New Bounds using the Solution of the Discrete Lyapunov Matrix Equation

Dong-Gi Lee, Gwang-Hee Heo, and Jong-Myung Woo

Abstract: In this paper, new results using bounds for the solution of the discrete Lyapunov matrix equation are proposed, and some of the existing works are generalized. The bounds obtained are advantageous in that they provide nontrivial upper bounds even when some existing results yield trivial ones.

Keywords: Lyapunov matrix inequalities, similarity transformation, bound estimates, stability.

1. INTRODUCTION

The Lyapunov matrix equation has played a fundamental role in the analysis of several control systems and design problems [1]. Thus, determining the exact solution of the Lyapunov matrix equation is essential in most applications. However, for certain applications such as system stability analysis, the exact solution is not required and reasonable bound estimates are used since obtaining the solution itself results in a very large computational burden as the dimension of system matrices is increased. Therefore, many researchers have been considerably attracted to the estimation problem for the Riccati and Lyapunov matrix equation [1-3, 4-7]. Also, recently the bound estimates for continuous Lyapunov matrix equation have been introduced by Fang *et al.* [1]. In this paper, the discrete bounds are presented based on the previous results [6, 7]. However, unfortunately all the results for the bound estimates are based on the assumption of $\lambda_1(A_q A_q^T) < 1$. Fang *et al.* [1] presented the upper bounds for continuous-time Lyapunov equation without using the common assumption that the largest eigenvalue of $A + A^T$ is negative definite, i.e., $\lambda_1(A + A^T) < 0$. Hence, the

objective of this paper is to extend this work to discrete bounds without the assumption that the system is asymptotically stable. Moreover, these bounds are compared to previous bounds investigated by many researchers for discrete-time Lyapunov matrix equation [4-7]. Bounds for the trace and the largest eigenvalues will be presented and special attention will be placed on the upper bounds for the trace due to their importance in robust stability and performance analysis.

2. NOTATIONS AND PRELIMINARIES

In this paper, the following notations will be used: $A_q \in \mathfrak{R}^{n \times n}$ is a real matrix, A_q^T denotes the matrix transpose, $tr(A_q)$ is the trace of A_q, $\lambda_i(A_q)$ denotes the eigenvalues of A_q, $(\lambda_i(A_q))$ are arranged in descending order when they are real, i.e., $\lambda_1(A_q) \geq \lambda_2(A_q) \geq \cdots \geq \lambda_n(A_q)$, $(\operatorname{Re}\lambda_i(A_q))$ are arranged in descending order, i.e., $\operatorname{Re}\lambda_1(A_q) \geq \operatorname{Re}\lambda_2(A_q) \geq \cdots \geq \operatorname{Re}\lambda_n(A_q)$.

Lemma 1 [13]: For symmetric positive semidefinite matrices A and B, with $1 \leq i$, $j \leq n$,

$$\lambda_{i+j-1}(AB) \leq \lambda_i(A)\lambda_j(B) \quad \text{if } i+j \leq n+1,$$

$$\lambda_{i+j-n}(AB) \geq \lambda_i(A)\lambda_j(B) \quad \text{if } i+j \geq n+1.$$

Lemma 2 [6]: For real symmetric matrices $A, B \geq 0$,

$$\sum_{l=1}^{k} \lambda_i(A+B) \leq \sum_{l=1}^{k}[\lambda_i(A) + \lambda_i(B)]$$

with equality when $k = n$.

Lemma 3 [6]: For symmetric $n \times n$ matrices A and B,

$$\sum_{i=1}^{k} \lambda_i(AB) \le \sum_{i=1}^{k} \lambda_i(A)\lambda_i(B).$$

Lemma 4 [7] (Rayleigh-Ritz Inequality): For any $x \in \Re^n$ and $A = A^T \in \Re^{n \times n}$,

$$\lambda_n(A)x^T x \le x^T A x \le \lambda_1(A)x^T x.$$

Lemma 5 [6]: Let $A \in \Re^{n \times n}$. Assume $A = T^T \Lambda T$ where T is orthogonal and Λ is diagonal with $0 \le \lambda_i(\Lambda) < 1$. Then

$$(I - A)^{-1} = I + A + A^2 + \cdots.$$

Now, let us consider the discrete Lyapunov equation

$$0 = A_q^T P A_q - P + Q, \tag{1}$$

where $A_q, P, Q \in \Re^{n \times n}, Q > 0$ and A_q is an asymptotically stable matrix. The discrete-time Lyapunov equation has unique positive definite solution P.

3. MAIN RESULTS

3.1. Discrete-time systems

Consider the linear shift-invariant discrete-time system

$$\begin{aligned} x(k+1) &= A_q x(k) + B_q u(k), \\ y(k) &= C_q x(k). \end{aligned} \tag{2}$$

where A_q, B_q, and C_q are given by

$$A_q = \begin{bmatrix} A_{q11} & A_{q12} \\ A_{q21} & A_{q22} \end{bmatrix}, \qquad B_q = \begin{bmatrix} B_{q1} \\ B_{q2} \end{bmatrix}, \qquad \text{and}$$

$$C_q = \begin{bmatrix} C_{q1} & C_{q2} \end{bmatrix}.$$

3.2. Estimate bounds for lyapunov matrix equation

Many researchers [6-10] have developed results of upper bounds for the discrete Lyapunov matrix equation. All the existing results are based on the assumption of $\lambda_1(A_q A_q^T) < 1$. However, the stability of A_q in discrete-time systems does not imply that $\lambda_1(A_q A_q^T)$ lies inside the unit circle. Similarly, Fang *et al.* [1] indicated a drawback in the assumption, that the stability of A does not guarantee that of $A + A^T$ for continuous-time systems. Hence, the objective of this paper is to extend the previous works to discrete-time systems and develop new results by removing this assumption.

Consider the algebraic Lyapunov matrix equation

$$P - A_q^T P A_q + Q = 0. \tag{3}$$

Since the previous works for upper bound estimates do not cover the case that $\lambda_1(A_q A_q^T)$ is not inside the unit circle, we should make the following modification. Using the similarity transformation, we set $\hat{P} = T^T P T$, $\hat{Q} = T^T Q T$, $\hat{A}_q = T^{-1} A_q T$. Then, the modified Lyapunov equation is obtained

$$\begin{aligned} (T^T P T) - (T^T A_q^T T^{-T})(T^T P T)(T^{-1} A_q T) \\ + (T^T Q T) = 0. \end{aligned} \tag{4}$$

Using (4) and Lemma 4, we obtain the following theorems.

Theorem 1: For the discrete Lyapunov equation (4),

$$tr(P) \le \frac{\lambda_1(E) tr(E^{-1} Q)}{1 - \lambda_1(\hat{A}_q \hat{A}_q^T)} \quad \text{if } \|\hat{A}_q\| < 1$$

where $\|\hat{A}_q\| = \sqrt{\lambda_1(\hat{A}_q \hat{A}_q^T)}$, $E = T^{-T} T^{-1}$ and T is the transformation matrix from (4).

Proof: From Lemma 4,

$$\lambda_n(A_q)x^T x \le x^T A_q x \le \lambda_1(A_q)x^T x \tag{5}$$

Then, the following inequality holds for any vector $x \in \Re^n$:

$$\begin{aligned} x^T \hat{P} x &= x^T \hat{A}_q^T \hat{P} \hat{A}_q x + x^T \hat{Q} x \\ &\le \lambda_1(\hat{P})x^T \hat{A}_q^T \hat{A}_q x + x^T \hat{Q} x, \end{aligned} \tag{6}$$

(6) implies

$$\hat{P} \le tr(\hat{P} \hat{A}_q^T \hat{A}_q) + \hat{Q}. \tag{7}$$

Then, using $tr(\hat{P} \hat{A}_q \hat{A}_q^T) \le \lambda_n(\hat{A}_q \hat{A}_q^T) tr(\hat{P})$, we obtain

$$tr(\hat{P})\left[1 - \lambda_1(\hat{A}_q \hat{A}_q^T)\right] \le tr(\hat{Q}), \tag{8}$$

(8) becomes

$$tr(\hat{P}) \le \frac{tr(\hat{Q})}{[1 - \lambda_1(\hat{A}_q \hat{A}_q^T)]}. \tag{9}$$

From $\hat{P} = T^T P T$, $\hat{Q} = T^T Q T$, $\hat{A}_q = T^{-1} A_q T$,

Since $tr(T^T P T) = tr(T T^T P) = tr(E^{-1} P)$, (9) is rewritten as follows:

$$tr(E^{-1}P) \le \frac{tr(E^{-1}Q)}{[1 - \lambda_1(\hat{A}_q \hat{A}_q^T)]} . \qquad (10)$$

From Lemma 1 and using $tr(E^{-1}P) \ge \lambda_n(E^{-1})tr(P)$, we have

$$tr(P) \le \lambda_1(E)\frac{tr(E^{-1}Q)}{[1 - \lambda_1(\hat{A}_q \hat{A}_q^T)]} . \qquad (11)$$

Note that $\lambda_n(E^{-1}) = \lambda_1^{-1}(E)$. This completes the proof. □

Theorem 2: Let P satisfy the decentralized discrete Lyapunov equation (4). Then we have

$$tr(P) \le \lambda_1(E)\sum_{i=1}^{k}\frac{\lambda_i(E^{-1}Q)}{\lambda_{n-i+1}(I - \hat{A}_q \hat{A}_q^T)},$$

where $i = 1,2,\ldots,k \le n$, $\lambda_1(\hat{A}_q \hat{A}_q^T) < 1$.

Proof: From Komaroff [6], the solution to (3), using integer l, is

$$P = \sum_0^\infty (A_q^T)^l Q A_q^l \qquad (12)$$
$$= Q + A_q^T Q A_q + (A_q^T)^2 Q A_q^2 + \cdots.$$

For notational convenience, set $A_q A_q^T = B$. Then

$$\lambda_i((A_q^T)^i Q A_q^l) = \lambda_i(QB^l). \qquad (13)$$

An application of Lemma 2 to (12), in view of (13), gives

$$\sum_l^k \lambda_i(\hat{P}) \le \sum_l^k [\lambda_i(\hat{Q}) + \lambda_i(\hat{Q}\hat{B}) + \lambda_i(\hat{Q}\hat{B}^2) + \cdots]$$
$$\le \sum_l^k \lambda_i(\hat{Q})[1 + \lambda_i(\hat{B}) + \lambda_i^2(\hat{B}) + \cdots], \qquad (14)$$

by Lemma 3. Assuming $\lambda_i(\hat{B}) < 1$, Lemma 5 may be employed in (14) to obtain

$$\sum_l^k \lambda_i(\hat{P}) \le \sum_l^k \lambda_i(\hat{Q})[1 - \lambda_i(\hat{A}_q \hat{A}_q^T)]^{-1}. \qquad (15)$$

Since $(I - \hat{A}_q \hat{A}_q^T)$ is symmetric and $\lambda_{n-i+1}(-\hat{A}_q \hat{A}_q^T) = -\lambda_i(\hat{A}_q \hat{A}_q^T)$, (15) is equivalent to the inequality of Theorem 2. This completes the proof. □

Theorem 4: For the decentralized discrete Lyapunov equation (4),

$$\lambda_k(P) \le \lambda_1(E)\left[\frac{1}{k}\sum_{i=1}^{k}\frac{\lambda_i(E^{-1}Q_0)}{[1 - \lambda_i(\hat{A}_q \hat{A}_q^T)]}\right]^k,$$

where $i = 1,2,\ldots,k \le n$.

Proof: Apply $k\lambda_k(\hat{P}) \le \sum_l \lambda_i(\hat{P})$ in (15). Then, the bound shown above is derived easily. This completes the proof. □

Theorem 4: Let the positive definite matrix P be the solution to (4). If $\sigma_1(\hat{A}_q) < 1$,

$$\lambda_i(P) \le \lambda_1(E)\left[\lambda_i\left(\frac{\lambda_1(E^{-1}Q)\hat{A}_q^T \hat{A}_q}{[1 - \sigma_1^2(\hat{A}_q)]} + E^{-1}Q\right)\right]$$

$; 1 \le i \le n$

$$tr(P) \le \frac{\lambda_1(E)\lambda_1(E^{-1}Q)}{[1 - \sigma_1^2(\hat{A}_q)]}tr(\hat{A}_q^T \hat{A}_q) + tr(E^{-1}Q).$$

Proof: (6) implies

$$\hat{P} \le \lambda_1(\hat{P})\hat{A}_q^T \hat{A}_q + \hat{Q}. \qquad (16)$$

From (16), we obtain

$$\lambda_1(\hat{P}) \le \frac{\lambda_1(\hat{Q})}{[1 - \lambda_1(\hat{A}_q \hat{A}_q^T)]} . \qquad (17)$$

Using (17) in (6), we have

$$\hat{P} \le \left(\frac{\lambda_1(\hat{Q})}{1 - \lambda_1(\hat{A}_q \hat{A}_q^T)}\right)\hat{A}_q \hat{A}_q^T + \hat{Q}. \qquad (18)$$

Then, we obtain the second inequality from (18). The first inequality can also be derived from (18). This completes the proof. □

Remark: The theorems presented above are based on [6, 7] and modified to cover the case that the common condition $\lambda_1(A_q A_q^T) < 1$ is not valid. By applying this modification, more generalized results are obtained.

3.3. Example

Example 1: A discrete-time model is obtained from its continuous-time model [14] by discretizing it using MATLAB function $c2d$ with the sampling period $h = 0.8$. The corresponding discrete-time system matrix is obtained as

$$A_q = \begin{bmatrix} 0.8563 & 0.2245 \\ -0.0001 & 0.5692 \end{bmatrix} \text{ with } Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \text{ Then,}$$

We obtain

$$\lambda_1(P) = 52.1890 , tr(P) = 53.4127 .$$

The eigenvalues of $A_q A_q^T$ are given by

$\lambda_1 = 1.4403$,

$\lambda_2 = 0.1135$.

Since $\lambda_1(A_q A_q^T)$ is unstable, this difficulty must be overcome. Then, similarity transformation matrix T is introduced. For this example,

$T = \begin{bmatrix} 0.9235 & 0.0765 \\ 1.2118 & -1.2118 \end{bmatrix}$, and the Jordan-transformed

matrix and its eigenvalues are obtained as

$\hat{A}_q \hat{A}_q^T = \begin{bmatrix} 0.9577 & 0 \\ 0 & 0.1708 \end{bmatrix}$,

$\lambda_1 = 0.9577$,

$\lambda_2 = 0.1708$.

Now, the assumption of $\lambda_1(A_q A_q^T) < 1$ is removed. Then, the upper bounds are given by the theorems described in main results.
The bound in Theorem 1 yields

$tr(P) \le 89.9098$.

By Theorem 2, we obtain

$tr(P) \le 80.1126$.

From Theorem 3, we have

$\lambda_1(P) \le 79.5743$.

The bounds in Theorem 4 yield

$\lambda_1(P) \le 79.5743$,

$tr(P) \le 94.9112$.

Example 2: A discrete-time model is obtained from its continuous-time model $A = \begin{bmatrix} -0.9 & 2 \\ 0 & -1.1 \end{bmatrix}$

by discretizing it using MATLAB function $c2d$ with the sampling period $h = 0.8$. The corresponding discrete-time system matrix is obtained as

Using the system matrix $A_q = \begin{bmatrix} 0.9139 & 0.1810 \\ 0 & 0.8958 \end{bmatrix}$

with $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then, we have

$\lambda_1(P) = 52.189$, $tr(P) = 53.413$.

The eigenvalues of $A_q A_q^T$ are given by

$\lambda_1 = 1.4163$,

$\lambda_2 = 0.1266$.

Since $\lambda_1(A_q A_q^T)$ is unstable, this difficulty should be overcome. Then, similarity transformation matrix T is introduced. For this example,

$T = \begin{bmatrix} 0.92326 & 0.07674 \\ 1.21410 & -1.21410 \end{bmatrix}$, and the Jordan-

transformed matrix and its eigenvalues are obtained as

$\hat{A}_q \hat{A}_q^T = \begin{bmatrix} 0.95767 & -6.0051*10^{-18} \\ -6.0051*10^{-18} & 0.18724 \end{bmatrix}$,

$\lambda_1 = 0.9567$,

$\lambda_2 = 0.18724$.

Now, the assumption of $\lambda_1(A_q A_q^T) < 1$ is removed. Then, the upper bounds are given by the theorems described in Main results.
The bound in Theorem 1 yields

$tr(P) \le 89.91$.

By Theorem 2, we obtain

$tr(P) \le 80.113$.

From Theorem 3, we have

$\lambda_1(P) \le 79.574$.

The bounds in Theorem 4 yield

$\lambda_1(P) \le 79.574$,

$tr(P) \le 94.911$.

The numerical results indicate the best values for each example. As shown above, the common assumption used for the bound estimation problem has been removed by applying similarity transformation to estimate bounds so that more generalized results can be obtained.

4. CONCLUSION

Investigation of stability analysis using bound estimates for the solution of discrete Lyapunov matrix equation is the topic of this paper. This issue is inspired by the work of Fang et al. [1]. Based on the upper bounds developed previously for discrete-time Lyapunov matrix equation [6-8], those bounds are extended and generalized with removal of the assumption of $\lambda_1(A_q A_q^T) < 1$. When applying similarity transformation to the theorems for discrete-time system, the inequalities for the upper bounds maintain their validity. The upper bound estimates are based on the solution of Lyapunov matrix equation for discrete-time system. The numerical results illustrated by the Examples demonstrate that the upper bounds for each system hold true.

REFERENCES

[1]   Y. Feng, K. A. Loparo, and X. Feng, "New estimates for solutions of Lyapunov equations," *IEEE Trans. on Automatic Control*, vol. 42, pp. 408-411, March 1997.

[2]   T. Mori, N. Fukuma, and M. Kuwahara, "Bounds in the Lyapunov matrix differential equation," *IEEE Trans. on Automatic Control*, vol. 32, no. 1, pp. 55-57, March 1987.

[3]   N. Komaroff, "Upper bounds for the eigenvalues of the solution of the Lyapunov

matrix equation," *IEEE Trans. on Automatic Control*, vol. 35, pp. 737-739, 1990.

[4] T. Mori, N. Fukuma, and M. Kuwahara, "Eigenvalue bounds for the discrete Lyapunov matrix equation," *IEEE Trans. on Automatic Control*, vol. 30, pp. 925-926, 1985.

[5] J. Garloff, "Bounds for the eigenvalues of the solution of the discrete Riccati and Lyapunov equation and the continuous Lyapunov equation," *International Journal of Control*, vol. 43, pp. 423-431, 1986.

[6] N. Komaroff, "Upper bounds for the eigenvalues of the solution of the discrete Lyapunov matrix equation," *IEEE Trans. on Automatic Control*, vol. 35, pp. 468-469, 1990.

[7] Chien-Hua Lee, "Upper and lower matrix bounds of the solution for the discrete Lyapunov equation," *IEEE Trans. on Automatic Control*, vol. 41, pp. 1338-1341, 1996.

[8] T. Mori, N. Fukuma, and M. Kuwahara, "On the discrete Lyapunov matrix equation," *IEEE Trans. on Automatic Control*, vol. 27, pp. 463-464, 1982.

[9] T. Mori, N. Fukuma, and M. Kuwahara, "On the discrete Riccati equation," *IEEE Trans. on Automatic Control*, vol. 32, pp. 828-829, 1987.

[10] M. T. Tran and M. E. Sawan, "A note on the discrete Lyapunov and Riccati matrix equations," *International Journal of Control*, vol. 23, pp. 87-88, 1978.

[11] N. Komaroff, "Lower bounds for the solution of the discrete algebraic Lyapunov equation," *IEEE Trans. on Automatic Control*, vol. 37, pp. 1017-1018, 1992.

[12] N. Komaroff and B. Shahian, "Lower summation bounds for the solution of the discrete Riccati and Lyapunov equation," *IEEE Trans. on Automatic Control*, vol. 37, pp. 1078-1080, 1992.

[13] M. Mrabti and A. Hmamed, "Bounds for the solution of the Lyapunov matrix equation-A unified approach," *Syst. Contr. Lett.*, vol. 18, pp. 73-81, 1992.

[14] P. Petkov, N. Christov, and M. Konstantinov, "A computational algorithm for pole placement assignment of linear multi-input systems," *IEEE Trans. on Automatic Control*, vol. 31, pp. 1044-1047, 1986.

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