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# MAXIMAL COLUMN RANKS AND THEIR PRESERVERS OF MATRICES OVER MAX ALGEBRA

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ABSTRACT. The maximal column rank of an $m$ by $n$ matrix $A$ over max algebra is the maximal number of the columns of $A$ which are linearly independent. We compare the maximal column rank with rank of matrices over max algebra. We also characterize the linear operators which preserve the maximal column rank of matrices over max algebra.

## 1. Introduction

There are many papers on the study of linear operators that preserve rank and maximal column rank of matrices over several semirings ([2]-[6]). Bapat, Pati and Song [2] obtained characterizations of linear operators that preserve the rank of matrices over max algebra.

Hwang, Kim and Song [5] defined a maximal column rank of a matrix over a semiring and compared it with rank over various semirings. And they obtained characterizations of the linear operators that preserve maximal column ranks of matrices over Boolean algebra.

In this paper, we investigate the relationships between rank and maximal column rank of matrices over max algebra. We also extend the study on known properties of linear operators preserving the rank of matrices over max algebra carry over to linear operators preserving maximal column rank.

The *max algebra* consists of the set $\mathbb{R}_{\max}$, where $\mathbb{R}_{\max}$ is the set of nonnegative real numbers, equipped with two binary operations, denoted by $\oplus$ and $\cdot$(and to be referred to addition and multiplication over max algebra), respectively. The operations are defined as $a \oplus b = \max\{a, b\}$ and $a \cdot b = ab$. That is, their sum is the maximum of $a$ and $b$ and their

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product is the usual product in the reals. There has been a great deal of interest in recent years in this max-algebra. This system allows us to express some nonlinear phenomena in the conventional algebra in a linear fashion. We refer to [1] for a description of such systems and their applications.

Let $\mathbb{M}_{m\times n}(\mathbb{R}_{\max})$ denote the set of all $m \times n$ matrices with entries in $\mathbb{R}_{\max}$. The $(i, j)$th entry of a matrix $A$ is denoted by $a_{ij}$. If $A = [a_{ij}]$ and $B = [b_{ij}]$ are $m \times n$ matrices over $\mathbb{R}_{\max}$, then the sum of $A$ and $B$ is denoted by $A \oplus B$, which is the $m \times n$ matrix with $a_{ij} \oplus b_{ij}$ as its $(i, j)$th entry. If $c \in \mathbb{R}_{\max}$, then $cA$ is the matrix $[ca_{ij}]$. If $A$ is an $m \times n$ matrix and $B$ is an $n \times p$ matrix, then their product is denoted by $A \otimes B$, which is the $m \times p$ matrix with $\max\{a_{ir}b_{rj} | r = 1, \dots, n\}$ as its $(i, j)$th entry. The zero matrix is denoted by $O$. The identity matrix of an appropriate order is denoted by $I$. And the transpose of $A = [a_{ij}]$, denoted by $A^t$, is defined in the usual way. That is, the $(i, j)$th entry of $A^t$ is $a_{ji}$ for all $i$ and $j$.

Let $S$ be a subset of $(\mathbb{R}_{\max})^n$, where $n$ is a positive integer. Then $S$ is called *linearly dependent* if there exists $\boldsymbol{x} \in S$ such that $\boldsymbol{x}$ is a linear combination of elements in $S - \{\boldsymbol{x}\}$ with scalars in $\mathbb{R}_{\max}$. Otherwise $S$ is *linearly independent*. Thus an independent set cannot contain a zero element.

## 2. Rank versus maximal column rank of matrices over max algebra

The *rank* or *factor rank*, $r(A)$, of a nonzero matrix $A \in \mathbb{M}_{m\times n}(\mathbb{R}_{\max})$ is defined as the least integer $k$ for which there exist $m \times k$ and $k \times n$ matrices $B$ and $C$ with $A = B \otimes C$. The rank of a zero matrix is zero. Also we can easily obtain that $0 \le r(A) \le \min\{m, n\}$. The *maximal column rank, $mc(A)$*, of $A \in \mathbb{M}_{m\times n}(\mathbb{R}_{\max})$ is the maximal number of the columns of which are linearly independent over $\mathbb{R}_{\max}$.

It follows that

$$0 \le r(A) \le mc(A) \le n \tag{2.1}$$

for all matrices over $\mathbb{R}_{\max}$.

The maximal column rank of a matrix may actually exceed its rank over $\mathbb{R}_{\max}$. For example, we consider a matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \in \mathbb{M}_{3\times 4}(\mathbb{R}_{\max}). \tag{2.2}$$

Then Example 2.5 (below) implies that $r(A) = 3$, but $mc(A) = 4$.

LEMMA 2.1. *For any $m \times n$ matrix $A$ over $\mathbb{R}_{\max}$, we have that $r(A) = 1$ if and only if $mc(A) = 1$.*

*Proof.* If $r(A) = 1$, then $A$ can be factored as

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \otimes \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix} = \begin{bmatrix} a_1b_1 & \cdots & a_1b_i & \cdots & a_1b_n \\ a_2b_1 & \cdots & a_2b_i & \cdots & a_2b_n \\ \vdots & & \vdots & & \vdots \\ a_mb_1 & \cdots & a_mb_i & \cdots & a_mb_n \end{bmatrix}.$$

If there exist nonzero $b_i$ and $b_j$ for some $i \neq j$, then $b_j = \frac{b_j}{b_i} b_i$. Hence $i$th and $j$th columns of $A$ are linearly dependent. This implies that any two columns of $A$ are linearly dependent. Therefore $mc(A) = 1$.

The converse is obvious from (2.1). □

Let $\beta(\mathbb{R}_{\max}, m, n)$ be the largest integer $k$ such that for all $A \in \mathbb{M}_{m\times n}(\mathbb{R}_{\max})$, $r(A) = mc(A)$ if $r(A) \leq k$. The matrix $A$ in (2.2) shows that $\beta(\mathbb{R}_{\max}, 3, 4) < 3$. In general $0 \leq \beta(\mathbb{R}_{\max}, m, n) \leq n$. We also obtain that

$$(2.3) \qquad r\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) = r(A) \quad \text{and} \quad mc\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) = mc(A)$$

for all $m \times n$ matrices $A$ over $\mathbb{R}_{\max}$.

LEMMA 2.2. *If $mc(A) > r(A)$ for some $p \times q$ matrix $A$ over $\mathbb{R}_{\max}$, then for all $m \geq p$ and $n \geq q$, $\beta(\mathbb{R}_{\max}, m, n) < r(A)$.*

*Proof.* Since $mc(A) > r(A)$ for some $p \times q$ matrix $A$, we have $\beta(\mathbb{R}_{\max}, p, q) < r(A)$ from the definition of $\beta$. Let $B = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$ be an $m \times n$ matrix containing $A$ as a submatrix. Then by (2.3),

$$r(B) = r(A) < mc(A) = mc(B).$$

So, $\beta(\mathbb{R}_{\max}, m, n) < r(A)$ for all $m \geq p$ and $n \geq q$. □

LEMMA 2.3. *For any $A \in \mathbb{M}_{2\times n}(\mathbb{R}_{\max})$ with $n \geq 2$, we have that $r(A) = 2$ if and only if $mc(A) = 2$.*

*Proof.* Let $r(A) = 2$. If $n = 2$, then (2.1) implies that $mc(A) = 2$. So we may assume that $n \geq 3$. Let

$$\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \text{ and } \begin{pmatrix} e \\ f \end{pmatrix}$$

be any three columns of $A$. Then we claim that the three columns are linearly dependent. To show this, we consider three cases.

Case 1) There are at least two zero elements in $\{a, b, c, d, e, f\}$. Then it is obvious that the three columns are linearly dependent.

Case 2) There is only one zero element in $\{a, b, c, d, e, f\}$. Then, without loss of generality, we may take $b = 0$ and $\max\{\frac{c}{d}, \frac{e}{f}\} = \frac{e}{f}$. Thus we have

$$\frac{e}{a}\begin{pmatrix} a \\ 0 \end{pmatrix} \oplus \frac{f}{d}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} e \oplus \frac{f}{d}c \\ f \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}.$$

So the given three columns are linearly dependent.

Case 3) There is no zero element in $\{a, b, c, d, e, f\}$. Then, without loss of generality, we may assume that $\frac{a}{b} \leq \frac{c}{d} \leq \frac{e}{f}$. Thus we obtain

$$\frac{d}{b}\begin{pmatrix} a \\ b \end{pmatrix} \oplus \frac{c}{e}\begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} \frac{d}{b}a \oplus c \\ d \oplus \frac{c}{e}f \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}.$$

Hence the given three columns are linearly dependent.

This shows that $mc(A) < 3$. Therefore $mc(A) = 2$ by (2.1).

The converse follows from (2.1) and Lemma 2.1. □

THEOREM 2.4. *For any $A \in \mathbb{M}_{m\times n}(\mathbb{R}_{\max})$ with $m \geq 2$ and $n \geq 2$, we have that $r(A) = 2$ implies $mc(A) = 2$ and conversely.*

*Proof.* Let $r(A) = 2$. Then $A$ can be factored as $A = B \otimes C$ for some $m \times 2$ matrix $B = [\boldsymbol{x}\ \boldsymbol{y}]$ and $2 \times n$ matrix $C$ with $r(B) = r(C) = 2$. If $n = 2$, then (2.1) implies that $mc(A) = 2$. So we can assume that $n \geq 3$. Then any column of $A$ has the form $a\boldsymbol{x} \oplus b\boldsymbol{y}$ with a column $\begin{pmatrix} a \\ b \end{pmatrix}$ of $C$. Let $a\boldsymbol{x} \oplus b\boldsymbol{y}$, $c\boldsymbol{x} \oplus d\boldsymbol{y}$ and $e\boldsymbol{x} \oplus f\boldsymbol{y}$ be any three columns of $A$. Then

$$\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \text{ and } \begin{pmatrix} e \\ f \end{pmatrix}$$

are columns of $C$ and hence they are linearly dependent by Lemma 2.3. Now we consider all three cases in the proof of Lemma 2.3. But it is sufficient to consider the case 3), that is, $\{a, b, c, d, e, f\}$ has no zero element with $\frac{a}{b} \leq \frac{c}{d} \leq \frac{e}{f}$. Then the proof of Lemma 2.3 implies that

$$c = \frac{d}{b}a \oplus \frac{c}{e}e \quad \text{and} \quad d = \frac{d}{b}b \oplus \frac{c}{e}f.$$

Thus we have

$$cx \oplus dy = \left(\frac{d}{b}a \oplus \frac{c}{e}e\right) x \oplus \left(\frac{d}{b}b \oplus \frac{c}{e}f\right) y = \frac{d}{b}(ax \oplus by) \oplus \frac{c}{e}(ex \oplus fy).$$

Therefore we have $mc(A) \leq 2$, which implies that $mc(A) = 2$ by (2.1).

The converse is obvious from (2.1) and Lemma 2.1. □

EXAMPLE 2.5. Consider the matrix in (2.2);

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \in \mathbb{M}_{3\times 4}(\mathbb{R}_{\max}).$$

Since all columns of $A$ are linearly independent over $\mathbb{R}_{\max}$, we have $mc(A) = 4$. And $r(A) \geq 3$ by Lemma 2.1 and Theorem 2.4. Hence $r(A) = 3$ from (2.1). □

THEOREM 2.6. *For $m \times n$ matrices over max algebra, we have the values of $\beta$ as follows;*

$$\beta(\mathbb{R}_{\max}, m, n) = \begin{cases} 1 & \text{if } \min\{m,n\} = 1, \\ 3 & \text{if } m \geq 3, \text{ and } n = 3, \\ 2 & \text{otherwise}. \end{cases}$$

*Proof.* If $\min\{m,n\} = 1$, then we have $\beta(\mathbb{R}_{\max}, m, n) = 1$ from Lemma 2.1. Consider the matrix $A \in \mathbb{M}_{3\times 4}(\mathbb{R}_{\max})$ in (2.2). Then $r(A) = 3$ and $mc(A) = 4$. Thus we have $\beta(\mathbb{R}_{\max}, m, n) \leq 2$ for all $m \geq 3$ and $n \geq 4$ by Lemma 2.2.

Suppose $m \geq 2$ and $n \geq 2$. Then we have $\beta(\mathbb{R}_{\max}, m, n) \geq 2$ for all $m \geq 2$ and $n \geq 2$ by Lemma 2.1 and Theorem 2.4. Moreover, for $A \in \mathbb{M}_{m\times 3}(\mathbb{R}_{\max})$ with $m \geq 3$, $r(A) = 3$ implies $mc(A) = 3$ from (2.1) and $mc(A) = 3$ implies $r(A) = 3$ from Lemma 2.1 and Theorem 2.4. Thus we have $\beta(\mathbb{R}_{\max}, m, 3) = 3$ for $m \geq 3$.

Therefore we have determined the values of $\beta$, as required. □

## 3. Linear operators that preserve maximal column rank of matrices over max algebra

In this section we obtain characterizations of the linear operators that preserve maximal column rank of matrices over max algebra.

A linear operator $T$ on $\mathbb{M}_{m\times n}(\mathbb{R}_{\max})$ is said to *preserve maximal column rank* if $mc(T(A)) = mc(A)$ for all $A \in \mathbb{M}_{m\times n}(\mathbb{R}_{\max})$. It *preserves* *maximal column rank r* if $mc(T(A)) = r$ whenever $mc(A) = r$. For the *rank preserver* and *rank r preserver,* they are defined similarly.

A square matrix $A$ over $\mathbb{R}_{\max}$ is called *monomial* if it has exactly one nonzero element in each row and column. Since $\mathbb{M}_{n\times n}(\mathbb{R}_{\max})$ is a semiring, we can consider the invertible members of its multiplicative monoid. The monomial matrices are precisely invertible matrices over $\mathbb{R}_{\max}$ [2].

LEMMA 3.1. *The maximal column rank of a matrix over $\mathbb{R}_{\max}$ is preserved under pre or post-multiplication by an invertible matrix.*

*Proof.* For the case of pre-multiplication, let $A$ be any matrix in $\mathbb{M}_{m\times n}(\mathbb{R}_{\max})$, and $U$ be an invertible matrix in $\mathbb{M}_{m\times m}(\mathbb{R}_{\max})$. Then $U$ is monomial. If $mc(A) = r$, then there exists $r$ linearly independent columns $\boldsymbol{a}_{i(1)}, \dots, \boldsymbol{a}_{i(r)}$ in $A$ which are maximal. Then $U\otimes\boldsymbol{a}_{i(1)}, \dots, U\otimes \boldsymbol{a}_{i(r)}$ are linearly independent columns of $U \otimes A$. Thus $mc(U \otimes A) \geq r$.

Conversely, if $mc(U \otimes A) = r$, then there exists $r$ linearly independent columns $\boldsymbol{b}_{i(1)}, \dots, \boldsymbol{b}_{i(r)}$ in $U \otimes A$ which are maximal. Then $U^{-1} \otimes \boldsymbol{b}_{i(1)}, \dots, U^{-1} \otimes \boldsymbol{b}_{i(r)}$ are linearly independent columns of $U^{-1} \otimes U \otimes A = A$. Hence $mc(A) \geq r$. Therefore we have $mc(A) = mc(U \otimes A)$.

For the case of post-multiplication, let $V$ be an invertible matrix in $\mathbb{M}_{n\times n}(\mathbb{R}_{\max})$. Then $V$ is monomial. Let $v_i$ be the nonzero entry of the $i$-th column of $V$. Then we have

$$A \otimes V = [\,\boldsymbol{a}_1\, \boldsymbol{a}_2 \cdots \boldsymbol{a}_n\,] \otimes V = [\,v_1\boldsymbol{a}_{i(1)}\, v_2\boldsymbol{a}_{i(2)}\, \cdots\, v_n\boldsymbol{a}_{i(n)}\,]$$

where, $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$ are the columns of $A$ and $\{i(1), \dots, i(n)\}$ is a permutation of $\{1, \dots, n\}$. If $\boldsymbol{a}_x, \boldsymbol{a}_y, \dots, \boldsymbol{a}_z$ are linearly independent columns of $A$, then $v_x\boldsymbol{a}_{i(x)}, v_y\boldsymbol{a}_{i(y)}, \dots, v_z\boldsymbol{a}_{i(z)}$ are linearly independent columns of $A \otimes V$, and conversely. Hence $mc(A) = mc(A \otimes V)$. □

We say that a linear operator $T$ is a *$(U, V)$-operator* if there exist monomial matrices $U \in \mathbb{M}_{m\times m}(\mathbb{R}_{\max})$ and $V \in \mathbb{M}_{n\times n}(\mathbb{R}_{\max})$ such that either $T(A) = U \otimes A \otimes V$ or $m = n$, $T(A) = U \otimes A^t \otimes V$ for all $A \in \mathbb{M}_{m\times n}(\mathbb{R}_{\max})$.

EXAMPLE 3.2. Let

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

be a matrix in $\mathbb{M}_{4\times 4}(\mathbb{R}_{\max})$. Then $mc(B) = 3$ since the first three columns of $B$ are linearly independent over $\mathbb{R}_{\max}$. But the maximal column rank of

$$B^t = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is 4 by (2.3) and Example 2.5. Thus the transposition operator does not preserve maximal column rank 3 on $\mathbb{M}_{m\times n}(\mathbb{R}_{\max})$ for $m = n \geq 4$.

THEOREM 3.3. ([2]) *If $T$ is a linear operator on $\mathbb{M}_{m\times n}(\mathbb{R}_{\max})$ with $m > 1$ and $n > 1$, then the following statements are equivalent:*

(1) *$T$ is a rank preserver;*
(2) *$T$ preserves the ranks of all rank one and rank two matrices;*
(3) *$T$ is a $(U, V)$-operator.*

THEOREM 3.4. *Suppose $T$ is a linear operator on $\mathbb{M}_{m\times n}(\mathbb{R}_{\max})$ with $m \geq 4$ and $n \geq 3$. Then the following statements are equivalent:*

(1) *$T$ preserves maximal column rank;*
(2) *$T$ preserves maximal column ranks 1, 2 and 3;*
(3) *There exist monomial matrices $U \in \mathbb{M}_{m\times m}(\mathbb{R}_{\max})$ and $V \in \mathbb{M}_{n\times n}(\mathbb{R}_{\max})$ such that $T(A) = U \otimes A \otimes V$ for all $A \in \mathbb{M}_{m\times n}(\mathbb{R}_{\max})$.*

*Proof.* Obviously (1) implies (2). Assume that $T$ preserves maximal column ranks 1, 2 and 3. Then $T$ preserves ranks 1 and 2 by Lemma 2.1 and Theorem 2.4. Theorem 3.3 implies that $T$ is a $(U, V)$-operator. But the transposition operator does not preserve maximal column rank 3 by Example 3.2. Hence $T$ has the form $T(A) = U \otimes A \otimes V$ for some monomial matrices $U \in \mathbb{M}_{m\times m}(\mathbb{R}_{\max})$ and $V \in \mathbb{M}_{n\times n}(\mathbb{R}_{\max})$. That is, (2) implies (3). Finally, if we assume (3), then $T$ preserves maximal column rank by Lemma 3.1. Hence (3) implies (1). □

We have assumed that $m \geq 4$ and $n \geq 3$ in the Theorem 3.4. For the other cases, the linear operators which preserve maximal column rank are the same as rank preservers in the Theorem 3.3. We show it in the following remark.

REMARK 3.5. Suppose $T$ is a linear operator on $\mathbb{M}_{m\times n}(\mathbb{R}_{\max})$ with $m \leq 3$ or $n \leq 2$. Then the following statements are equivalent:

(1) $T$ preserves maximal column rank;
(2) $T$ preserves maximal column ranks 1 and 2;
(3) $T$ is a $(U, V)$-operator.

*Proof.* It is obvious that (1) implies (2). Assume that $T$ preserves maximal column ranks 1 and 2. Then $T$ preserves ranks 1 and 2 by Theorem 2.6. Thus $T$ is a $(U, V)$-operator by Theorem 3.3. That is, (2) implies (3). Assume that $T$ is a $(U, V)$-operator. Then for any $A \in \mathbb{M}_{m\times n}(\mathbb{R}_{\max})$, there exist monomial matrices $U \in \mathbb{M}_{m\times m}(\mathbb{R}_{\max})$ and $V \in \mathbb{M}_{n\times n}(\mathbb{R}_{\max})$ such that either $T(A) = U \otimes A \otimes V$ or $m = n$, $T(A) = U \otimes A^t \otimes V$. For the case $T(A) = U \otimes A \otimes V$, $T$ preserves all maximal column ranks by Lemma 3.1. For the case $m = n$ and $T(A) = U \otimes A^t \otimes V$, we have $m = n \leq 3$ from the conditions on $m$ and $n$. But Theorem 2.6 implies that $r(A) = mc(A) \leq 3$ for $m = n \leq 3$. Then $T(A) = U \otimes A^t \otimes V$ preserves all ranks by Theorem 3.3 and hence it preserves all maximal column ranks for $m = n \leq 3$. Therefore (3) implies (1). □

Thus we have characterized the linear operators that preserve the maximal column rank of matrices over max algebra.

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