

## PARANORMAL CONTRACTIONS AND INVARIANT SUBSPACES

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ABSTRACT. It is shown that if a paranormal contraction  $T$  has no nontrivial invariant subspace, then it is a proper contraction. Moreover, the nonnegative operator  $Q = T^{2*}T^2 - 2T^*T + I$  also is a proper contraction. If a quasihyponormal contraction has no nontrivial invariant subspace then, in addition, its defect operator  $D$  is a proper contraction and its self-commutator is a trace-class strict contraction. Furthermore, if one of  $Q$  or  $D$  is compact, then so is the other, and  $Q$  and  $D$  are strict contraction.

### 1. Introduction

By an operator we mean a bounded linear transformation of a nonzero complex Hilbert space  $\mathcal{H}$  into itself. Let  $\mathcal{B}[\mathcal{H}]$  denote the algebra of all operators on  $\mathcal{H}$ . For an arbitrary operator  $T$  in  $\mathcal{B}[\mathcal{H}]$  set, as usual,  $|T| = (T^*T)^{\frac{1}{2}}$  and  $[T^*, T] = T^*T - TT^* = |T|^2 - |T^*|^2$  (the self-commutator of  $T$ ) so that  $T^*[T^*, T]T = |T^2|^2 - |T|^4$ , and consider the following standard definitions:  $T$  is hyponormal if  $[T^*, T]$  is nonnegative (i.e.,  $|T^*|^2 \leq |T|^2$ ; equivalently,  $\|T^*x\| \leq \|Tx\|$  for every  $x$  in  $\mathcal{H}$ ), quasihyponormal if  $T^*[T^*, T]T$  is nonnegative (i.e.,  $|T|^4 \leq |T^2|^2$ ), paranormal if  $\|Tx\|^2 \leq \|T^2x\|\|x\|$  for every  $x$  in  $\mathcal{H}$ , and normaloid if  $r(T) = \|T\|$  (where  $r(T)$  denotes the spectral radius of  $T$ ). These classes are related by *proper* inclusion:

Hyponormal  $\subset$  Quasihyponormal  $\subset$  Paranormal  $\subset$  Normaloid.

A contraction is an operator  $T$  such that  $\|T\| \leq 1$  (i.e.,  $\|Tx\| \leq \|x\|$  for every  $x$  in  $\mathcal{H}$ ; equivalently,  $T^*T \leq I$ ). A proper contraction is an

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operator  $T$  such that  $\|Tx\| < \|x\|$  for every nonzero  $x$  in  $\mathcal{H}$  (equivalently,  $T^*T < I$ ). A strict contraction is an operator  $T$  such that  $\|T\| < 1$  (i.e.,  $\sup_{0 \neq x} (\|Tx\|/\|x\|) < 1$  or, equivalently,  $T^*T \prec I$ , which means that  $T^*T \leq \gamma I$  for some  $\gamma \in (0, 1)$ ). Again, these are related by *proper* inclusion:

Strict Contraction  $\subset$  Proper Contraction  $\subset$  Contraction.

An operator  $T$  on  $\mathcal{H}$  is uniformly stable if the power sequence  $\{T^n\}_{n \geq 1}$  converges uniformly to the null operator (i.e.,  $\|T^n\| \rightarrow 0$ ). It is strongly stable if  $\{T^n\}_{n \geq 1}$  converges strongly to the null operator (i.e.,  $\|T^n x\| \rightarrow 0$  for every  $x$  in  $\mathcal{H}$  — notation:  $T^n \xrightarrow{s} O$ ).  $T$  is uniformly stable if and only if  $r(T) < 1$ . It is clear that uniform stability implies strong stability. If  $T$  is a strongly stable contraction, then it is usual to say that  $T$  is a  $\mathcal{C}_0$ -contraction. If  $T^*$  is a strongly stable contraction then  $T$  is a  $\mathcal{C}_0$ -contraction. On the other extreme, if a contraction  $T$  is such that  $\|T^n x\| \not\rightarrow 0$  for every nonzero vector  $x$  in  $\mathcal{H}$ , then it is said to be a  $\mathcal{C}_1$ -contraction. Dually, if a contraction  $T$  is such that  $\|T^{*n} x\| \not\rightarrow 0$  for every nonzero vector  $x$  in  $\mathcal{H}$ , then it is a  $\mathcal{C}_1$ -contraction. These are the Nagy-Foiaş classes of contractions (see [16], p.72]. All combinations are possible leading to classes  $\mathcal{C}_{00}$ ,  $\mathcal{C}_{01}$ ,  $\mathcal{C}_{10}$  and  $\mathcal{C}_{11}$ . In particular,  $T$  and  $T^*$  are both strongly stable contractions if and only if  $T$  is a  $\mathcal{C}_{00}$ -contraction. Uniformly stable contractions are of class  $\mathcal{C}_{00}$ .

It was recently proved in [9] that *if a hyponormal contraction  $T$  has no nontrivial invariant subspace, then  $T$  is a proper contraction and its self-commutator  $[T^*, T]$  is a strict contraction*. This was extended in [7] to the class

$$\mathcal{U} = \{T \in \mathcal{B}[\mathcal{H}]: 0 \leq |T^2| - |T|^2\},$$

which properly includes the class of quasihyponormal operators and is properly included in the class of paranormal operators, as follows: *if a contraction  $T$  in  $\mathcal{U}$  has no nontrivial invariant subspace, then both  $T$  and the nonnegative operator  $|T^2| - |T|^2$  are proper contractions*. In this paper we extend those results to the class of paranormal operators. It is proved that if a paranormal contraction  $T$  has no nontrivial invariant subspace, then  $T$  is still a proper contraction and so is the nonnegative operator  $|T^2|^2 - 2|T|^2 + I$ .

## 2. An invariant subspace theorem for paranormal contractions

Paranormal operators constitute a class of operators that has been much investigated in current literature over the past three decades. They share some important properties inherited from hyponormal operators, besides being normaloid. Let us single out a few of these properties. If  $T$  is a paranormal operator, then the following holds.

1. Nonzero eigenvalues of  $T$  are normal eigenvalues (i.e., if  $\ker(\lambda I - T) \neq \{0\}$  for some  $\lambda \neq 0$ , then  $\ker(\lambda I - T) \subseteq \ker(\bar{\lambda}I - T^*)$ ) [3], [5].
2. If the spectrum of  $T$  is countable, then  $T$  is normal (in particular, compact paranormal operators are normal) [13].
3.  $T$  satisfies Weyl's Theorem (so that  $\sigma_{00}(T) = \sigma_0(T)$ , where  $\sigma_{00}(T)$  and  $\sigma_0(T)$  denote the sets of isolated points of the spectrum of  $T$  that are eigenvalues of  $T$  of finite algebraic and geometric multiplicities, respectively) [3].
4. If  $T$  is a completely nonunitary contraction, then  $T$  is of class  $\mathcal{C}_0$  (i.e., every completely nonunitary coparanormal contraction is strongly stable) [12].
5. If  $T$  is an injective pure contraction and the defect operator  $(1 - T^*T)^{\frac{1}{2}}$  is of Hilbert-Schmidt class  $\mathcal{B}_2[\mathcal{H}]$ , then  $T$  is a  $\mathcal{C}_{10}$ -contraction [5], [6].
6.  $T$  is not supercyclic [2].

Here is an alternative definition of paranormal operators. Take any  $T$  in  $\mathcal{B}[\mathcal{H}]$  and, for each  $\lambda > 0$ , set

$$Q_\lambda = |T^2|^2 - 2\lambda|T|^2 + \lambda^2I$$

in  $\mathcal{B}[\mathcal{H}]$ . An operator  $T$  is paranormal if and only if  $Q_\lambda$  is nonnegative for all  $\lambda > 0$  (cf. [1], also see [15]). We shall say that an operator  $T$  is of class  $\mathcal{Q}_\lambda$  if  $Q_\lambda$  is nonnegative; that is, for each  $\lambda > 0$  set

$$\mathcal{Q}_\lambda = \{T \in \mathcal{B}[\mathcal{H}] : 0 \leq |T^2|^2 - 2\lambda|T|^2 + \lambda^2I\}.$$

Thus  $T$  is paranormal if and only if  $T \in \bigcap_{\lambda > 0} \mathcal{Q}_\lambda$  so that every paranormal operator is of class  $\mathcal{Q}_1$ . Put  $\mathcal{Q} = \mathcal{Q}_1$  for short.

**LEMMA 1.** *If  $T$  is a contraction of class  $\mathcal{Q}_1$ , then the nonnegative operator  $Q = |T^2|^2 - 2|T|^2 + I$  is a contraction whose power sequence  $\{Q^n\}_{n \geq 1}$  converges strongly to an orthogonal projection  $P$ , and  $TP = O$ .*

*Proof.* Take any  $x$  in  $\mathcal{H}$  and any nonnegative integer  $n$ . If  $T \in \mathcal{Q}_1$ , then  $O \leq Q$ . Let  $R = Q^{\frac{1}{2}}$  be the unique nonnegative square root of  $Q$ . Recall that  $\| |T|x \| = \|Tx\|$  for every  $x \in \mathcal{H}$ , for all  $T$  in  $\mathcal{B}[\mathcal{H}]$ . If, in addition,  $T$  is a contraction, then

$$\begin{aligned} \langle Q^{n+1}x; x \rangle &= \|R^{n+1}x\|^2 = \langle QR^n x; R^n x \rangle \\ &= \| |T|^2 R^n x \|^2 - 2\| |T| R^n x \|^2 + \|R^n x\|^2 \\ &\leq \|R^n x\|^2 - \|TR^n x\|^2 \\ &\leq \|R^n x\|^2 = \langle Q^n x; x \rangle. \end{aligned}$$

Thus  $R$  (and so  $Q$ ) is a contraction (set  $n = 0$ ), and  $\{Q^n\}_{n \geq 1}$  is a decreasing sequence of nonnegative contractions. Since a bounded monotone sequence of self-adjoint operators converges strongly, the weak limit of any weakly convergent power sequence is idempotent, and since the set of all nonnegative operators is weakly (thus strongly) closed, it follows that  $\{Q^n\}_{n \geq 1}$  converges strongly to an orthogonal projection (i.e., to a nonnegative idempotent), say,  $P$ . Moreover,

$$\begin{aligned} \sum_{n=0}^m \|TR^n x\|^2 &\leq \sum_{n=0}^m (\|R^n x\|^2 - \|R^{n+1} x\|^2) \\ &= \|x\|^2 - \|R^{m+1} x\|^2 \leq \|x\|^2 \end{aligned}$$

for all nonnegative integers  $m$  and every  $x$  in  $\mathcal{H}$ . Therefore,  $\|TR^n x\| \rightarrow 0$  as  $n \rightarrow \infty$ , and hence

$$TPx = T \lim_n Q^n x = \lim_n TR^{2n} x = 0,$$

for every  $x \in \mathcal{H}$ , so that  $TP = O$ . □

REMARK 1. Take an arbitrary  $\lambda$  in  $(0, 1)$ . If  $T$  is a contraction of class  $\mathcal{Q}_\lambda$ , then

$$Q_\lambda = |T^2|^2 - 2\lambda|T|^2 + \lambda^2 I \quad \text{is a strict contraction.}$$

Indeed, take any  $x \in \mathcal{H}$  and let  $R_\lambda$  be the unique nonnegative square root of  $Q_\lambda$ . Since  $T \in \mathcal{Q}_\lambda$  is a contraction, it follows that

$$\begin{aligned} \|R_\lambda^{n+1} x\|^2 &= \|T^2 R_\lambda^n x\|^2 - 2\lambda \|TR_\lambda^n x\|^2 + \lambda^2 \|R_\lambda^n x\|^2 \\ &\leq (1 - 2\lambda) \|TR_\lambda^n x\|^2 + \lambda^2 \|R_\lambda^n x\|^2. \end{aligned}$$

If  $\lambda$  lies in  $(0, \frac{1}{2}]$ , then  $\|R_\lambda^{n+1} x\|^2 \leq (1 - 2\lambda) \|R_\lambda^n x\|^2 + \lambda^2 \|R_\lambda^n x\|^2 = (1 - \lambda)^2 \|R_\lambda^n x\|^2$  so that  $\|R_\lambda^{n+1}\| \leq (1 - \lambda) \|R_\lambda^n\|$ . If  $\lambda$  lies in  $(\frac{1}{2}, 1)$ , then  $\|R_\lambda^{n+1} x\|^2 \leq \lambda^2 \|R_\lambda^n x\|^2$  so that  $\|R_\lambda^{n+1}\| \leq \lambda \|R_\lambda^n\|$ . In both cases

$\|R_\lambda^n\| \rightarrow 0$ ; that is,  $R_\lambda$  is uniformly stable. Thus  $\|Q_\lambda^n\| \rightarrow 0$  or, equivalently,  $r(Q_\lambda) < 1$ . Since  $Q_\lambda$  is nonnegative,  $\|Q_\lambda\| = r(Q_\lambda)$ .

Note: If  $T = O$ , then  $Q_\lambda = \lambda^2 I$  is power unbounded for every  $\lambda > 1$ .

**THEOREM 1.** *If a paranormal contraction  $T$  has no nontrivial invariant subspace, then it is a proper contraction and the nonnegative operator  $Q = |T^2|^2 - 2|T|^2 + I$  is a strongly stable contraction.*

*Proof.* (i) Take any  $T \in \mathcal{B}[\mathcal{H}]$  and set

$$\mathcal{M} = \{x \in \mathcal{H} : \|Tx\| = \|T\|\|x\|\}.$$

It is well known (see e.g. [9]) that  $\|Tx\| = \|T\|\|x\|$  if and only if  $|T|^2 x = \|T\|^2 x$ . Thus  $\mathcal{M} = \ker(\|T\|^2 I - |T|^2)$ , which is a closed linear manifold of  $\mathcal{H}$ , so that  $\mathcal{M}$  is a subspace of  $\mathcal{H}$ . Take an arbitrary  $x$  in  $\mathcal{M}$ . If  $T$  is a paranormal operator (i.e., if  $\|Tx\|^2 \leq \|T^2 x\|\|x\|$  for every  $x \in \mathcal{H}$ ), then

$$\|T\|\|Tx\|\|x\| = \|Tx\|^2 \leq \|T^2 x\|\|x\| \leq \|T\|\|Tx\|\|x\|,$$

and hence  $\|T(Tx)\| = \|T\|\|Tx\|$  so that  $Tx$  lies in  $\mathcal{M}$ . Therefore, if  $T$  is paranormal, then  $\mathcal{M}$  is an invariant subspace for  $T$ . Now suppose the paranormal operator  $T$  is a contraction:  $\|Tx\| \leq \|x\|$  for every  $x \in \mathcal{H}$ . If  $T$  is a strict contraction, then it is trivially a proper contraction. If  $T$  is a nonstrict contraction, then  $\|T\| = 1$  and, consequently,  $\mathcal{M} = \{x \in \mathcal{H} : \|Tx\| = \|x\|\}$ . If, in addition,  $T$  has no nontrivial invariant subspace, then the invariant subspace  $\mathcal{M}$  is trivial: either  $\mathcal{M} = \{0\}$  or  $\mathcal{M} = \mathcal{H}$ ; but if  $\mathcal{M} = \mathcal{H}$ , then  $T$  is an isometry, and isometries have nontrivial invariant subspaces. Thus  $\mathcal{M} = \{0\}$  so that  $\|Tx\| < \|x\|$  for every nonzero  $x$  in  $\mathcal{H}$ ; that is,  $T$  is a proper contraction.

(ii) Let  $T \neq O$  be a paranormal contraction. According to Lemma 1 the nonnegative operator  $Q$  is a contraction,  $Q^n \xrightarrow{s} P$ , and  $TP = O$  so that  $PT^* = O$  (recall:  $P$  is self-adjoint). If  $T$  has no nontrivial invariant subspace, then  $T^*$  has no nontrivial invariant subspace as well. Since  $\ker P$  is a nonzero invariant subspace for  $T^*$  whenever  $PT^* = O$  and  $T \neq O$ , it follows that  $\ker P = \mathcal{H}$ . Hence  $P = O$ , and therefore  $Q^n \xrightarrow{s} O$ .  $\square$

### 3. Proper contractions and strongly stable contractions

In general, proper contractions and strongly stable contractions are not related. It is true that a compact proper contraction is of class  $\mathcal{C}_{00}$  (reason: the concepts of proper and strict contraction coincide for compact operators [9], and a strict contraction is, of course, uniformly

stable) but there exist compact contractions of class  $\mathcal{C}_{00}$  that are not proper (for instance, uniformly stable compact nonproper contractions — sample:  $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$  is a nilpotent nonproper contraction on  $\mathbb{C}^2$ ). On the other hand, there exist (noncompact) proper contractions of class  $\mathcal{C}_{11}$  (example: the bilateral weighted shift  $T = \text{shift}(\{1 - (|k| + 2)^{-2}\}_{k \in \mathbb{Z}})$  on  $\ell^2$  is a proper contraction of class  $\mathcal{C}_{11}$  [9]). However, as we shall see next, every paranormal strongly stable operator is a proper contraction, and every quasinormal proper contraction is of class  $\mathcal{C}_{00}$ . (An operator  $T$  is quasinormal if it commutes with  $T^*T$ ; normal operators are quasinormal, and quasinormal operators are hyponormal.) Although they have a straightforward application to Theorem 1, the results below seem to be important in their own right.

LEMMA 2. *Take any  $T$  in  $\mathcal{B}[\mathcal{H}]$  and consider the following assertions.*

- (a)  *$T$  is strongly stable.*
- (b)  *$T$  is a proper contraction.*
- (c)  *$T$  is a  $\mathcal{C}_{00}$ -contraction.*

*If  $T$  is paranormal, then (a) implies (b) and (c). If  $T$  is quasinormal, then these assertions are pairwise equivalent.*

*Proof.* (i) Take an arbitrary  $x$  in  $\mathcal{H}$ . It was shown in [10] (also see [8], p.78) that  $\|Tx\|^n \leq \|T^n x\| \|x\|^{n-1}$  for every  $n \geq 1$  whenever  $T$  is hyponormal by using only the fact that hyponormal operators are paranormal. Thus, if  $T$  is paranormal, then

$$\frac{\|Tx\|^n}{\|x\|^n} \leq \frac{\|T^n x\|}{\|x\|}$$

for every nonzero vector  $x$  in  $\mathcal{H}$  and every positive integer  $n$ . Therefore (a) implies (b) whenever  $T$  is paranormal. Moreover, every strongly stable operator is completely nonunitary, and completely nonunitary paranormal contractions are of class  $\mathcal{C}_{.0}$  [12]. Thus, if  $T$  is a strongly stable paranormal operator, then  $T$  is a (proper) contraction of class  $\mathcal{C}_{00}$ , and hence (a) also implies (c) whenever  $T$  is paranormal.

(ii) If  $T$  is a quasinormal contraction, then the strong limits of  $\{T^{*n} T^n\}_{n \geq 1}$  and  $\{T^n T^{*n}\}_{n \geq 1}$  are projections [11]. If  $T$  is a contraction and if the strong limits of  $\{T^{*n} T^n\}_{n \geq 1}$  and  $\{T^n T^{*n}\}_{n \geq 1}$  are projections, then

$$T = B \oplus S_- \oplus S_+ \oplus U,$$

where  $B$  is a  $\mathcal{C}_{00}$ -contraction,  $S_+$  is a unilateral shift,  $S_-$  is a backward unilateral shift, and  $U$  is unitary; where any of the above direct summands may be missing [11] (also see [8], p.83). If, in addition,  $T$  is a

proper contraction, then any direct summand of it is again a proper contraction, and hence  $T = B$  (reason:  $S_+$ ,  $S_-^*$  and  $U$  are isometries, thus nonproper contractions, and proper contractiveness is  $*$ -invariant [9]). Hence (b) implies (c) whenever  $T$  is quasinormal. Tautologically, (c) implies (a) for every operator and, since quasinormal operators are paranormal, (a) implies (b) for quasinormal operators according to part (i).  $\square$

REMARK 2. Lemma 2 says that (b) implies (c), and hence (b) implies (a), for a quasinormal operator. However, (b) does not imply (a), and hence (b) does not imply (c), even for a hyponormal operator. In fact, the unilateral weighted shift  $T = \text{shift}(\{(k+1)^{\frac{1}{2}}(k+2)^{-1}(k+3)^{\frac{1}{2}}\}_{k \geq 0})$  on  $\ell_+^2$  is a hyponormal proper contraction but not strongly stable; it is a  $\mathcal{C}_{10}$ -contraction (the positive increasing weight sequence converges to 1 and  $\|T^n x\| \rightarrow \|A^{\frac{1}{2}} x\|$  for every  $x$  in  $\ell_+^2$ , where the operator  $A$  is a positive diagonal on  $\ell_+^2$ , namely,  $A = \text{diag}(\{(k+1)(k+2)^{-1}\}_{k \geq 0})$  — cf. [11] or [8] pp.51–52).

Since nonnegative operators are paranormal, Lemma 2 yields the following corollary of Theorem 1.

COROLLARY 1. *If a paranormal contraction  $T$  has no nontrivial invariant subspace, then both  $T$  and  $Q = |T^2|^2 - 2|T|^2 + I$  are proper contractions.*

#### 4. Quasihyponormal contractions

If the hypothesis of paranormality is replaced with the stronger hypothesis of quasihyponormality in Corollary 1, then the defect operator is also a proper contraction. This will be shown in Corollary 2 below after recalling the following relationship between the defect operator and the self-commutator. Take any operator  $T$  in  $\mathcal{B}[\mathcal{H}]$ . Set  $Q = |T^2|^2 - 2|T|^2 + I = T^{2*}T^2 - 2T^*T + I$  and  $D = I - |T|^2 = I - T^*T$ . Thus  $T^*DT = |T|^2 - |T^2|^2$ , and hence

$$Q = D - T^*DT.$$

Clearly,  $T$  is a contraction (a proper contraction) if and only if  $D$  is nonnegative (positive). If  $T$  is a contraction, then  $D^{\frac{1}{2}}$  is the defect operator of  $T$ . Consider the self-commutator of  $T$ , viz.  $[T^*, T] = T^*T -$

$TT^*$ . Since  $T^*[T^*, T]T = |T^2|^2 - |T|^4$ , it follows that

$$Q = T^*[T^*, T]T + D^2.$$

These are self-adjoint operators in  $\mathcal{B}[\mathcal{H}]$  for every  $T$ . Recall: if  $D$  is of Schatten class  $\mathcal{B}_p[\mathcal{H}]$  for some  $p \geq 1$ , then  $T$  has a nontrivial invariant subspace (see e.g. [14], p.107). Does  $T$  have a nontrivial invariant subspace if  $D$  is just compact?

**COROLLARY 2.** *If a quasihyponormal contraction  $T$  has no nontrivial invariant subspace, then  $T$ ,  $Q$  and  $D$  are proper contractions,  $[T^*, T]$  is a strict contraction of trace class  $\mathcal{B}_1[\mathcal{H}]$ , and  $|T^2| - |T|^2$  is a strict contraction of Hilbert-Schmidt class  $\mathcal{B}_2[\mathcal{H}]$ . Moreover, if one of  $Q$  or  $D$  is compact, then so is the other, and  $Q$  and  $D$  are strict contractions too.*

*Proof.* If an operator has no nontrivial invariant subspace, then it is quasiinvertible (or a quasiaffinity; that is, injective with a dense range) because  $\ker(T)$  and  $\text{ran}(T)^\perp$  are invariant subspaces for  $T$ . Therefore, if  $T$  is quasihyponormal ( $O \leq T^*[T^*, T]T$ ) and has no nontrivial invariant subspace ( $T$  is quasiinvertible), then  $T$  is hyponormal ( $O \leq [T^*, T]$ ). Thus, if a quasihyponormal contraction  $T$  has no nontrivial invariant subspace, then  $[T^*, T]$  is trace-class (by the Berger-Shaw Theorem; see e.g. [4], p.152) and is a strict contraction [9]. Moreover,  $|T^2| - |T|^2$  (which is nonnegative whenever  $T$  is quasihyponormal) is a Hilbert-Schmidt strict contraction [7] (i.e.,  $(|T^2| - |T|^2)^2$  is a trace-class strict contraction).

Recall that if  $A$  and  $B$  are operators on  $\mathcal{H}$  such that

$$A^*A \leq B,$$

then

$$\|Ax\|^2 = \langle A^*Ax; x \rangle \leq \langle Bx; x \rangle \leq \|Bx\| \|x\|$$

for every  $x \in \mathcal{H}$ , and therefore  $A$  is a proper contraction whenever  $B$  is a proper contraction, and  $A$  is compact whenever  $B$  is compact.

If  $T$  is a quasihyponormal contraction without a nontrivial invariant subspace, then both  $T$  and  $Q$  are proper contractions (by Corollary 1 once  $T$  is paranormal). Since  $O \leq T^*[T^*, T]T$  ( $T$  is quasihyponormal) and  $D^*D = D^2$  ( $D$  is self-adjoint),

$$D^*D = Q - T^*[T^*, T]T \leq Q.$$

Hence  $D$  is a proper contraction because  $Q$  is a proper contraction, and  $D$  is compact whenever  $Q$  is compact. Since  $Q = D - T^*DT$ , it



follows at once that  $Q$  is compact whenever  $D$  is compact. Suppose  $Q$  (or, equivalently,  $D$ ) is compact so that  $Q$  and  $D$  are compact proper contractions. But the concepts of proper and strict contraction coincide for compact operators [9]. Therefore,  $Q$  and  $D$  are strict contractions.  $\square$

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