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SOME SEQUENCES RESEMBLING HOFSTADTER'S

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ABSTRACT. A number of variants of Hofstadter's original sequence have been investigated. Here we investigate a collection of similarly defined such sequences which give rise to intriguingly different results.

1. Introduction

In [2], p.138, Hofstadter introduced the sequence $H(n)$, vaguely reminiscent of Fibonacci's, defined by: $H(1) = H(2) = 1$ and, for $k \geq 3, H(k) = H(k - H(k-1)) + H(k - H(k-2))$. In [1] Problem 2, p.14, the authors introduced, as a computer exercise, the sequences $Y(k)$ and $Z(k)$ given by $Y(k) = Z(k) = 1 (k = 1, 2, 3)$ and $Y(k) = Y(k - Y(k-1)) + Y(k - Y(k-2)) + Y(k - Y(k-3))$ and $Z(k) = Z(k - Z(k-1)) + Z(k - Z(k-3))$. The former sequence "stops" almost immediately (if we regard Y as being undefined for negative arguments) since to evaluate $Y(6)$ we have to call upon the value $Y(-1)$. Likewise the Z sequence stops because $Z(165)$ calls upon $Z(-37)$. It appears to be unknown whether $H(n)$ is undefinable for some value of n. More promising variations of the sequence H are those investigated by Tanny in [3] and Conolly, see [4], pp.127–138. Calling these sequences $T(n)$ and $C(n)$ ($C(n)$ is called $F(n)$ in [4]), we have the definitions: $T(0) = T(1) = T(2) = C(1) = C(2) = 1$ and, for $k \geq 3$, $T(k) = T(k-1-T(k-1)) + T(k-2-T(k-2)); C(k) = C(k - C(k-1)) + C(k-1-C(k-2))$. These sequences look very similar except (so it would seem) that for each power 2^α of 2 there are $\alpha + 1$ integers m for which $C(m) = 2^\alpha$ and $\alpha + 2$ integers m for which $T(m) = 2^\alpha$.

Initially unaware of these investigations, the first author investigated both sequence C and the following "companion" sequences A, B and D defined as follows:

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(I) $A(k) = B(k) = D(k) = 1 (k = 1, 2, 3)$ and,
(II)$A(k) = A(k-1-A(k-2)) + A(k-2-A(k-3))$,
$B(k) = B(k-B(k-1)) + B(k-2-B(k-3))$,
$D(k) = D(k-D(k-1)) + D(k-1-D(k-2)) + D(k-2-D(k-3))$,
for $k \geq 4$.

We exhibit some of their values below.

What is perhaps most interesting is that these sequences A, B, C, D behave in totally different ways.

Now it would be easy to generate and investigate many more such sequences. We finish by looking at just one since its behaviour is also rather curious. It is defined by $E(1) = E(2) = E(3) = 1$ and $E(k) = E(k - E(k-1)) + E(k-1-E(k-3))$, for $k \geq 4$.

2. Properties

The sequence C.

$n = 1$	2	3	4	5	6	7	8	9	10
$C(0+n) = 1$	1	2	2	3	4	4	4	5	6
$C(10+n) = 6$	7	8	8	8	8	9	10	10	11
$C(20+n) = 12$	12	12	13	14	14	15	16	16	16
$C(30+n) = 16$	16	17	18	18	19	20	20	20	21
$C(40+n) = 22$	22	23	24	24	24	24	25	26	26
$C(50+n) = 27$	28	28	28	29	30	30	31	32	32
$C(60+n) = 32$	32	32	32	33	34	34	35	36	36
$C(70+n) = 36$	37	38	38	39	40	40	40	40	41
$C(80+n) = 42$	42	43	44	44	44	45	46	46	47
$C(90+n) = 48$	48	48	48	48	49	50	50	51	52

The most obvious (apparent) property of sequence C which is not mentioned in [4] (but is proved true in [3], Proposition 2.1, for sequence T) is given by

THEOREM 1. *For all* $n, C(n) = C(n-1) + r$ *where* $r = 0$ *or* 1.

Proof. The result claimed is true for the first few values of n. Suppose that the result claimed is true for all $n < k$ and consider $C(k) - C(k-1)$ which, after slight rearrangement, may be written,

$$\{C(k - C(k-1)) - C(k-1-C(k-2))\} + \{C(k-1-C(k-2)) - C(k-2-C(k-3))\}. \tag{1}$$

Now, by the induction hypothesis, we know that $C(k-1) = C(k-2)+s$ and $C(k-2) = C(k-3)+t$ where $s, t \in \{0,1\}$. If $s = 1$ then the two arguments in the first bracket are equal, hence the first bracket is 0. Likewise, for the second bracket if $t = 1$. On the other hand, if $t = 0$ (and $s = 1$) we see that the arguments in the second bracket differ by 1. Hence, by the induction assumption, $C(k-1-C(k-2)) - C(k-2-C(k-3))$ is equal to 0 or 1. Consequently $C(k) - C(k-1) = 0$ or 1 in the cases $(s,t) = (1,1), (1,0), (0,1)$. It remains only to consider the case where $s = t = 0$. In this case we have $C(k-1) = C(k-2) = C(k-3)$. But then,

$$\begin{aligned} 0 &= C(k-1) - C(k-2) \\ &= \{C(k-1-C(k-2)) - C(k-2-C(k-3))\} \\ &\quad + \{C(k-2-C(k-3)) - C(k-3-C(k-4))\}. \end{aligned} \tag{2}$$

Since, by hypothesis, $C(k-3) = C(k-4)+w$ where $w = 0$ or 1, we see that the only possibility from (2) is that each bracket is 0. Returning to (1) we see that $C(k) - C(k-1) \leq 1$ since the second bracket of (1) is 0 when $C(k-1) = C(k-2) = C(k-3)$.

Immediate consequences of Theorem 1 also proved, in [3], Corollary 2.2, for the sequence T, include (i) the C sequence is non-decreasing and "hits" every positive integer; (ii) each odd integer is hit exactly once; (iii) each even integer is hit at least twice. Deeper properties of $C(n)$ and $T(n)$ can be found in [3] and [4]. □

The sequence A. We can dispose of the main property of the A sequence quickly: Since $A(891) = 945$ we find that $A(893)$ cannot be evaluated!

The sequence B.

$n = 1$	2	3	4	5	6	7	8	9	10
$B(0+n) = 1$	1	1	2	2	3	3	4	4	5
$B(10+n) = 5$	6	6	7	7	8	8	9	9	10
$B(20+n) = 10$	11	11	12	12	13	13	14	14	15
$B(30+n) = 15$	16	16	17	17	18	18	19	19	20
$B(40+n) = 20$	21	21	22	22	23	23	24	24	25
$B(50+n) = 25$	26	26	27	27	28	28	29	29	30

Sequence B generates a seemingly obvious pattern. We leave the reader the straightforward task of verifying that, for $n \geq 1$, $B(2n) = B(2n+1) = n$.

Our main interest in this paper is in

The sequence D.

$n =$	1	2	3	4	5	6	7	8	9	10
$D(0+n) =$	1	1	1	3	3	3	5	5	7	5
$D(10+n) =$	7	7	9	9	9	11	11	13	11	15
$D(20+n) =$	13	17	13	17	15	19	17	19	17	21
$D(30+n) =$	19	23	19	23	21	25	23	25	25	27
$D(40+n) =$	27	27	29	29	31	29	33	31	35	31
$D(50+n) =$	37	33	39	33	41	35	43	35	43	37
$D(60+n) =$	45	39	45	39	47	41	49	41	49	43
$D(70+n) =$	51	45	51	45	53	47	55	47	57	49
$D(80+n) =$	59	49	59	51	61	53	61	55	63	57
$D(90+n) =$	63	57	65	59	67	59	67	61	69	63
$D(100+n) =$	69	63	71	65	73	65	73	67	75	69
$D(110+n) =$	75	71	77	73	77	75	79	77	79	79
$D(120+n) =$	81	81	81	83	83	85	83	87	85	89

Here is a later part of this sequence.

$n =$	1	2	3	4	5	6	7	8	9	10
$D(340+n) =$	219	231	221	233	223	233	225	235	227	235
$D(350+n) =$	229	237	231	237	233	239	235	239	237	241
$D(360+n) =$	239	241	241	243	243	243	245	245	247	245
$D(370+n) =$	249	247	251	247	253	249	255	249	257	251
$D(380+n) =$	259	251	261	253	263	253	265	255	267	255

(For the significance of the underlined terms, see below.)

Sequence D appears to have a number of interesting properties. Amongst these are the followings.

(i) For each $t > 0$, $D(t+2) - D(t) \in \{0, 2\}$. If true then:

(ii) For each $n > 0$, $D(n) \leq n$. Consequently $D(n)$ is defined for all n.

(iii) No three successive *alternate* terms are equal. If true then:

(iv) No integer appears more than four times in the sequence.

(v) No *four* successive terms are equal.

(vi) The only cases where *three* successive terms are equal occurs where the terms are 3^k for some k - and then the first occurrence of 3^k is at the $(1 + 3 + \cdots + 3^k)$th term.

(vii) The difference $D(2t) - D(2t-1)$ changes sign around those values of n for which both $D(2n) - D(2n-1)$ and $D(2n+2) - D(2n+1)$ are both 0, these values of n being $3, 6, 21, 60, 183, 546, 1641, 4920$ (so that each is 3 times its predecessor plus or minus 3).

(viii) For no *three* successive pairs $D(k)$, $D(k-1)$ is their difference 0.

We prove (i), (ii), (iii), (iv), (v), part of (vii) and (viii), leaving (vi) and the rest of (vii) as conjectures - which we hope the theorems below will assist in proving. Given the relative complexity of some of the proofs below, it is not surprising that, as yet, we can see no way of establishing these interesting conjectures.

REMARK. These conjectures have now been established in a submitted paper *"On the behaviour of a form of meta-Fibonacci sequences"* by Joseph Callaghan, John J. Chew, III and Stephen M. Tanny.

We begin, then, with

THEOREM 2. *For each positive integer t, the differences $D(t+2)-D(t)$ can only take the value 0 or 2.*

Proof. This is similar to that of Theorem 1. We see that $D(n)-D(n-2)\in\{0,2\}$ holds for the first few terms of the D sequence. Suppose this difference is valid for all $n<k$. Consider the difference $D(k)-D(k-2)$ which we rewrite as

$$\begin{aligned} &\{D(k-D(k-1))-D(k-2-D(k-3))\}\\ &+\ \{D(k-1-D(k-2))-D(k-3-D(k-4))\}\\ &+\ \{D(k-2-D(k-3))-D(k-4-D(k-5))\}. \end{aligned} \tag{3}$$

{We call this kind of rewriting a *pairing.*} If $D(k-1)=D(k-3)+2$ then the two arguments in the first bracket are equal - hence the first bracket has value 0. Likewise for the other two brackets. On the other hand, if $D(k-1)=D(k-3)$ then the arguments in the first bracket differ by 2 and the bracket itself can take value 0 or 2. Let $V(m)$ denote the difference $D(m-1)-D(m-3)$. The above reasoning shows us that $D(k)-D(k-2)$ has value 0 or 2 if $(V(k),V(k-1),V(k-2))=(2,2,2)$ or $(2,2,0)$ or $(2,0,2)$ or $(0,2,2)$. Let us consider the case $(V(k),V(k-1),V(k-2))=(2,0,0)$. This means that $V(k-1)=D(k-2)-D(k-4)=0$. But $D(k-2)-D(k-4)=\{D(k-2-D(k-3))-D(k-4-D(k-5))\}+\{D(k-3-D(k-4))-D(k-5-D(k-6))\}+\{D(k-4-D(k-5))-D(k-6-D(k-7))\}$ and, by our previous arguments, each bracket has value 0 or 2. Hence each bracket has value 0. In particular $D(k-2-D(k-3))-D(k-4-D(k-5))=0$. That is, the final bracket of (3) is 0. It follows that, in the (2,0,0) case, $D(k)-D(k-2)$ has value 0 or 2.

Likewise, in the case $(V(k), V(k-1), V(k-2)) = (0,2,0)$ we know that $0 = D(k-1) - D(k-3) = \{D(k-1-D(k-2)) - D(k-3-D(k-4))\} + \{D(k-2-D(k-3)) - D(k-4-D(k-5))\} + \{D(k-3-D(k-4)) - D(k-5-D(k-6))\}$ from which we deduce that $D(k-2-D(k-3)) - D(k-4-D(k-5)) = 0$ and then that (3) has value 0 or 2.

An identical argument deals with the case $(V(k), V(k-1), V(k-2)) = (0,0,2)$. Finally, if $(V(k), V(k-1), V(k-2)) = (0,0,0)$ we deduce from $V(k) = 0$, that, in (3), both $D(k-1-D(k-2)) - D(k-3-D(k-4))$ and $D(k-2-D(k-3)) - D(k-4-D(k-5))$ are 0 - and hence that $D(k) - D(k-2) = 0$ or 2. There follows immediately. □

COROLLARY 3. *For all positive integers n we have $D(n) < n$. (Hence $D(n)$ is well defined for all positive integers n).*

Making much use of the "pairing method" introduced above we next prove:

THEOREM 4. *In the D sequence no three successive (alternate) terms are equal.*

Proof. Suppose to the contrary that for some integer k we have $D(k) = D(k+2) = D(k+4) = a$, (such a triple we shall call a *trio*) and that k is the least integer for which such a trio occurs. Then, taking $D(k-1) = b$, we see that, if $D(k+1) = b$ then $D(k-3) = b-2$ and $D(k+3) = b+2$ (by Theorem 2 and the definition of k). Similarly we must have $D(k-2) = a-2$. We have three cases which we denote by:

$k+n$	$n=$	-3	-2	-1	0	1	2	3	4
case(i)		$b-2$	$a-2$	b	a	b	a	$b+2$	a
case(ii)		?	$a-2$	b	a	$b+2$	a	$b+2$	a
case(iii)		?	$a-2$	b	a	$b+2$	a	$b+4$	a

In case (i) we have $0 = D(k+4) - D(k+2) = \{D(k+4-D(k+3)) - D(k+2-D(k+1))\} + \{D(k+3-D(k+2)) - D(k+1-D(k))\} + \{D(k+2-D(k+1)) - D(k-D(k-1))\}$, that is

$$(4) \quad \begin{aligned} 0 = &\{D(k+4-(b+2)) - D(k+2-b)\} \\ &+ \{D(k+3-a) - D(k+1-a)\} \\ &+ \{D(k+2-b) - D(k-b)\}. \end{aligned}$$

Here each bracket must be 0. In an identical way we obtain $2 = \{D(k+3-a) - D(k+1-a)\} + \{D(k+2-b) - D(k-b)\} + \{D(k+1-a) - D(k-1-(a-2))\}$.

This is impossible since, from (4) each of the first two brackets is 0 and, trivially, so is the third.

In case (iii) we find that $2 = D(k+3) - D(k+1) = \{D(k+3-a) - D(k+1-a)\} + \{D(k+2-(b+2)) - D(k-b)\} + \{D(k+1-a) - D(k-1-(a-2))\}$, so that, this time, $D(k+3-a)) - D(k+1-a) = 2$. However $a-(b+4) = D(k+4) - D(k+3) =$ (on eliminating two pairs of terms) $D(k+4-D(k+3)) - D(k+1-D(k)) = D(k+4-(b+4)) - D(k+1-a)$ whereas $(b+4) - a = D(k+3) - D(k+2) = D(k+3-D(k+2)) - D(k-D(k-1)) = D(k+3-a) - D(k-b)$ from which the contradiction $D(k+3-a)) - D(k+1-a) = 0$ follows. Thus only case (ii) remains. In that case we argue as follows: first, $a-(b+2) = D(k+4) - D(k+3) = D(k+4-D(k+3)) - D(k+1-D(k)) = D(k+4-(b+2)) - D(k+1-a)$. Similarly, $(b+2)-a = D(k+3) - D(k+2) = D(k+3-a) - D(k-b)$ whilst $a-(b+2) = D(k+2) - D(k+1) = D(k+2-(b+2)) - D(k-1-(a-2))$ and $a-b = D(k) - D(k-1) = D(k-b) - D(k-3-D(k-4))$.

These equalities imply that $D(k+2-b) = D(k-b)$ and

$$D(k+3-a) = D(k+1-a). \tag{5}$$

Next $0 = D(k+2) - D(k) = \{D(k+2-D(k+1)) - D(k-D(k-1))\} + \{D(k+1-D(k)) - D(k-1-D(k-2))\} + \{D(k-D(k-1)) - D(k-2-D(k-3))\} = 0 + 0 + \{D(k-b)) - D(k-2-D(k-3))\}$.

From this it follows that $D(k-3) \neq b$ for, if it were, we could deduce that $D(k-b) = D(k-2-b)$ and then (4) would show the existence of an earlier trio than the a, a, a we started with. Hence, by Theorem 2, $D(k-3) = b-2$.

Next $2 = D(k+1) - D(k-1) = \{D(k+1-a) - D(k-1-(a-2))\} + \{D(k-b)) - D(k-2-(b-2))\} + \{D(k-1-(a-2))) - D(k-3-D(k-4))\}$.

We deduce that $D(k-4) \neq a-4$ (since the last bracket cannot be 0). Hence $D(k-4) = a-2$ which, in turn, implies $D(k+1-a) = D(k-1-a) + 2$. Finally, looking at $2 = D(k) - D(k-2)$ in the same way we find $D(k-5) = b-4$ {since $D(k-5) = b-2$ would again give the trio $D(k+2-b), D(k-b), D(k-2-b)$.} Thus, so far our case(ii) sequence looks like:

$$\ldots, b-4, a-2, b-2, a-2, b, a, b+2, a, (b+2, a, \ldots).$$

We notice that the "b"s appear to have begun descending by 2 every (alternate) step whereas the "a"s descend likewise, but only after repeating. □

We now show that once this pattern sets in it persists. That is we show, for case (ii), LEMMA 5. *Using the numbering introduced above, for all non negative integers t, the values of $D(k-(4t-2)), D(k-(4t-1)), D(k-4t), D(k-(4t+1))$ are given by $a-2t, b-4t+2, a-2t, b-4t$.*

Proof. We prove this not in blocks of four successive values of D but in blocks of eight successive values of D. That is, we prove, for all non negative integers s that the values of $D(k-(8s-2)), D(k-(8s-1)), D(k-8s), D(k-(8s+1)), D(k-(8s+2)), D(k-(8s+3)), D(k-(8s+4)), D(k-(8s+5))$ are $a-4s, b-8s+2, a-4s, b-8s, a-4s-2, b-8s-2, a-4s-2, b-8s-4$. Certainly these values are correct for $s=0$. Suppose them valid for all $s \leq n$.

First note that $2 = D(k-(8n))-D(k-(8n+2)) = \{D(k-[8n]-D(k-[8n+1]))-D(k-[8n+2]-D(k-[8n+3]))\}+\{D(k-[8n+1]-D(k-[8n+2]))-D(k-[8n+3]-D(k-[8n+4]))\}+\{D(k-[8n+2]-D(k-[8n+3]))-D(k-[8n+4]-D(k-[8n+5]))\} = \{D(k-[8n]-(b-8n))-D(k-[8n+2]-(b-8n-2))\}+\{D(k-[8n+1]-(a-4n-2))-D(k-[8n+3]-(a-4n-2))\}+\{D(k-[8n+2]-(b-8n-2))-D(k-[8n+4]-(b-8n-4))\}$.

Since the first and third brackets are trivially 0 the second bracket shows that $D(k-4n+1-a)-D(k-4n-1-a)) = 2$. Next: $D(k-[8n+6])$ cannot have value $a-4n-2$ since that would give rise to three successive (alternate) equal values $D(k-[8n+r]), r = 6,4,2$. Hence $D(k-[8n+6]) = a-4n-4$.

To determine the value of $D(k-(8n+7))$ we look at $0 = D(k-(8n+2))-D(k-(8n+4)) = \{D(k-[8n+2]-D(k-[8n+3]))-D(k-[8n+4]-D(k-[8n+5]))\}+\{D(k-[8n+3]-D(k-[8n+4]))-D(k-[8n+5]-D(k-[8n+6]))\}+\{D(k-[8n+4]-D(k-[8n+5]))-D(k-[8n+6]-D(k-[8n+7]))\}$.

If $D(k-[8n+7]) = b-8n-4$ then the last bracket's being 0 would give $D(k-[8n+4]-(b-[8n+4])) = D(k-[8n+6]-(b-[8n+4]))$ i.e. $D(k-b) = D(k-b-2)$ thus giving trio trio $D(k-b-2), D(k-b)$ and $D(k-b+2)$ again.

To determine the value of $D(k-(8n+8))$ we correspondingly look at $2 = D(k-(8n+3))-D(k-(8n+5)) = \{D(k-[8n+3]-D(k-[8n+4]))-D(k-[8n+5]-D(k-[8n+6]))\}+\{D(k-[8n+4]-D(k-[8n+5]))-D(k-[8n+6]-D(k-[8n+7]))\}+\{D(k-[8n+5]-D(k-[8n+6]))-D(k-[8n+7]-D(k-[8n+8]))\}$.

The first two brackets are 0 by the previous step. Hence the final bracket has value 2. Therefore $D(k-[8n+6]) \neq D(k-[8n+8])+2$, that is, $D(k-[8n+8]) = D(k-[8n+6]) = a-4n-4$.

To determine the value of $D(k-(8n+9))$ we correspondingly look at $2 = D(k-(8n+4))-D(k-(8n+6)) = \{D(k-[8n+4]-D(k-[8n+$

$5])) - D(k - [8n+6] - D(k - [8n+7]))\} + \{D(k - [8n+5] - D(k - [8n+6])) - D(k - [8n+7] - D(k - [8n+8]))\} + \{D(k - [8n+6] - D(k - [8n+7])) - D(k - [8n+8] - D(k - [8n+9]))\}$. The previous step shows that the middle bracket has value 2. Hence the final bracket has value 0. If $D(k - [8n+9]) = b - 8n - 6$ the final bracket above would again give $D(k-b) = D(k-b-2)$ and hence the earlier trio $D(k-b-2), D(k-b)$ and $D(k-b+2)$.

As with $D(k-[8n+6])$ we see that $D(k-(8n+10)) \neq a-4n-4$ and so $D(k-[8n+10]) = a-4n-6$. The cases $D(k-(8n+r)), r = 11, 12, 13$ can be dealt with similarly thus completing the induction part of the proof of Theorem 4. To finish Theorem 4 entirely we only need to observe that the pattern $\ldots, b-6, b-4, b-2, b, b+2, \ldots$ in alternating terms is not present from the start of the sequence D. This completes the proof of Theorem 4. □

An immediate corollary is

COROLLARY 6. *No integer occurs in the D sequence more than four times.*

We now prove

THEOREM 7. *For no value of k do we have $D(k) = D(k+1) = D(k+2) = D(k+3)$. (Such a quadruplet we shall call a foursome.)*

Proof. Suppose these values are equal, to a, say, and that k is the first such integer for which this happens. We therefore have $D(k-1) = a-2$ and $D(k+4) = a+2$, by Theorem 4. Thus $D(k-2) = a-2$.

Now using the same techniques as in Theorem 4 we have: First: $0 = D(k+3) - D(k+1)$ implies that $D(k+3-a) - D(k+1-a) = 0$ and $D(k+1-a) = D(k-1-D(k-2))$. Hence $D(k-2) \neq a$; for otherwise $D(k+r-a), r = -1, 1, 3$, would be a trio contrary to Theorem 4.

Next $0 = D(k+3) - D(k+2) = D(k+3-D(k+2)) - D(k-D(k-1)) = D(k+3-a) - D(k-(a-2))$ and $0 = D(k+2) - D(k+1) = D(k+2-D(k+1)) - D(k-1-D(k-2)) = D(k+2-a) - D(k-1-(a-2))$. Therefore

$$D(k+3-a) = D(k+2-a) = D(k+1-a). \tag{6}$$

Next $0 = D(k+2) - D(k)$ implies that $D(k-D(k-1)) = D(k-2-D(k-3)) = 0$. If $D(k-3) = a-2$ we should have $D(k-(a-2)) = D(k-2-(a-2))$ which, together with (6) would provide an earlier foursome. Hence $D(k-3) = a-4$.

Next $2 = D(k+1) - D(k-1) = D(k-1-D(k-2)) - D(k-3-D(k-4))$.

It follows that $D(k-4) \neq D(k-2)-2$. Hence $D(k-4) = D(k-2) = a-2$ and, hence, $D(k+1-a) = D(k-1-a)+2$. Finally, using the last equality, we get $2 = D(k) - D(k-2) = \{D(k-(a-2)) - D(k-2-(a-4))\} + \{D(k-1-(a-2)) - D(k-3-(a-2))\} + \{D(k-2-(a-4)) - D(k-4-D(k-5))\} = 0+2+0$. Hence $D(k-5) \neq a-4$: for, if so, we get $D(k+2-a) = D(k-a)$ and thus obtain an earlier foursome again. Hence $D(k-5) = a-6$. The values $D(k+r)$ where $r = -5,-4,-3,-2,-1,0,1,2$ are precisely those we used (if we put $b = a-2$) to prove that case (ii) of Theorem 4 was impossible. Hence we can conclude that Theorem 7 is also proved. □

It is now easy to prove that

THEOREM 8. *There is no value of k for which $D(k+5) - D(k+4), D(k+3) - D(k+2), D(k+1) - D(k)$ are all 0.*

Proof. Suppose that $\dots, D(k), \dots, D(k+5), \dots$ *do* have the property mentioned. By Theorems 2 and 4 we see that this subsequence must be of the form $\dots, a, a, a+2, a+2, a+4, a+4, \dots$ Then, $2 = D(k+5) - D(k+3) = \{D(k+5-[a+4]) - D(k+3-[a+2])\} + \{D(k+4-[a+2]) - D(k+2-a)\} + \{D(k+3-[a+2]) - D(k+1-a)\} = 0+0+0$, an obvious contradiction

Note that: (i) *two* pairs $D(k) = D(k+1); D(k+2) = D(k+3)$ *can* occur; (ii) one may likewise prove that up to *four* {but not *five*} successive pairs $D(k), D(k+1)$ with constant (non-zero) difference may occur. □

In relation to Note (i) we have

THEOREM 9. *If $D(k) = D(k+1)$ and $D(k+2) = D(k+3)$ then one of $D(k-2) - D(k-1)$ and $D(k+4) - D(k+5)$ has value 2, the other having value -2.*

Informally : around a 0-0 pair of successive differences the difference $D(t+1) - D(t)$ changes sign.

Proof. In the subsequence $\dots D(k-2), D(k-1), a, a, a+2, a+2, D(k+4), D(k+5), \dots$ we must have $\dots (D(k-2), D(k-1)) = (a-2, a)$ or $(a, a-2)$ and $(D(k+4), D(k+5)) = (a+2, a+4)$ or $(a+4, a+2)$ by Theorems 2 and 4. The subsequence

$$\dots, a-2, a, a, a, a+2, a+2, a+2, a+4, \dots$$

is not possible since $0 = D(k+4) - D(k+3) = D(k+4-[a+2]) - D(k+1-a)$ whereas $2 = D(k+2) - D(k+1) = D(k+2-a) - D(k-1-[a-2])$.

On the other hand the subsequence

$$a, a-2, a, a, a+2, a+2, a+4, a+2, \dots$$

is not possible since $2 = D(k+4) - D(k+3) = D(k+4-[a+2]) - D(k+1-a)$ whereas $0 = D(k+3) - D(k+2) = D(k+3-[a+2]) - D(k-[a-2])$.

Note The question about $D(k+1) - D(k)$ changing signs infinitely often can therefore be rephrased as: do consecutive pairs $D(k) = D(k+1), D(k+2) = D(k+3)$ occur infinitely often? At this moment we are unable to see how to do this. □

The sequence E.

It is a pity that the definition of the E sequence is slightly unsymmetrical but that is more than compensated for by the interesting pattern it developes – and the fact that this pattern can be proved to persist – once one has passed the 18th term! To help see the pattern it is best to arrange the terms of the sequence in blocks of eight:

The terms

$n =$ 1	2	3	4	5	6	7	8
$E(0+n) =$ 1	1	1	2	2	4	3	4
$E(8+n) =$ 4	8	5	6	5	8	7	12
$E(16+n) =$ 6	14	6	13	8	15	8	19
$E(24+n) =$ 8	20	6	21	8	23	8	27
$E(32+n) =$ 8	28	6	29	8	31	8	35
$E(40+n) =$ 8	36	6	37	8	39	8	43
$E(48+n) =$ 8	44	6	45	8	47	8	51
$E(56+n) =$ 8	52	6	53	8	55	8	59
$E(64+n) =$ 8	60	6	61	8	63	8	67
$E(72+n) =$ 8	68	6	69	8	71	8	75
$E(80+n) =$ 8	76	6	77	8	79	8	83
$E(88+n) =$ 8	84	6	85	8	87	8	91
$E(96+n) =$ 8	92	6	93	8	95	8	99

We shall prove the following

THEOREM 10. *For each integer* $n \geq 3$ *we have:* $E(8n-5) = 6; E(8n-4) = 8n-11; E(8n-3) = 8; E(8n-2) = 8n-9; E(8n-1) = 8; E(8n) = 8n-5; E(8n+1) = 8; E(8n+2) = 8n-4$.

Proof. The given equalities are seen to be true for $n = 3$. Suppose them all true for every $n \leq k$ and suppose that $n = k+1$. Taking them in turn we get: (i) $E(8(k+1)-5) = E(8k+3) = E(8k+3-E(8k+2)) + E(8k+2-E(8k)) = E(8k+3-(8k-4)) + E(8k+2-(8k-5)) = E(7) + E(7) = 6$.

(ii) $E(8(k+1)-4) = E(8k+4) = E(8k+4-E(8k+3)) + E(8k+3-E(8k+1)) = E(8k+4-6) + E(8k+3-8) = E(8k-2) + E(8k-5) = 8k-9+6 = 8k-3 = 8(k+1)-11$.

(iii) $E(8(k+1)-3) = E(8k+5) = E(8k+5-E(8k+4)) + E(8k+4-E(8k+2)) = E(8k+5-(8k-3)) + E(8k+4-(8k-4)) = E(8) + E(8) = 8$.

(iv) $E(8(k+1)-2) = E(8k+6) = E(8k+6-E(8k+5)) + E(8k+5-E(8k+3)) = E(8k+6-8) + E(8k+5-6) = E(8k-2) + E(8k-1) = 8k-9+8 = 8k-1 = 8(k+1)-9$.

(v) $E(8(k+1)-1) = E(8k+7) = E(8k+7-E(8k+6)) + E(8k+6-E(8k+4)) = E(8k+7-(8k-1)) + E(8k+6-(8k-3)) = E(8) + E(9) = 8$.

(vi) $E(8(k+1)) = E(8k+8) = E(8k+8-E(8k+7)) + E(8k+7-E(8k+5)) = E(8k+8-8) + E(8k+7-8) = E(8k) + E(8k-1) = 8k-5+8 = 8k+3 = 8(k+1)-5$.

(vii) $E(8(k+1)+1) = E(8k+9) = E(8k+9-E(8k+8)) + E(8k+8-E(8k+6)) = E(8k+9-(8k+3)) + E(8k+8-(8k-1)) = E(6) + E(9) = 8$.

(viii) $E(8(k+1)+2) = E(8k+10) = E(8k+10-E(8k+9)) + E(8k+9-E(8k+7)) = E(8k+10-8) + E(8k+9-8) = E(8k+2) + E(8k+1) = 8k-4+8 = 8k+4 = 8(k+1)-4$.

This completes the induction. □

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