dx + \int_0^T (I + II)(s)ds. \end{aligned}$$

If we apply Hölder inequality and Young's inequality to $\int_0^T (I + II)(t)dt$,

$$\begin{aligned} &\int_0^T |I|(s)ds \\ &\leq p(M_T) \int_0^T \left(\int_{\mathbf{T}^3} (\operatorname{div} \mathbf{u})^2 |\mathbf{u}|^\alpha dx \right)^{\frac{1}{2}} \left(\int_{\mathbf{T}^3} |\mathbf{u}|^{\alpha+2} dx \right)^{\frac{\alpha}{2(\alpha+2)}} ds \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{2} \int_0^T \int_{\mathbf{T}^3} |\mathbf{u}|^{\alpha+2}(s, x) dx dt \\ &\quad + \frac{p(M_T)^2}{2} \left(\int_0^T \int_{\mathbf{T}^3} |\mathbf{u}|^{\alpha+2}(t, x) dx dt \right)^{\frac{\alpha}{\alpha+2}} T^{\frac{1}{\alpha+2}}, \end{aligned}$$

and

$$\begin{aligned} &\int_0^T |II|(t) dt \\ &\leq \alpha p(M_T) \int_0^T \left(\int_{\mathbf{T}^3} (|\mathbf{u}|^{\alpha-2} |\nabla |\mathbf{u}|^2|^2)(t, x) dx \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_{\mathbf{T}^3} |\mathbf{u}|^{\alpha+2} dx \right)^{\frac{\alpha}{2(\alpha+2)}} dt \\ &\leq \frac{\alpha}{4} \int_0^T \int_{\mathbf{T}^3} (|\mathbf{u}|^{\alpha-2} |\nabla |\mathbf{u}|^2|^2)(t, x) dx dt \\ &\quad + \alpha p(M_T)^2 \left(\int_0^T \int_{\mathbf{T}^3} |\mathbf{u}|^{\alpha+2} dx dt \right)^{\frac{\alpha}{\alpha+2}} T^{\frac{2}{\alpha}}. \end{aligned}$$

Hence we get the following inequality

$$\begin{aligned} &\frac{m_T}{\alpha + 2} \sup_{0 < t < T} \int_{\mathbf{T}^3} |\mathbf{u}(t, x)|^{\alpha+2} dx \\ &\quad + \frac{1}{2} \int_{Q_T} |\nabla \mathbf{u}|^2 |\mathbf{u}|^\alpha dx dt + \frac{\alpha}{4} \int_{Q_T} |\mathbf{u}|^{\alpha-2} |\nabla |\mathbf{u}|^2|^2 dx dt \\ &\leq \frac{M}{\alpha + 2} \| \mathbf{u}_0 \|_{L^\infty(Q_T)}^{\alpha+2} + \frac{p(M_T)^2}{2} \left(\int_{Q_T} |\mathbf{u}|^{\alpha+2} dx dt \right)^{\frac{\alpha}{\alpha+2}} T^{\frac{1}{\alpha+2}} \\ (5.1) \quad &+ \alpha p(M_T)^2 \left(\int_{Q_T} |\mathbf{u}|^{\alpha+2} dx dt \right)^{\frac{\alpha}{\alpha+2}} T^{\frac{2}{\alpha}}. \end{aligned}$$

It follows from the Hölder’s inequality and Sobolev inequality that

$$\begin{aligned} &\int_{Q_T} |\mathbf{u}(t, x)|^{\frac{5}{3}(\alpha+2)} dx dt \\ &\leq \int_{Q_T} (|\mathbf{u}|^{\alpha+2})^{\frac{2}{3}} (|\mathbf{u}|^{\alpha+2}) dx dt \\ &\leq \int_0^T \left(\int_{\mathbf{T}^3} |\mathbf{u}|^{\alpha+2} dx \right)^{\frac{2}{3}} \left(\int_{\mathbf{T}^3} |\mathbf{u}|^{3(\alpha+2)} dx \right)^{\frac{1}{3}} dt \\ &\leq \left(\sup_{0 < t < T} \int_{\mathbf{T}^3} |\mathbf{u}|^{\alpha+2} dx \right)^{\frac{2}{3}} \int_0^T \left(\int_{\mathbf{T}^3} (|\mathbf{u}|^{\frac{\alpha+2}{2}})^6 dx \right)^{\frac{1}{3}} dt \end{aligned}$$

$$\begin{aligned} &\leq C \left(\sup_{0 < t < T} \int_{\mathbf{T}^3} |\mathbf{u}|^{\alpha+2} dx \right)^{\frac{2}{3}} \left[\int_{Q_T} |\nabla |\mathbf{u}|^{\frac{\alpha+2}{2}}|^2 dx dt + \int_{Q_T} |\mathbf{u}|^{\alpha+2} dx dt \right] \\ &\leq C(\alpha + 2)^2 \left(\sup_{0 < t < T} \int_{\mathbf{T}^3} |\mathbf{u}|^{\alpha+2} dx \right)^{\frac{2}{3}} \\ &\times \left[\int_{Q_T} |\mathbf{u}|^{\alpha-2} |\nabla |\mathbf{u}|^2|^2 dx dt + \int_{Q_T} |\mathbf{u}|^{\alpha+2} dx dt \right]. \end{aligned}$$

Applying (5.1) to the last term of the above inequality, we have the reverse Hölder inequality

$$\begin{aligned} &\int_{Q_T} |\mathbf{u}|^{\frac{5}{3}(\alpha+2)} \\ &\leq C(M, M_T, m_T, T, |\mathbf{u}_0|_{L^\infty(\mathbf{T}^3)}) (\alpha + 2)^{\frac{8}{3}} \left(\int_{Q_T} |\mathbf{u}|^{\alpha+2} dx dt \right)^{\frac{5}{3}} \\ (5.2) \quad &+ C(M, M_T, m_T, T, |\mathbf{u}_0|_{L^\infty(\mathbf{T}^3)}) (\alpha + 2)^{\frac{8}{3}}. \end{aligned}$$

Set $r = \frac{5}{3}$. For each positive integer $k \geq 2$, take positive real number α_k satisfying that $\alpha_k + 2 = r^k$. Then (5.2) can be written in the form

$$\int_{Q_T} |\mathbf{u}|^{r^{k+1}} dx dt \leq C_1 k^{c_2} \left(\int_{Q_T} |\mathbf{u}|^{r^k} dx dt \right)^r + C_1 k^{c_2}.$$

Here C_1 and c_2 are the positive constants greater than 1 depending only on M, M_T, m_T, T and $|\mathbf{u}_0|_{L^\infty(\mathbf{T}^3)}$.

Now, we iterate the above reverse Hölder inequality to get

$$\begin{aligned} &\int_{Q_T} |\mathbf{u}|^{r^{k+1}} dx dt \\ &\leq C_1 k^{c_2} \left(\int_{Q_T} |\mathbf{u}|^{r^k} dx dt \right)^r + C_1 k^{c_2} \\ &\leq 2^{\frac{2}{3}} C_1^{1+r} r^{c_2(k+r(k-1))} \left(\int_{Q_T} |\mathbf{u}|^{r^{k-1}} dx dt \right)^{r^2} + 2^{\frac{2}{3}} C_1^{1+r} r^{c_2(k+r(k-1))} \\ &\vdots \\ &\leq 2^{\frac{2}{3} \cdot (k-2)} C_1^{1+\dots+r^{k-2}} r^{c_2(k+r(k-1)+\dots+r^{k-2} \cdot 2)} \left(\int_{Q_T} |\mathbf{u}|^{r^2} dx dt \right)^{r^{k-1}} \end{aligned}$$

$$\begin{aligned}
 &+2^{\frac{2}{3}} \cdot (k-2) C_1^{1+r+r^2+\dots+r^{k-2}} r^{c_2(k+r(k-1)+r^2(k-2)+\dots+r^{k-2} \cdot 2)} \\
 &+2^{\frac{2}{3}} \cdot (k-3) C_1^{1+r+\dots+r^{k-3}} r^{c_2(k+r(k-1)+r^2(k-2)+\dots+r^{k-3} \cdot 3)} \\
 &\vdots \\
 &+C_1 r^{c_2 k}.
 \end{aligned}$$

Here we used the inequality

$$(a + b)^s \leq 2^{s-1} (a^s + b^s),$$

for all positive real numbers a, b and $s > 0$. Thus we have

$$\begin{aligned}
 \int_{Q_T} |\mathbf{u}|^{r^{k+1}} &\leq 2^{(\frac{2}{3}) \cdot (k-2)} C_1^{\sum_{l=0}^{k-2} r^l} r^{c_2(\sum_{l=0}^{k-2} (k-l)r^l)} \left(\int_{Q_T} |\mathbf{u}|^{r^2} dx dt \right)^{r^{k-1}} \\
 &\quad + (k-1) 2^{(\frac{2}{3}) \cdot (k-2)} C_1^{\sum_{l=0}^{k-2} r^l} r^{c_2(\sum_{l=0}^{k-2} (k-l)r^l)}.
 \end{aligned}$$

Thus

$$\begin{aligned}
 &\| \mathbf{u} \|_{L^{r^{k+1}}(Q_T)} \\
 &\leq 2^{(\frac{2}{3}) \cdot (k-2) \cdot r^{-(k+1)}} C_1^{\sum_{l=0}^{k-2} r^{l-s-1}} r^{c_2(\sum_{l=0}^{k-2} (k-l)r^{l-s-1})} \| \mathbf{u} \|_{L^{r^2}(Q_T)} \\
 &\quad + (k-1) r^{-(k+1)} 2^{(\frac{2}{3}) \cdot (k-2) \cdot r^{-(k+1)}} C_1^{\sum_{l=0}^{k-2} r^{l-s-1}} r^{c_2(\sum_{l=0}^{k-2} (k-l)r^{l-s-1})}.
 \end{aligned}$$

From the direct calculations, we observe that

$$\begin{aligned}
 \sum_{l=0}^{k-2} r^{l-k-1} &= \sum_{l=3}^{k+1} r^{-l} \leq \sum_{l=0}^{\infty} r^{-l} < +\infty, \\
 \sum_{l=0}^{k-2} (s-l)r^{l-s-1} &= \sum_{l=3}^{k+1} (l-1)r^{-l} \leq \sum_{l=0}^{\infty} lr^{-l} < +\infty, \\
 (k-1)r^{-(k+1)} &\leq \exp(r^{-(k+1)} \ln(k-1)) \leq \limsup_{k \rightarrow \infty} \exp \frac{\ln k}{r^{k+1}} < +\infty
 \end{aligned}$$

and

$$2^{\frac{2}{3}(k-2)r^{-(k+1)}} \leq \limsup_{k \rightarrow \infty} 2^{\frac{2}{3} \frac{k}{r^{k+1}}} < +\infty.$$

Let $a = \sum_{l=0}^{\infty} r^{-l}$, $b = \sum_{l=0}^{\infty} lr^{-l}$, $c = \limsup_{k \rightarrow \infty} \exp \frac{\ln k}{r^{k+1}}$ and $d = \limsup_{k \rightarrow \infty} 2^{\frac{2}{3} \frac{k}{r^{k+1}}}$.

If k goes to ∞ , we have the following inequality

$$\sup_{(t,x) \in Q_T} |\mathbf{u}(t, x)| \leq d C_1^a r^{c_2 b} \| \mathbf{u} \|_{L^{r^2}(Q_T)} + cd C_1^a r^{c_2 b}.$$

This completes our proof. □

Combining the Lemma 2.1 and Lemma 5.1, we have the following theorem.

THEOREM 5.2. *Suppose $0 < m_T$ and $M_T < +\infty$. Then*

$$\sup_{(t,x) \in Q_T} |\mathbf{u}(x,t)| \leq C(m_T, M_T, T, |\mathbf{u}_0|_{L^\infty(\mathbf{T}^3)}),$$

for some positive constant $C(m_T, M_T, T, |\mathbf{u}_0|_{L^\infty(\mathbf{T}^3)})$.

We apply the result of Theorem 4.1. and Theorem 5.2 to Section 3. Then it is easy to see that

$$\int_0^T \|\nabla^2 P\mathbf{u}\|_{L^2}^2 + \|\nabla G\|_{L^2}^2 dt < C.$$

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Hi Jun Choe
Department of Mathematics
Yonsei University
Seoul 120-749, Korea
E-mail: choe@yonsei.ac.kr

Bum Ja Jin
Department of Mathematics
Seoul National University
Seoul 151-747, Korea
E-mail: bumjajin@hanmail.net