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ALGEBRAIC NUMBERS, TRANSCENDENTAL NUMBERS AND ELLIPTIC CURVES DERIVED FROM INFINITE PRODUCTS

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ABSTRACT. Let k be an imaginary quadratic field, $\mathfrak{h}$ the complex upper half plane, and let $\tau \in \mathfrak{h} \cap k$, $p = e^{\pi i \tau}$. In this article, using the infinite product formulas for g_2 and g_3, we prove that values of certain infinite products are transcendental whenever τ are imaginary quadratic. And we derive analogous results of Berndt-Chan-Zhang ([4]). Also we find the values of $\prod_{n=1}^{\infty}(\frac{1-p^{2n-1}}{1+p^{2n-1}})^8$ and $p\prod_{n=1}^{\infty}(1+p^{2n})^{12}$ when we know $j(\tau)$. And we construct an elliptic curve $E: y^2 = x^3 + 3x^2 + (3 - \frac{j}{256})x + 1$ with $j = j(\tau) \neq 0$ and $P = (16^2 p^2 \prod_{n=1}^{\infty}(1+p^{2n})^{24}, 0) \in E$.

§1. Introduction

The elliptic modular function $j(\tau)$ is transcendental for algebraic number τ in the upper half plane which is not an imaginary quadratic irrationality. On the other hand, it is known from the theory of complex multiplication that $j(\tau)$ is algebraic for any imaginary quadratic τ ([16], [21]).

Ramanujan [19] introduced the following functions:

$$P(z) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)z^n,$$
$$Q(z) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)z^n,$$

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$$R(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) z^n,$$

with $\sigma_k(n) = \sum_{d|n} d^k$ and $z \in \mathbb{C}$.

In 1967 Mahler conjectured that $J(z) = \dfrac{1728Q(z)^3}{Q(z)^3 - R(z)^2}$ is transcendental for any algebraic value of z, $0 < |z| < 1$. It was proved by Barré-Sirieix, Diaz, Gramain, Philibert ([1]). The algebraic independence of the numbers $J(z)$, $J'(z)$, $J''(z)$ was shown in [18]. The result is a consequence of the main result in [18] on numbers z, $P(z)$, $Q(z)$, $R(z)$.

Meanwhile in [13], [14] and [15], we dealt with certain algebraic integers as values of elliptic functions constructed from Weierstrass $\wp$-function by using infinite products.

In this article, by means of infinite product formulas of the Weierstrass $\wp$-function we prove that $\eta(\tau)^4 \frac{Q(p^2)}{R(p^2)}$ and $\dfrac{J'(\tau)}{\eta(\tau)^4}$ are algebraic numbers (Theorem 10) with $\eta(\tau)$ the Dedekind eta-function, and we consider the zeros of the Weierstrass $\wp$-function and infinite products (Theorem 2). In [13] we derived analogous results of Berndt-Chan-Zhang ([4]), which would be a generalization in the case m even. Here we derive similar results (Theorem 9) of [13]. Furthermore, we prove that certain values of infinite product are algebraic integers (Corollary 5, Theorem 8).

Also, we justify that $[\mathbb{Q}(\prod_{n=1}^{\infty}(\dfrac{1-p^{2n-1}}{1+p^{2n-1}})^8) : \mathbb{Q}(j(\tau))] \le 6$ (resp. $[\mathbb{Q}(p^2 \prod_{n=1}^{\infty}(1+p^{2n})^{24}) : \mathbb{Q}(j(\tau))] \le 3$), and we get the polynomials

$$M(x) = x^6 - 3x^5 + \left(6 - \frac{j}{256}\right) x^4 + (-7 + \frac{j}{128})x^3 + \left(6 - \frac{j}{256}\right) x^2 - 3x + 1$$

and

$$N(z) = z^3 + 3z^2 + (3 - \frac{j}{256})z + 1$$

satisfying $M(\prod_{n=1}^{\infty}(\dfrac{1-p^{2n-1}}{1+p^{2n-1}})^8) = 0$ and $N(16^2 p^2 \prod_{n=1}^{\infty}(1+p^{2n})^{24}) = 0$ when $\tau \in \mathfrak{h} \cap k$ and $j = j(\tau) \neq 0$. From this we construct an elliptic curve

$$E : y^2 = x^3 + 3x^2 + (3 - \frac{j}{256})x + 1$$

with a point $P = (16^2 p^2 \prod_{n=1}^{\infty}(1+p^{2n})^{24}, 0) \in E$ (Theorem 11).

Finally, we consider the equations of Fricke functions (Theorem 12). Throughout the article we adopt the following notations:

- k an imaginary quadratic field
- $\mathfrak{h}$ the complex upper half plane
- $\tau \in \mathfrak{h} \cap k$
- $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$
- $p = e^{\pi i \tau}$
- $\overline{\mathbb{Q}}$ the set of all algebraic numbers in $\mathbb{C}$
- $\wp(z) := \wp(z, \Lambda_\tau) = \frac{1}{z^2} + \sum_{\omega \in \Lambda_\tau - \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right)$, a Weierstrass $\wp$-function (relative to Λ_τ)
- $G_k(\Lambda_\tau) := G_k(\tau) = \sum_{\omega \in \Lambda_\tau - \{0\}} \frac{1}{\omega^k}$, the Eisenstein series with weight k
- $g_2(\tau) = 60G_4(\tau)$
- $g_3(\tau) = 140G_6(\tau)$
- $\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2 = (2\pi)^{12}\eta(\tau)^{24}$
- $j(\tau) = 1728\dfrac{g_2(\tau)^3}{\Delta(\tau)}$
- E an elliptic curve
- $\rho = \prod_{n=1}^\infty \left(\dfrac{1-p^{2n-1}}{1+p^{2n-1}}\right)^8$
- $\kappa = 16^2 p^2 \prod_{n=1}^\infty (1+p^{2n})^{24}$.

§2. Values of infinite products

PROPOSITION 1. ([8, p.86], [13], [17, p.140]) *Let* $\tau \in k \cap \mathfrak{h}$.

(a) $\wp\left(\dfrac{\tau}{2}\right) = -\dfrac{\pi^2}{3} \prod_{n=1}^\infty (1-p^{2n})^4 \big(\prod_{n=1}^\infty (1+p^{2n-1})^8$
$+16p \prod_{n=1}^\infty (1+p^{2n})^8\big)$.

(b) $\wp\left(\dfrac{\tau+1}{2}\right) = -\dfrac{\pi^2}{3} \prod_{n=1}^\infty (1-p^{2n})^4 \big[\prod_{n=1}^\infty (1+p^{2n-1})^8$
$-32p \prod_{n=1}^\infty (1+p^{2n})^8\big]$.

(c) $\wp\left(\dfrac{1}{2}\right) = \dfrac{\pi^2}{3} \prod_{n=1}^\infty (1-p^{2n})^4 \big(2\prod_{n=1}^\infty (1+p^{2n-1})^8$
$-16p \prod_{n=1}^\infty (1+p^{2n})^8\big)$.

(d) $g_2(\tau) = \dfrac{4\pi^4}{3} \prod_{n=1}^\infty (1-p^{2n})^8 \big[\prod_{n=1}^\infty (1+p^{2n-1})^{16}$
$-16p \prod_{n=1}^\infty (1+p^{n})^8 + 256p^2 \prod_{n=1}^\infty (1+p^{2n})^{16}\big]$.

(e) $g_3(\tau)=\frac{8\pi^6}{27}\prod_{n=1}^{\infty}(1-p^{2n})^{12}\big(\prod_{n=1}^{\infty}(1+p^{2n-1})^{24}$
$-24p\prod_{n=1}^{\infty}(1+p^{2n-1})^{16}(1+p^{2n})^8$
$-384p^2\prod_{n=1}^{\infty}(1+p^{2n-1})^8(1+p^{2n})^{16}$
$+4096p^3\prod_{n=1}^{\infty}(1+p^{2n})^{24}\big).$

(f) $j(\tau)=\frac{1}{p^2}\big[\prod_{n=1}^{\infty}(1+p^{2n-1})^{16}-16p\prod_{n=1}^{\infty}(1+p^n)^8$
$+256p^2\prod_{n=1}^{\infty}(1+p^{2n})^{16}\big]^3.$

Jacobi ([23, p.470]) showed that

$$\prod_{n=1}^{\infty}(1+p^{2n-1})^8-\prod_{n=1}^{\infty}(1-p^{2n-1})^8=16p\prod_{n=1}^{\infty}(1+p^{2n})^8.$$

By the Jacobi relation, we derive from Proposition 1 that

$$\begin{aligned}&\wp\left(\frac{\tau}{2}\right)\\ (1)\qquad &=\frac{\pi^2}{3}\prod_{n=1}^{\infty}(1-p^{2n})^4\left(-2\prod_{n=1}^{\infty}(1+p^{2n-1})^8+\prod_{n=1}^{\infty}(1-p^{2n-1})^8\right),\end{aligned}$$

$$\begin{aligned}&\wp\left(\frac{\tau+1}{2}\right)\\ (2)\qquad &=\frac{\pi^2}{3}\prod_{n=1}^{\infty}(1-p^{2n})^4\left(\prod_{n=1}^{\infty}(1+p^{2n-1})^8-2\prod_{n=1}^{\infty}(1-p^{2n-1})^8\right),\end{aligned}$$

$$\begin{aligned}&\wp\left(\frac{1}{2}\right)\\ (3)\qquad &=\frac{\pi^2}{3}\prod_{n=1}^{\infty}(1-p^{2n})^4\left(\prod_{n=1}^{\infty}(1+p^{2n-1})^8+\prod_{n=1}^{\infty}(1-p^{2n-1})^8\right),\end{aligned}$$

$$\begin{aligned}&g_2(\tau)\\ &=\frac{4\pi^4}{3}\prod_{n=1}^{\infty}(1-p^{2n})^8\Big[\prod_{n=1}^{\infty}(1+p^{2n-1})^{16}+\prod_{n=1}^{\infty}(1-p^{2n-1})^{16}\\ (4)\qquad &\quad-\prod_{n=1}^{\infty}(1+p^{2n-1})^8(1-p^{2n-1})^8\Big],\end{aligned}$$

$$g_3(\tau) = \frac{4\pi^6}{27}\prod_{n=1}^{\infty}(1-p^{2n})^{12}\Big(-2\prod_{n=1}^{\infty}(1+p^{2n-1})^{24}$$

$$(5) \qquad + 3\prod_{n=1}^{\infty}(1+p^{2n-1})^{16}(1-p^{2n-1})^{8}$$

$$+ 3\prod_{n=1}^{\infty}(1+p^{2n-1})^{8}(1-p^{2n-1})^{16} - 2\prod_{n=1}^{\infty}(1-p^{2n-1})^{24}\Big),$$

$$j(\tau) = \frac{1}{p^2}\Big[\prod_{n=1}^{\infty}(1+p^{2n-1})^{16} + \prod_{n=1}^{\infty}(1-p^{2n-1})^{16} - \prod_{n=1}^{\infty}(1+p^{2n-1})^{8}(1-p^{2n-1})^{8}\Big]^3,$$

$$\wp''(\frac{\tau}{2}) = 6\wp(\frac{\tau}{2})^2 - \frac{1}{2}g_2(\tau) \quad ([2, \text{p.}332])$$

$$(6) \qquad = 2\pi^4\prod_{n=1}^{\infty}(1-p^{2n})^{8}(1+p^{2n-1})^{8} \times \Big(\prod_{n=1}^{\infty}(1+p^{2n-1})^{8} - \prod_{n=1}^{\infty}(1-p^{2n-1})^{8}\Big),$$

$$\wp''(\frac{\tau+1}{2}) = 2\pi^4\prod_{n=1}^{\infty}(1-p^{2n})^{8}(1-p^{2n-1})^{8} \times \Big(\prod_{n=1}^{\infty}(1-p^{2n-1})^{8} - \prod_{n=1}^{\infty}(1+p^{2n-1})^{8}\Big),$$

$$\wp''(\frac{1}{2}) = 2\pi^4\prod_{n=1}^{\infty}(1-p^{2n})^{8}(1-p^{2n-1})^{8}(1+p^{2n-1})^{8}.$$

By using different expressions of these infinite products, Borcherds studied the denominator functions of generalized Kac-Moody algebra ([5], [6]).

Since $\Delta(\tau)$ is nonzero and $\prod_{n=1}^{\infty}(1+p^{2n})(1-p^{2n-1})(1+p^{2n-1}) = 1$, we see that $\prod_{n=1}^{\infty}(1-p^{2n})$, $\prod_{n=1}^{\infty}(1+p^{2n})$, $\prod_{n=1}^{\infty}(1+p^{2n-1})$, $\prod_{n=1}^{\infty}(1-p^{2n-1})$ are nonzero. So, by (1)-(6), we end up with the following:

THEOREM 2. *Let $\tau \in k \cap \mathfrak{h}$.*

(a) $\wp\left(\frac{\tau}{2}\right) = 0 \iff \rho = 2 \iff \wp\left(\frac{1}{2}\right) = -\wp\left(\frac{\tau+1}{2}\right) = \pi^2\prod_{n=1}^{\infty}(1-p^{2n})^4(1+p^{2n-1})^8$.

(b) $\wp\left(\dfrac{\tau+1}{2}\right)=0 \Longleftrightarrow \rho=\frac{1}{2} \Longleftrightarrow \wp\left(\dfrac{1}{2}\right)=-\wp\left(\dfrac{\tau}{2}\right)$
$=\pi^2\prod_{n=1}^{\infty}(1-p^{2n})^4(1-p^{2n-1})^8$.

(c) $\wp\left(\dfrac{1}{2}\right)=0 \Longleftrightarrow \rho=-1 \Longleftrightarrow \wp\left(\dfrac{\tau}{2}\right)=-\wp\left(\dfrac{\tau+1}{2}\right)$
$=\pi^2\prod_{n=1}^{\infty}(1-p^{2n})^4(1-p^{2n-1})^8$.

(d) $g_2(\tau)=j(\tau)=0 \Longleftrightarrow \rho^3=-1$ *with* $\rho\neq -1$.

(e) $g_3(\tau)=0 \Longleftrightarrow \rho=\frac{1}{2}, -1$ *or* 2.

(f) $\wp''\left(\dfrac{1}{2}\right)$, $\wp''\left(\dfrac{\tau}{2}\right)$ *and* $\wp''\left(\dfrac{\tau+1}{2}\right)$ *are all nonzero and distinct.*

Here we refer to [21, p.63] for (f).

Note that Eichler and Zagier [11] found the values of z which are zeros of $\wp(z,\tau)$ $(z\in\mathbb{C})$, i.e., the zeros of $\wp(z,\tau)(\tau\in\mathfrak{h}, z\in\mathbb{C})$ are given by

$$z=m+\frac{1}{2}+n\tau\pm\left(\frac{\log(5+2\sqrt{6})}{2\pi i}+144\pi i\sqrt{6}\int_{\tau}^{i\infty}(t-\tau)\frac{\Delta(t)}{E_6(t)^{3/2}}dt\right)$$

$(m,n\in\mathbb{Z})$, where $E_6(t)$ and $\Delta(t)$ $(t\in\mathfrak{h})$ denote the normalized Eisenstein series of weight 6 and unique normalized cusp form of weight 12 on $SL_2(\mathbb{Z})$, respectively, and the integral is to be taken over the line $t=\tau+i\mathbb{R}_+$ in $\mathfrak{h}$.

§3. Algebraic and transcendental numbers

Let $\alpha=\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}$ with b mod d and $|\alpha|$ be the determinant of α, and let

$$\phi_\alpha(\tau):=|\alpha|^{12}\frac{\Delta\left(\alpha\binom{\tau}{1}\right)}{\Delta\left(\binom{\tau}{1}\right)}=|\alpha|^{12}d^{-12}\frac{\Delta(\alpha\tau)}{\Delta(\tau)}. \tag{7}$$

Then we recall the following fact.

Proposition 3. ([16]) *For any* $\tau\in k\cap\mathfrak{h}$, *the value* $\phi_\alpha(\tau)$ *is an algebraic integer, which divides* $|\alpha|^{12}$.

First, we consider

$$\frac{\Delta(\tau)}{\Delta(\frac{\tau}{2})}=\frac{(2\pi)^{12}p^2\prod_{n=1}^{\infty}(1-p^{2n})^{24}}{(2\pi)^{12}p\prod_{n=1}^{\infty}(1-p^{n})^{24}}=p\prod_{n=1}^{\infty}(1+p^n)^{24}$$

and

$$\frac{\Delta(\frac{\tau}{2})}{\Delta(\tau)} = \frac{(2\pi)^{12} p \prod_{n=1}^{\infty}(1-p^n)^{24}}{(2\pi)^{12} p^2 \prod_{n=1}^{\infty}(1-p^{2n})^{24}} = p^{-1} \frac{1}{\prod_{n=1}^{\infty}(1+p^n)^{24}}.$$

Put $\alpha_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $\alpha_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

By (7) we get

$$\phi_{\alpha_1}(\frac{\tau}{2}) = 2^{12} \frac{\Delta(\tau)}{\Delta(\frac{\tau}{2})} \left(= 2^{12} \frac{\eta(\tau)^{24}}{\eta(\frac{\tau}{2})^{24}} \right), \tag{8}$$

from which and Proposition 3 we see that $\sqrt{2} p^{\frac{1}{24}} \prod_{n=1}^{\infty}(1+p^n)$ is an algebraic integer.

Using Proposition 3, we get that both

$$\sqrt{a} \frac{\eta(\frac{a\tau+b}{d})}{\eta(\tau)} \quad \text{and} \quad \sqrt{d} \frac{\eta(\tau)}{\eta(\frac{a\tau+b}{d})} \tag{9}$$

are algebraic integers.

Similarly, we get the following properties.

Proposition 4. ([13]) *Let $\tau \in k \cap \mathfrak{h}$. Then the following assertions hold*:

(a) $\sqrt{2} p^{\frac{1}{24}} \prod_{n=1}^{\infty}(1+p^n)$, $p^{-\frac{1}{24}} \frac{1}{\prod_{n=1}^{\infty}(1+p^n)}$, $p^{-\frac{1}{24}} \prod_{n=1}^{\infty}(1-p^{2n-1})$, $\sqrt{2}\prod_{n=1}^{\infty}(1+p^n)(1-p^{2n-1})$, $p^{-\frac{1}{24}} \prod_{n=1}^{\infty}(1+p^{2n-1})$ *and* $\sqrt{2}\prod_{n=1}^{\infty}(1+p^n)(1+p^{2n-1})$ *are algebraic integers.*

(b) $\frac{3}{\pi^2}\frac{\wp(\frac{\tau}{2})}{\eta(\tau)^4}$, $\frac{3}{\pi^2}\frac{\wp(\frac{\tau+1}{2})}{\eta(\tau)^4}$, $\frac{3}{\pi^2}\frac{\wp(\frac{1}{2})}{\eta(\tau)^4}$, $\frac{3}{4\pi^4}\frac{g_2(\tau)}{\eta(\tau)^8}$, $\frac{27}{\pi^6}\frac{g_3(\tau)}{\eta(\tau)^{12}}$, $\frac{\wp(\frac{\tau}{2}) - \wp(\frac{1}{2})}{\pi^2\eta(\tau)^4}$, $\frac{\wp(\frac{\tau+1}{2}) - \wp(\frac{1}{2})}{\pi^2\eta(\tau)^4}$ *and* $\frac{\wp(\frac{\tau+1}{2}) - \wp(\frac{\tau}{2})}{\pi^2\eta(\tau)^4}$ *are algebraic integers.*

In general, set $\alpha_t = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ and $\alpha_l = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$ with $a \in \mathbb{Z}^+$. Then we get again by (7) that

$$\begin{aligned} \phi_{\alpha_t}(\frac{\tau}{2})^{1/24} &= \sqrt{a} p^{\frac{1}{24}(a-1)} \prod_{n=1}^{\infty}(\sum_{m=0}^{a-1} p^{mn}), \\ \phi_{\alpha_l}(\frac{a\tau}{2})^{1/24} &= p^{-\frac{1}{24}(a-1)} \frac{1}{\prod_{n=1}^{\infty}(\sum_{m=0}^{a-1} p^{mn})} \end{aligned} \tag{10}$$

are algebraic integers.

In 1949, Gelfond and Schneider independently solved the Hilbert 7-th problem concerning the transcendence of $2^{\sqrt{2}}$. They actually proved the following strong transcendence criterion. For α, $\beta \in \overline{\mathbb{Q}}$ with $\alpha \neq 0, 1$ and $\beta \notin \mathbb{Q}$, α^β is transcendental ([20], [22]). Thus, for $\tau \in k \cap \mathfrak{h}$, the Gelfond-Schneider theorem yields that $e^{\pi\alpha} = (-1)^{-i\alpha}$ is transcendental whenever $i\alpha$ is algebraic of degree at least 2 over $\mathbb{Q}$. This leads us to the fact that

$$p = e^{\pi i \tau} \text{ is a transcendental number.} \tag{11}$$

COROLLARY 5. *Let $\tau \in k \cap \mathfrak{h}$.*

(a) *If $a \in \mathbb{Q} - \{-\frac{1}{24}\}$ then $p^{-a}\prod_{n=1}^{\infty}(1+p^n)$, $p^a\prod_{n=1}^{\infty}(1-p^{2n-1})$ and $p^a\prod_{n=1}^{\infty}(1+p^{2n-1})$ are transcendental numbers.*
(b) *If $a \in \mathbb{Q} - \{0\}$ then $p^a\prod_{n=1}^{\infty}(1+p^n)(1+p^{2n-1})$ and $p^a\prod_{n=1}^{\infty}(1+p^n)(1-p^{2n-1})$ are transcendental numbers.*
(c) *If $b \in \mathbb{Q} - \{\frac{1}{24}(a-1)\}$ then $p^b\prod_{n=1}^{\infty}(\sum_{m=0}^{a-1}p^{mn})$ is a transcendental number with $a \in \mathbb{Z}^+$.*
(d) *Assume that $g_2(\tau)$ and $g_3(\tau)$ are nonzero. If $\wp(\frac{\tau}{2})$ is transcendental, so are $\wp(\frac{\tau+1}{2})$, $\wp(\frac{1}{2})$, $g_2(\tau)$, $g_3(\tau)$ and $\Delta(\tau)$. And if $\wp(\frac{\tau}{2})$ is algebraic then so are $\wp(\frac{\tau+1}{2})$, $\wp(\frac{1}{2})$, $g_2(\tau)$, $g_3(\tau)$ and $\Delta(\tau)$.*

    If $f(\tau), g(\tau) \in \overline{\mathbb{Q}}(\wp(\frac{\tau}{2}), \wp(\frac{\tau+1}{2}), \wp(\frac{1}{2}), g_2(\tau), g_3(\tau), \Delta(\tau))$ and these are of the same even weight, then $\frac{f(\tau)}{g(\tau)}$ is an algebraic number.

Proof. (a), (b), (c) Immediate from Proposition 4, (10) and (11).

(d) Since $g_3(\tau)$ is nonzero, we see by (a), (b), (c), and (e) of Theorem 2 that $\wp(\frac{\tau+1}{2})$, $\wp(\frac{\tau}{2})$ and $\wp(\frac{1}{2})$ are nonzero. By (1), (2), (3) and Proposition 4, we derive that

$$\frac{\wp(\frac{\tau+1}{2})}{\wp(\frac{\tau}{2})} = \frac{2\rho-1}{2-\rho}, \quad \frac{\wp(\frac{1}{2})}{\wp(\frac{\tau}{2})} = \frac{1+\rho}{-2+\rho} \tag{1'}$$

are algebraic numbers, and so is $\dfrac{\wp(\frac{1}{2})}{\wp(\frac{\tau+1}{2})}$. By the assumption, (4), (5) and $(1')$, we get that

$$
\begin{aligned}
&g_2(\tau)\\
=& -4\left[\wp\left(\frac{1}{2}\right)\wp\left(\frac{\tau}{2}\right)+\wp\left(\frac{\tau+1}{2}\right)\wp\left(\frac{\tau}{2}\right)+\wp\left(\frac{1}{2}\right)\wp\left(\frac{\tau+1}{2}\right)\right]\\
=& -4\left[\frac{1+\rho}{-2+\rho}\wp\left(\frac{\tau}{2}\right)^2+\frac{2\rho-1}{2-\rho}\wp\left(\frac{\tau}{2}\right)^2+\frac{2\rho-1}{2-\rho}\cdot\frac{1+\rho}{-2+\rho}\wp\left(\frac{\tau}{2}\right)^2\right]\\
=& 12\frac{\rho^2-\rho+1}{(\rho-2)^2}\wp\left(\frac{\tau}{2}\right)^2,\\
&g_3(\tau)\\
=& -4\frac{(2\rho-1)(\rho+1)}{(\rho-2)^2}\wp\left(\frac{\tau}{2}\right)^3,\\
&\Delta(\tau)\\
=& \frac{432}{(\rho-2)^6}\left\{4(\rho^2-\rho+1)^3-(\rho-2)^2(2\rho-1)^2(\rho+1)^2\right\}\wp\left(\frac{\tau}{2}\right)^6\\
=& 2^4\cdot 3^6\frac{(\rho-1)^2\rho^2}{(\rho-2)^6}\wp\left(\frac{\tau}{2}\right)^6.
\end{aligned}
$$

Using the above relation, we can readily check the final statement. □

Let

$$\phi(\tau) := \phi(e^{\pi i\tau}) = \frac{\eta(\frac{\tau+1}{2})^2}{\eta(\tau+1)} = \prod_{n=1}^{\infty}(1+p^{2n-1})^2(1-p^{2n}) = \theta_3(0,\tau).$$

Here we refer to [8, p.86] for the last equality. Berndt, Chan and Zhang showed in [4] the following proposition by using three of Ramanujan's modular equations, values for certain class invariants of Ramanujan, representations for quotients of values of ϕ in terms of class invariants and the theta-transformation formula.

PROPOSITION 6. ([4]) *Let m and n be positive integers. Then $\phi(mni)/\phi(ni)$ is algebraic. Furthermore, if m is odd, then $\sqrt{2m}\phi(mni)/\phi(ni)$ is an algebraic integer dividing $2\sqrt{m}$, while if m is even, then $2\sqrt{m}\phi(mni)/\phi(ni)$ is an algebraic integer dividing $4\sqrt{m}$.*

In [13] we derived certain analogues of these results purely in terms of infinite products, which would be a generalization in case m even.

PROPOSITION 7. ([13]) *Let τ be any imaginary quadratic and r, s, u, v be positive integers such that $(r,s) = (u,v) = 1$. Then $4\sqrt{rv}\dfrac{\phi(\frac{r}{s}\tau)}{\phi(\frac{u}{v}\tau)}$ is an algebraic integer. Furthermore, $2\sqrt{r}\dfrac{\phi(\frac{r}{s}\cdot\frac{u}{v}\tau)}{\phi(\frac{u}{v}\tau)}$ and $2\sqrt{v}\dfrac{\phi(\frac{r}{s}\tau)}{\phi(\frac{u}{v}\cdot\frac{r}{s}\tau)}$ are algebraic integers.*

The same argument used in [13] to prove that $4\sqrt{rv}\dfrac{\phi(\frac{r}{s}\tau)}{\phi(\frac{u}{v}\tau)}$ is an algebraic integer can be applied to the following theorem.

THEOREM 8. *Let $\tau \in k \cap \mathfrak{h}$.*

(a) Let θ, θ_2, θ_3 be the Jacobi theta functions ([8], [9]). Then $\theta_3(\tau) = \alpha\eta(\tau)$, $\theta(\tau) = \alpha'\eta(\tau)$, $\theta_2(\tau) = \eta(2\tau)\alpha'' = \dfrac{\eta(\tau)}{\sqrt{2}}\alpha'''$, where α, α', α'', α''' are algebraic integers. Furthermore, we see that $a\sqrt{d}\dfrac{\theta(\frac{a\tau+b}{d})}{\theta(\tau)}$, $\sqrt{a'}d'\dfrac{\theta(\tau)}{\theta(\frac{a'\tau+b'}{d'})}$, $ad'\sqrt{a'd}\dfrac{\theta(\frac{a\tau+b}{d})}{\theta(\frac{a'\tau+b'}{d'})}$, $ad'\sqrt{a'd}\dfrac{\theta_2(\frac{a\tau+b}{d})}{\theta_2(\frac{a'\tau+b'}{d'})}$ are algebraic integers for $a, a'd, d'$ positive integers and b, b' integers.

(b) Let us define

$$a(\tau) := a(p^2) = \sum_{n,m=-\infty}^{\infty} p^{2(n^2+nm+m^2)},$$

$$b(\tau) := b(p^2) = \sum_{n,m=-\infty}^{\infty} \omega^{n-m}p^{2(n^2+nm+m^2)},$$

$$c(\tau) := c(p^2) = \sum_{n,m=-\infty}^{\infty} p^{2((n+\frac{1}{3})^2+(n+\frac{1}{3})(m+\frac{1}{3})+(m+\frac{1}{3})^2)},$$

where $\omega := e^{\frac{2\pi i}{3}}$.

Then $a(\tau) = \frac{\beta}{9}b(\tau)$, $c(\tau) = \frac{\beta'}{9}b(\tau)$, $a(\tau) = \beta''c(\tau)$ and $b(\tau) = \beta'''c(\tau)$, where β, β', β'', β''' are algebraic integers. Furthermore, $m\sqrt{md}\dfrac{b(\frac{m\tau+n}{d})}{b(\tau)}$, $d'\sqrt{m'd'}\dfrac{b(\tau)}{b(\frac{m'\tau+n'}{d'})}$, $md'\sqrt{mm'dd'}\dfrac{b(\frac{m\tau+n}{d})}{b(\frac{m'\tau+n'}{d'})}$, $md'\sqrt{mm'dd'}\dfrac{c(\frac{m\tau+n}{d})}{c(\frac{m'\tau+n'}{d'})}$ are algebraic integers with m, m', d, d' positive integers and n, n' integers.

Proof. (a) We know from [8] that $\theta_3(\tau) := \theta_3(p^2) = \prod_{n=1}^{\infty}(1+p^{2n-1})^2(1-p^{2n}) = \dfrac{\eta(\frac{\tau+1}{2})^2}{\eta(\tau+1)}$. Since

$$\frac{\theta_3(\tau)}{\eta(\tau)} = \prod_{n=1}^{\infty}\frac{(1+p^{2n-1})^2(1-p^{2n})}{p^{1/12}(1-p^{2n})} = p^{-\frac{1}{12}}\prod_{n=1}^{\infty}(1+p^{2n-1})^2,$$

we get that there exists an algebraic integer α satisfying $\theta_3(\tau) = \alpha\eta(\tau)$.

By [9], we have the identities

$$\theta(\tau) := \theta(p^2) = \frac{\eta(\tau)^2}{\eta(2\tau)} \quad \text{and} \quad \theta_2(\tau) := \theta_2(p^2) = \frac{2\eta(4\tau)^2}{\eta(2\tau)}. \tag{12}$$

And we deduce from (9) and (12) that

$$\begin{aligned}
\frac{\theta(\tau)}{\eta(\tau)} &= \frac{\eta(\tau)^2}{\eta(2\tau)\eta(\tau)} = \frac{\eta(\tau)}{\eta(2\tau)},\\
\frac{\theta_2(\tau)}{\eta(2\tau)} &= 2\frac{\eta(4\tau)^2}{\eta(2\tau)^2} = \left(\sqrt{2}\frac{\eta(4\tau)}{\eta(2\tau)}\right)^2,\\
\sqrt{2}\frac{\theta_2(\tau)}{\eta(\tau)} &= 2\sqrt{2}\frac{\eta(4\tau)^2}{\eta(\tau)\eta(2\tau)} = \left(2\frac{\eta(4\tau)}{\eta(\tau)}\right)\left(\sqrt{2}\frac{\eta(4\tau)}{\eta(2\tau)}\right)
\end{aligned}$$

are algebraic integers. Also, it follows from (12) that

$$\begin{aligned}
a\sqrt{d}\frac{\theta(\frac{a\tau+b}{d})}{\theta(\tau)} &= \left(\begin{vmatrix} a & b\\ 0 & d\end{vmatrix}^{\frac12} d^{-\frac12}\frac{\eta\left(\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}\tau\right)}{\eta(\tau)}\right)^2\\
&\quad\cdot\left(\begin{vmatrix} d & -2b\\ 0 & a\end{vmatrix}^{\frac12} a^{-\frac12}\frac{\eta\left(\begin{pmatrix} d & -2b\\ 0 & a\end{pmatrix}\tau\right)}{\eta(\frac{2a\tau+2b}{d})}\right),\\
\sqrt{a'd'}\frac{\theta(\tau)}{\theta(\frac{a'\tau+b'}{d'})} &= \left(\begin{vmatrix} a' & 2b'\\ 0 & d'\end{vmatrix}^{\frac12} d'^{-\frac12}\frac{\eta(\frac{a'\tau+2b'}{d'})}{\eta(2\tau)}\right)\\
&\quad\cdot\left(\begin{vmatrix} d' & -b'\\ 0 & a'\end{vmatrix}^{\frac12} a'^{-\frac12}\frac{\eta(\tau)}{\eta(\frac{a'\tau+b'}{d'})}\right)^2,
\end{aligned}$$

$ad'\sqrt{a'd}\dfrac{\theta(\frac{a\tau+b}{d})}{\theta(\frac{a'\tau+b'}{d'})}$ are algebraic integers.

Similarly, we conclude that $ad'\sqrt{a'd}\dfrac{\theta_2(\frac{a\tau+b}{d})}{\theta_2(\frac{a'\tau+b'}{d'})}$ is an algebraic integer.

(b) J. Borwein, P. Borwein and Garvan showed in [7] that $b(\tau) = \frac{\eta^3(\tau)}{\eta(3\tau)}$, $c(\tau) = \frac{\eta^3(3\tau)}{\eta(\tau)}$ and $a(\tau)^3 = b(\tau)^3 + c(\tau)^3$. Thus we get

$$\frac{b(\tau)}{c(\tau)} = \left(\frac{\eta(\tau)}{\eta(3\tau)}\right)^4 \quad \text{and} \quad \frac{c(\tau)}{b(\tau)} = \frac{1}{9}\left(\sqrt{3}\frac{\eta(3\tau)}{\eta(\tau)}\right)^4.$$

Put $\alpha_3 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ and $\alpha_4 = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$. By using (7), we obtain that $\frac{b(\tau)}{c(\tau)}$ and $\frac{9c(\tau)}{b(\tau)}$ are algebraic integers. Since $a(p^2)^3 = b(p^2)^3 + c(p^2)^3$, we conclude that $\frac{a(\tau)}{c(\tau)}$ and $\frac{9a(\tau)}{b(\tau)}$ are algebraic integers. Also, we get that

$$m\sqrt{md}\frac{b(\frac{m\tau+n}{d})}{b(\tau)} = \left(\sqrt{m}\frac{\eta(\frac{m\tau+n}{d})}{\eta(\tau)}\right)^3 \cdot \left(\sqrt{d}\frac{\eta(3\tau)}{\eta(\frac{3m\tau+3n}{d})}\right),$$

$$d'\sqrt{m'd'}\frac{b(\tau)}{b(\frac{m'\tau+n'}{d'})}, \quad md'\sqrt{mm'dd'}\frac{b(\frac{m\tau+n}{d})}{b(\frac{m'\tau+n'}{d'})}, \quad md'\sqrt{mm'dd'}\frac{c(\frac{m\tau+n}{d})}{c(\frac{m'\tau+n'}{d'})}$$

are algebraic integers. □

Duverney, Ke. Nishioka, Ku. Nishioka, and Shiokawa proved in [10] that the Rogers-Ramanujan continued fraction $RR(z)$ is transcendental for any algebraic number z with $0 < |z| < 1$. Meanwhile, we consider in Theorem 9 some examples of the Rogers-Ramanujan continued fraction for the case of transcendental numbers z.

Let

$$\begin{aligned}
F_1(p^2) &:= \frac{p^{\frac{2}{5}}}{1} \, \frac{p^2}{+\,1} \, \frac{p^4}{+\,1} \, \frac{p^6}{+\,1} \, + \cdots, \\
F_2(p^2) &:= \frac{1}{1} \, \frac{p^2}{+\,1+p^2} \, \frac{p^4}{+\,1+p^4} \, \frac{p^6}{+\,1+p^6} \, + \cdots, \\
F_3(p^2) &:= \frac{p}{1+p^2} \, \frac{p^4}{+\,1+p^6} \, \frac{p^8}{+\,1+p^{10}} \, \frac{p^{12}}{+\,1+p^{14}} \, + \cdots, \\
F_4(p^2) &:= \frac{1}{1+} \, \frac{p^2+p^4}{1} \, + \, \frac{p^4+p^8}{1} \, + \, \frac{p^6+p^{12}}{1} \, + \cdots
\end{aligned}$$

be the continued fractions as in [10].

THEOREM 9. *If* $l_1 \neq -\frac{4}{5}$, $l_2 \neq \frac{1}{4}$, $l_3 \neq 0$, $l_4 \neq \frac{2}{3}$ *are rational numbers then* $p^{l_1}\left(\frac{1}{F_1(p^2)} - F_1(p^2) - 1\right)$, $p^{l_2}F_2(p^2)$, $p^{l_3}\left(\frac{1}{F_3(p^2)} + F_3(p^2)\right)$, $p^{l_4}F_4(p^2)$ *are transcendental numbers.*

Furthermore,

$$p^{-\frac{4}{5}}\left(\frac{1}{F_1(p^2)} - F_1(p^2) - 1\right), 2p^{\frac{1}{4}}F_2(p^2), 2\left(\frac{1}{F_3(p^2)} + F_3(p^2)\right)$$

and $2\sqrt{2}p^{\frac{2}{3}}F_4(p^2)$ *are algebraic integers.*

Proof. First, we consider the value of continued fraction $F_1(p^2)$. It follows from [3] and [10] that $\frac{1}{F_1(p^2)} - F_1(p^2) - 1 = p^{\frac{4}{5}}\frac{\eta(\frac{1}{5}\tau)}{\eta(5\tau)}$.

Set $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 25 \end{pmatrix}$. Then we obtain by (9) that

$$p^{-\frac{4}{5}}\left(\frac{1}{F_1(p^2)} - F_1(p^2) - 1\right) = \frac{\eta(\frac{1}{5}\tau)}{\eta(5\tau)} \tag{13}$$

is an algebraic integer. By (11) and (13) we are led to the fact that

$$p^{l_1}\left(\frac{1}{F_1(p^2)} - F_1(p^2) - 1\right)$$

is a transcendental number except for the case $l_1 = -\frac{4}{5}$.

Secondly, it follows from [3, p.221], [8], [9], [10], [12, p.186] that

$$\begin{aligned} F_2(p^2) &= \frac{\theta_2(p)}{2p^{\frac{1}{4}}\theta_3(p^2)} \\ &= \frac{1}{2p^{\frac{1}{4}}}\frac{2\eta(2\tau)^2}{\eta(\tau)}\frac{\eta(4\tau)^2\eta(\tau)^2}{\eta(2\tau)^5} \\ &= \frac{1}{p^{\frac{1}{4}}}\frac{\eta(4\tau)^2\eta(\tau)}{\eta(2\tau)^3} \\ &= \frac{1}{2p^{\frac{1}{4}}}\left(\sqrt{2}\frac{\eta(4\tau)}{\eta(2\tau)}\right)^2\left(\frac{\eta(\tau)}{\eta(2\tau)}\right). \qquad (14) \end{aligned}$$

Thus, by (11) and (14) we derive that $2p^{\frac{1}{4}}F_2(p^2)$ is an algebraic integer and $p^{l_2}F_2(p^2)$ is a transcendental number provided that $l_2 \neq \frac{1}{4}$.

In a similar way we get that

$$
\begin{aligned}
\frac{1}{F_3(p^2)} + F_3(p^2) &= \frac{2\theta_3(p^2)}{\theta_2(p^4)} \quad ([3, \text{p.}221], [10]) \\
&= \frac{2\theta_3(p^2)}{\theta_2(p^4)} \\
&= \frac{2\eta(2\tau)^5}{\eta(4\tau)^2\eta(\tau)^2} \frac{\eta(4\tau)}{2\eta(8\tau)^2} \\
&= \frac{\eta(2\tau)^5}{\eta(4\tau)\eta(\tau)^2\eta(8\tau)^2} \\
&= \frac{1}{2}\left(\frac{\eta(2\tau)}{\eta(4\tau)}\right)\left(\sqrt{2}\frac{\eta(2\tau)}{\eta(\tau)}\right)^2\left(\frac{\eta(2\tau)}{\eta(8\tau)}\right)^2.
\end{aligned}
$$

So $2\left(\frac{1}{F_3(p^2)} + F_3(p^2)\right)$ is an algebraic integer; hence $p^{l_3}\left(\frac{1}{F_3(p^2)} + F_3(p^2)\right)$ is a transcendental number unless $l_3 = 0$.

Finally, we deduce from [3, p.345] and [10] that

$$
F_4(p^2) = \frac{\eta(\tau)\eta(6\tau)^3}{p^{\frac{2}{3}}\eta(2\tau)\eta(3\tau)^3} = \frac{1}{2\sqrt{2}p^{\frac{2}{3}}}\left(\frac{\eta(\tau)}{\eta(2\tau)}\right)\left(\sqrt{2}\frac{\eta(6\tau)}{\eta(3\tau)}\right)^3.
$$

Hence we conclude by (9) and (11) that $2\sqrt{2}p^{\frac{2}{3}}F_4(p^2)$ is an algebraic integer and $p^{l_4}F_4(p^2)$ is a transcendental number when $l_4 \neq \frac{2}{3}$. □

Since $j(\tau)$ is an algebraic integers

$$
\frac{1}{27} - \frac{64}{j(\tau)} = \frac{g_3(\tau)^2}{g_2(\tau)^3} \tag{15}
$$

is an algebraic number.

THEOREM 10. *Let "′" denote the derivation $z\frac{d}{dz}$ and $J(p^2) =: J(\tau)$. Assume that $g_2(\tau)$ and $g_3(\tau)$ are nonzero. Then $\eta(\tau)^4\frac{Q(p^2)}{R(p^2)}$ and $\frac{J'(\tau)}{\eta(\tau)^4}$ are algebraic numbers.*

Proof. By Proposition 4, we note that

$$\frac{Q(p^2)}{R(p^2)} = \frac{1}{\eta(\tau)^4} \frac{\frac{3}{4\pi^4}\frac{g_2(\tau)}{\eta(\tau)^8}}{\frac{27g_3(\tau)}{8\pi^6\eta(\tau)^{12}}}.$$

Since $\frac{3}{4\pi^4}\frac{g_2(\tau)}{\eta(\tau)^8}$ and $\frac{27g_3(\tau)}{8\pi^6\eta(\tau)^{12}}$ are algebraic numbers, $\eta(\tau)^4\dfrac{Q(p^2)}{R(p^2)}$ is an algebraic number. We see from [9] that $\dfrac{J'(\tau)}{J(\tau)} = -\dfrac{R(p^2)}{Q(p^2)}$. By the above, we get that

$$\frac{J'(\tau)}{\eta(\tau)^4} = -\left(\eta(\tau)^4\frac{Q(p^2)}{R(p^2)}\right)^{-1} J(\tau)$$

is an algebraic number. □

§4. Elliptic curves and infinite products

In this section, we assume that $j = j(\tau) \neq 0$ and $R(p^2) \neq 0$. It follows from (4), (5) and the definition of $j(\tau)$ that there exists a polynomial

$$\begin{aligned} M(x) = x^6 - 3x^5 + \left(6 - \frac{j}{256}\right)x^4 + (-7 + \frac{j}{128})x^3 \\ + \left(6 - \frac{j}{256}\right)x^2 - 3x + 1, \end{aligned} \tag{16}$$

with $j = j(\tau)$ such that $M\left(\prod_{n=1}^{\infty}\left(\frac{1-p^{2n-1}}{1+p^{2n-1}}\right)^8\right) = 0$. We then derive by (16) that

$$[\mathbb{Q}(\prod_{n=1}^{\infty}\left(\frac{1-p^{2n-1}}{1+p^{2n-1}}\right)^8) : \mathbb{Q}(j(\tau))] \leq 6.$$

Since

$$\prod_{n=1}^{\infty}\left(\frac{1-p^{2n-1}}{1+p^{2n-1}}\right) = \frac{\theta_4^4(0,\tau)}{\theta_3^4(0,\tau)}$$

([2, p.362]), we have $[\mathbb{Q}(\frac{\theta_4^{32}(0,\tau)}{\theta_3^{32}(0,\tau)}) : \mathbb{Q}(j(\tau))] \leq 6$.

Put $\delta = \rho + \frac{1}{\rho}$. Then we obtain

$$M(\rho) = \rho^3\left(\delta^3 - 3\delta^2 + (3 - \frac{j}{256})\delta - 1 + \frac{j}{128}\right) = 0.$$

From the Jacobi's relation we have

$$\begin{aligned}&\prod_{n=1}^{\infty}(1+p^{2n-1})^{16} + \prod_{n=1}^{\infty}(1-p^{2n-1})^{16}\\ &= 16^2p^2\prod_{n=1}^{\infty}(1+p^{2n})^{16} + \frac{2}{\prod_{n=1}^{\infty}(1+p^{2n})^8}\end{aligned}$$

and hence

$$\begin{aligned}&16^2p^2\prod_{n=1}^{\infty}(1+p^{2n})^{24}\\ &= \left(\prod_{n=1}^{\infty}(1+p^{2n-1})^{16} + \prod_{n=1}^{\infty}(1-p^{2n-1})^{16}\right)\prod_{n=1}^{\infty}(1+p^{2n})^8 - 2\\ &= \frac{\left(\prod_{n=1}^{\infty}(1+p^{2n-1})^{16} + \prod_{n=1}^{\infty}(1-p^{2n-1})^{16}\right)\prod_{n=1}^{\infty}(1+p^{2n})^8}{\prod_{n=1}^{\infty}(1+p^{2n-1})^8\prod_{n=1}^{\infty}(1-p^{2n-1})^8\prod_{n=1}^{\infty}(1+p^{2n})^8} - 2\\ &= (\rho + \frac{1}{\rho}) - 2.\end{aligned}$$

Let $\kappa = 16^2p^2\prod_{n=1}^{\infty}(1+p^{2n})^{24}$.

Since $M(\rho) = 0$, there exists a polynomial

$$N(z) = z^3 + 3z^2 + (3 - \frac{j}{256})z + 1$$

with $j = j(\tau)$ satisfying $N(\kappa) = 0$. This implies that

$$[\mathbb{Q}(p^2\prod_{n=1}^{\infty}(1+p^{2n})^{24}) : \mathbb{Q}(j)] \leq 3.$$

Hence we can construct an elliptic curve

$$E : y^2 = x^3 + 3x^2 + (3 - \frac{j}{256})x + 1 \tag{16'}$$

with $j = j(\tau)$ satisfying $P = (16^2p^2\prod_{n=1}^{\infty}(1+p^{2n})^{24}, 0) \in E$. Here, the discriminant of E is

$$\Delta(E) = \frac{j^2(-1728+j)}{262144} \quad \text{and} \quad j(E) = \frac{1728j}{-1728+j}.$$

We summarize the above as follows.

THEOREM 11. *Let $R(p^2) \neq 0$.*

(a) $[\mathbb{Q}(\prod_{n=1}^{\infty}\left(\frac{1-p^{2n-1}}{1+p^{2n-1}}\right)^8) : \mathbb{Q}(j(\tau))] \leq 6$ *and* $[\mathbb{Q}(p^2\prod_{n=1}^{\infty}(1+p^{2n})^{24}) : \mathbb{Q}(j)] \leq 3$.

(b) *Let $E : y^2 = x^3 + 3x^2 + (3 - \frac{j}{256})x + 1$ be an elliptic curve. Then $P = (16^2p^2\prod_{n=1}^{\infty}(1+p^{2n})^{24}, 0) \in E$, $\Delta(E) = \frac{j^2(-1728+j)}{262144}$ and $j(E) = \frac{1728j}{-1728+j}$ with $j = j(\tau)$.*

§5. Other examples with infinite products

In general, by (16) we find the values of $\prod_{n=1}^{\infty}(\frac{1-p^{2n-1}}{1+p^{2n-1}})^8$ when $j(\tau)$ is known.

Put $\prod_{n=1}^{\infty}(\frac{1-p^{2n-1}}{1+p^{2n-1}})^8 = \rho$. By (1′), it follows that

$$\frac{\wp(\frac{\tau+1}{2})}{\wp(\frac{\tau}{2})} = \frac{2\rho-1}{2-\rho} \quad \text{and} \quad \frac{\wp(\frac{1}{2})}{\wp(\frac{\tau}{2})} = \frac{1+\rho}{-2+\rho}.$$

Let

$$f_0(z;\tau) = -2^7 3^5 \frac{g_2(\tau)g_3(\tau)}{\Delta(\tau)}\wp(z,\Lambda_\tau)$$

be the *first Weber function* for $\tau \in \mathfrak{h}$ and $z \in k$. For a fixed integer $N > 1$ and r, s in $\mathbb{Z}$ not both divisible by N, let

$$f_{r,s}(\tau) = f_0(\frac{r\tau+s}{N};\tau)$$

be the *Fricke function.*

By (1), (4) and (5) we obtain the identity for $f_{1,0}$:

$$\begin{aligned}
&f_{1,0}\\
=\;& f_0(\frac{\tau}{2};\tau)\\
=\;& -2^7 3^5 \frac{g_2(\tau)g_3(\tau)}{\Delta(\tau)}\wp(\frac{\tau}{2})\\
=\;& -\frac{1}{2p^2}\Bigg[\{\prod_{n=1}^{\infty}(1+p^{2n-1})^{16} + \prod_{n=1}^{\infty}(1-p^{2n-1})^{16}\\
&- \prod_{n=1}^{\infty}(1+p^{2n-1})^8(1-p^{2n-1})^8\}\\
&\cdot\{-2\prod_{n=1}^{\infty}(1+p^{2n-1})^{24} - 2\prod_{n=1}^{\infty}(1-p^{2n-1})^{24}
\end{aligned}$$

$$+\ 3\prod_{n=1}^{\infty}(1+p^{2n-1})^{16}(1-p^{2n-1})^{8}$$
$$+3\prod_{n=1}^{\infty}(1+p^{2n-1})^{8}(1-p^{2n-1})^{16}\}$$
$$\cdot\{-2\prod_{n=1}^{\infty}(1+p^{2n-1})^{8}+\prod_{n=1}^{\infty}(1-p^{2n-1})^{8}\}\Big].$$

By the Jacobi relation, we derive that

$$\frac{1}{p^2}=16^2\prod_{n=1}^{\infty}(1+p^{2n})^{16}\Big/\Big[\prod_{n=1}^{\infty}(1+p^{2n-1})^{16}$$
$$-\ 2\prod_{n=1}^{\infty}(1+p^{2n-1})^{8}(1-p^{2n-1})^{8}+\prod_{n=1}^{\infty}(1-p^{2n-1})^{16}\Big].$$

Since $\prod_{n=1}^{\infty}(1+p^{2n-1})(1-p^{2n-1})(1+p^{2n})=1$, we end up with the following:

$$f_{1,0}$$
$$=\ -2^7\Big[\{\prod_{n=1}^{\infty}(1+p^{2n-1})^{16}+\prod_{n=1}^{\infty}(1-p^{2n-1})^{16}$$
$$-\ \prod_{n=1}^{\infty}(1+p^{2n-1})^{8}(1-p^{2n-1})^{8}\}$$
$$\cdot\{-2\prod_{n=1}^{\infty}(1+p^{2n-1})^{24}-2\prod_{n=1}^{\infty}(1-p^{2n-1})^{24}$$
$$+\ 3\prod_{n=1}^{\infty}(1+p^{2n-1})^{16}(1-p^{2n-1})^{8}$$
$$+\ 3\prod_{n=1}^{\infty}(1+p^{2n-1})^{8}(1-p^{2n-1})^{16}\}$$
$$\cdot\{-2\prod_{n=1}^{\infty}(1+p^{2n-1})^{8}+\prod_{n=1}^{\infty}(1-p^{2n-1})^{8}\}\Big]\Big/$$
$$\Big[\prod_{n=1}^{\infty}(1+p^{2n-1})^{16}\prod_{n=1}^{\infty}(1-p^{2n-1})^{16}(\prod_{n=1}^{\infty}(1+p^{2n-1})^{8}$$
$$-\ \prod_{n=1}^{\infty}(1-p^{2n-1})^{8})^{2}\Big].$$

We are also able to express $j(\tau)$ as

$$j(\tau) = 2^8 \left[\prod_{n=1}^{\infty} (1+p^{2n-1})^{16} + \prod_{n=1}^{\infty} (1-p^{2n-1})^{16} - \prod_{n=1}^{\infty} (1+p^{2n-1})^{8} (1-p^{2n-1})^{8} \right]^3 \Big/ \left[\prod_{n=1}^{\infty} (1+p^{2n-1})^{16} (1-p^{2n-1})^{16} \cdot \{ \prod_{n=1}^{\infty} (1+p^{2n-1})^{8} - \prod_{n=1}^{\infty} (1-p^{2n-1})^{8} \}^2 \right].$$

We can apply (1), (4), and (5) to $\frac{f_{1,0}}{j} = -2 \cdot 3^2 \frac{g_3}{g_2} \wp(\frac{\tau}{2})$. Then we can use

$$\frac{f_{1,0}}{f_{0,1}} = \frac{\wp(\frac{\tau}{2})}{\wp(\frac{1}{2})}, \quad \frac{f_{1,0}}{f_{1,1}} = \frac{\wp(\frac{\tau}{2})}{\wp(\frac{\tau+1}{2})}.$$

Thus we get the following:

$$f_{1,0} = -\frac{1}{2} \frac{(-2+3\rho+3\rho^2-2\rho^3)(-2+\rho)}{(1-\rho+\rho^2)^2} \cdot j,$$
$$f_{0,1} = -\frac{1}{2} \frac{(-2+3\rho+3\rho^2-2\rho^3)(1+\rho)}{(1-\rho+\rho^2)^2} \cdot j,$$
$$f_{1,1} = -\frac{1}{2} \frac{(-2+3\rho+3\rho^2-2\rho^3)(1-2\rho)}{(1-\rho+\rho^2)^2} \cdot j.$$

We denote by $\mathbb{F}_{N,\mathbb{C}}$ the *field of modular functions of level N*. As is known([16]),

$$\mathbb{F}_{1,\mathbb{C}} = \mathbb{C}(j) \quad \text{and} \quad \mathbb{F}_{N,\mathbb{C}} = \mathbb{F}_{1,\mathbb{C}}(f_{r,s})_{\text{all } r,s} = \mathbb{C}(j, f_{r,s})_{\text{all } r,s}.$$

From the identities of $j(\tau)$, $f_{1,0}$, $f_{0,1}$ and $f_{1,1}$ we see that

$$\mathbb{F}_{2,\mathbb{C}} = \mathbb{C}(p^{-\frac{1}{3}} \prod_{n=1}^{\infty} (1+p^{2n-1})^8, p^{-\frac{1}{3}} \prod_{n=1}^{\infty} (1-p^{2n-1})^8).$$

We summarize the above in the following theorem.

THEOREM 12. (a) $\dfrac{f_{1,1}}{f_{1,0}} = \dfrac{\wp(\frac{\tau+1}{2})}{\wp(\frac{\tau}{2})} = \dfrac{2\rho-1}{2-\rho}$,

$$\frac{f_{0,1}}{f_{1,0}} = \frac{\wp(\frac{1}{2})}{\wp(\frac{\tau}{2})} = \frac{1+\rho}{-2+\rho}.$$

(b)

$$
\begin{aligned}
f_{1,0} &= -\frac{1}{2}\frac{(-2+3\rho+3\rho^2-2\rho^3)(-2+\rho)}{(1-\rho+\rho^2)^2}\cdot j,\\
f_{0,1} &= -\frac{1}{2}\frac{(-2+3\rho+3\rho^2-2\rho^3)(1+\rho)}{(1-\rho+\rho^2)^2}\cdot j,\\
f_{1,1} &= -\frac{1}{2}\frac{(-2+3\rho+3\rho^2-2\rho^3)(1-2\rho)}{(1-\rho+\rho^2)^2}\cdot j.
\end{aligned}
$$

(c) $\mathbb{F}_{2,\mathbb{C}} = \mathbb{C}(p^{-\frac{1}{3}}\prod_{n=1}^{\infty}(1+p^{2n-1})^8, p^{-\frac{1}{3}}\prod_{n=1}^{\infty}(1-p^{2n-1})^8)$.

EXAMPLE 13. We consider the cases of complex multiplication. We know from [21] all elliptic curves defined over $\mathbb{Q}$ with complex multiplication by an order $R = \mathbb{Z}+fR_k$ of conductor f in a quadratic imaginary field $K = \mathbb{Q}(\sqrt{-D})$ of discriminant $-D$. By using (15), (16) and (16′), we get the following:

Case $D = 7$, $f = 1$: $j(E) = -3^3 5^3$, $M(x) = x^6 - 3x^5 + \frac{4911}{256}x^4 - \frac{4271}{128}x^3 + \frac{4911}{256}x^2 - 3x + 1$, $E : y^2 = x^3 + 3x^2 + \frac{4143}{256}x + 1$.

By using Mathematica 4.0, we write the value of $16^2 p^2 \prod_{n=1}^{\infty}(1+p^{2n})^{24}$ with $p = e^{\pi\sqrt{-1}(\frac{1+\sqrt{-7}}{2})}$ as follows: $-\frac{1}{16}$ or $\frac{1}{32}(-47-45\sqrt{-7})$ or $\frac{1}{32}(-47+45\sqrt{-7})$.

And by using Theorem 12, we get the following:

D	f	$j(E)$	ρ	$f_{1,0}$	$f_{0,1}$	$f_{1,1}$
7	1	$-3^3 5^3$	$\frac{1-3\sqrt{-7}}{32}$	8505	$\frac{-8505+1215\sqrt{-7}}{2}$	$\frac{-8505-1215\sqrt{-7}}{2}$
			$\frac{1-3\sqrt{-7}}{2}$	$\frac{-8505+1215\sqrt{-7}}{2}$	$\frac{-8505-1215\sqrt{-7}}{2}$	8505
			$\frac{31-3\sqrt{-7}}{32}$	$\frac{-8505-1215\sqrt{-7}}{2}$	8505	$\frac{-8505+1215\sqrt{-7}}{2}$
			$\frac{1+3\sqrt{-7}}{32}$	8505	$\frac{-8505-1215\sqrt{-7}}{2}$	$\frac{-8505+1215\sqrt{-7}}{2}$
			$\frac{1+3\sqrt{-7}}{2}$	$\frac{-8505-1215\sqrt{-7}}{2}$	$\frac{-8505+1215\sqrt{-7}}{2}$	8505
			$\frac{31+3\sqrt{-7}}{32}$	$\frac{-8505+1215\sqrt{-7}}{2}$	8505	$\frac{-8505-1215\sqrt{-7}}{2}$

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References

[1] K. Barré-Sirieix, G. Diaz, F. Gramain, G. Philibert, *Une preuve de la conjecture de Mahler-Manin*, Invent. Math. **124** (1996), 1–9.

[2] H. Bateman, *Higher transcendental functions, vol 2*, McGRAW-HILL Company, 1953.

[3] B. C. Berndt, *Ramanujan's Notebooks Part III*, Springer-Verlag, 1991.

[4] B. C. Berndt, H. H. Chan and L.-C. Zhang, *Ramanujan's remarkable product of theta-functions* **40** (1997), Proc. Edinburgh Math. Soc., 583–612.

[5] R. E. Borcherds, *Automorphic forms on $O_{s+2,2}(R)^+$ and generalized Kac-Moody algebras*, Proceedings of the International Congress of Mathematics **1, 2** (1995), 744–752.

[6] ———, *Automorphic forms on $O_{s+2,2}(R)^+$ and infinite products*, Invent. Math. **120** (1995), no. 1, 161–213.

[7] J. M. Borwein, P. B. Borwein and F. G. Garvan, *Some cubic modular identities of Ramanujan*, Trans. Amer. Math. Soc. **343** (1994), 35–47.

[8] K. Chandrasekharan, *Elliptic Functions*, Springer-Verlag, Grundlehren der mathematischen wissenschaften 281, 1985.

[9] D. Duverney, Ke. Nishioka, Ku. Nishioka and I. Shiokawa, *Transcendence of Jacobi's theta series*, Proc. Japan Acad. Ser. A Math. Sci. **72** (1996), 202–203.

[10] ———, *Transcendence of Rogers-Ramanujan continued fraction and reciprocal sums of Fibonacci numbers*, Proc. Japan Acad. Ser. A Math. Sci. **73** (1997), 140–142.

[11] M. Eichler and D. Zagier, *On the zeros of the Weierstrass $\wp$-function*, Math. Ann. **258** (1982), 399–407.

[12] L. K. Hua, *Introduction to number theory*, Springer-Verlag, 1982.

[13] D. Kim and J. K. Koo, *Algebraic integer as values of elliptic functions*, Acta Arith. **100** (2001), 105–116.

[14] ———, *Integrability as values of cusp forms in imaginary quadratic*, Commun. Korean Math. Soc. **16** (2001), no. 4, 585–594.

[15] ———, *Transcendental numbers as values of elliptic functions*, Bull. Korean Math. Soc. **37** (2000), no. 4, 675–683.

[16] S. Lang, *Elliptic Functions*, Addison-Wesley, 1973.

[17] H. Mckean and V. Moll, *Elliptic Curves*, Cambridge University Press, 1997.

[18] Y. Nesterenko, *Modular functions and transcendence problems*, Math. Sb. **187** (1996), 65–96.

[19] S. Ramanujan, *On certain arithmetical functions*, Trans. Camb. Phil. Soc. **22** (1916), 159–184.

[20] C. L. Siegel, *Transcendental numbers*, Princeton Univ. Press, 1949.

[21] J. H. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*, Springer-Verlag, New York, 1994.

[22] M. Waldschmidt, *Nombres transcendents*, Lecture Notes in Math. 402, Springer-Verlag, New York, 1974.

[23] E. T. Whittaker and G. N. Watson, *A course of modern analysis*, Cambridge Press, 1978.

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