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Teaching and Learning Models for Mathematics using Mathematica (I)

Kim, Hyang Sook
School of Computer Aided Science, Inje University, 607 Eobang-dong, Gimhae, Gyeongnam 621-749, Korea; Email: mathkim@ijnc.inje.ac.kr

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In this paper, we give examples of models we have created for use in university mathematics courses. We explain the concept of linear transformation, investigate the roles of each component of 2×2 and 3×3 transformation matrices, consider the relation between sound and trigonometry, visualize the Riemann sum, the volume of surfaces of revolution and the area of unit circle. This paper illustrates how one can use Mathematica to visualize abstract mathematical concepts, thus enabling students to understand mathematics problems effectively in class. Development of these kinds of teaching and learning models can stimulate the students' curiosity about mathematics and increase their interest.

Keywords: technology, Mathematica, linear transformation, sound, trigonometry, Riemann sum, volume.
ZDM classification: U74
MSC2000 classification: 97U70

I. INTRODUCTION

When one studies a mathematical problem nowadays, tools such as calculators and computers are available for students and teachers. Students can be actively engaged in reasoning, communicating, problem solving, and making connections with mathematics and other disciplines (Kim 2001a). Learning mathematical properties and principles can be enhanced through visualization using computer graphics. Complicated calculations that are time-consuming by paper and pencil can be carried out instantaneously using Mathematica[1] or some other mathematical software packages. Some writers have asserted that the computational medium is affecting not only the means by which

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The Mathematica code described in this paper is available from the authors. However, for the sake of brevity and readability, most of the code has been appeared in Appendix. Moreover, we used the Mathematica of version 4.0.2 and License number L2976-755.

mathematics is transmitted and learned, but also the content and context in which it is important (Kaput 1998).

For years people have been developing software to use in mathematics class. Packages have been varied greatly in their ease of use, their motivational appeal, their level of intellectual demand and their cross-platform accessibility (Maeder 1996). The National Research Council of the United States taking a problem-solving view of mathematics tells us that "research on learning shows that most students cannot learn mathematics effectively by only listening and imitating" (NRC 1989). Technology brings to students and teachers the opportunity to individualize learning — to generate illustrative examples, to follow interesting topics to the desired depth, to choose their own problems and then gather the tools for solving them.

Young (1986) has compared the advent of the computer to the introduction of Arabic numerals or the invention of the calculus in its impact "... computers have posed new problems for research [in Mathematics], supplied new tools to solve old problems, and introduced new research strategies". He points out similarities with those earlier "revolutions" by means of which "hard problems became easy and solvable not only by an intellectual elite but by a multiplicity of people without special mathematical talent; problems arose that had not been previously visualized, and their solutions changed the entire level of the field". Information technologies have transformed the workplace, but not yet the schools. The various teaching-learning models, which were made by mathematical software, need to be developed further (Watson 1998). In this paper, we give examples of models we have created to use in university mathematics courses. We explain the concept of linear transformation, investigate the roles of each component of 2×2 and 3×3 transformation matrices, consider the relation between sound and trigonometry, visualize the Riemann sum, the volume of surfaces of revolution and the area of unit circle. We give interesting, visual, meaningful and effective models for teaching the above subjects, which are obtained by the powerful functions of Mathematica (Kaput 1998) such as animation of graphics, variety of visualization and speed of computation. This paper illustrates how one can use Mathematica to visualize abstract mathematical concepts, thus enabling students' effective understanding in the mathematics classroom. Development of these kinds of teaching and learning models can stimulate the students' curiosity about Mathematics and increase their interest.

II. MAIN

We believe that finding various methods for teaching abstract mathematical concepts is the main road of teaching mathematics with technology.

Model 1: Linear Transformations

1. *Plane linear transformations* (*cf.* Kim 2001b)

A plane transformation $T : E^2 \to E^2$ is represented by

$$(1.1) \qquad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

and the coefficient matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

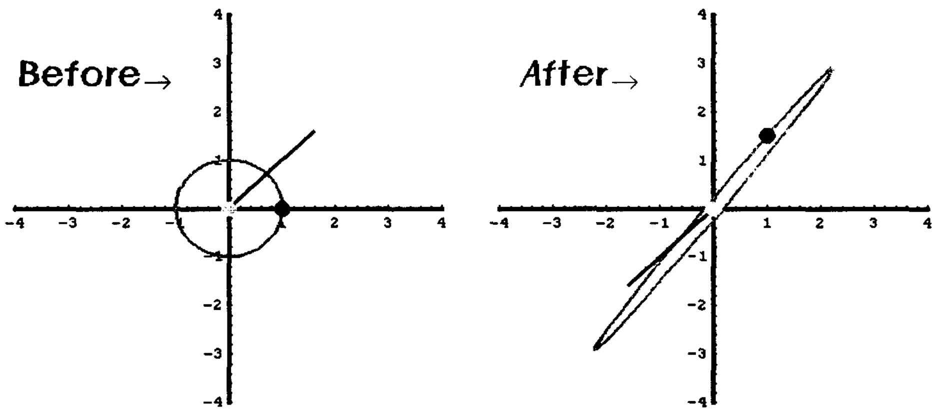
is said to be the *representation matrix* of *T*. In order to investigate the properties of *T* and the roles of each component of *T*, we define the Module function trans [*x*_, *y*_, *z*_] (See the Program [1-1] in Appendix), where x is a 2×2 matrix and *y*, *z* are curves. We shall study how the curves are transformed by the transformation matrix *x*. Using this tool, we can to visualize various aspects of linear transformations: the identity transformation, reflections, rotations, homotheties, and inverse transformations.

We take as our curves, the unit circle centred at the origin and the line $y = x$.

$$\mathtt{x} = \begin{pmatrix} \mathtt{1} & \mathtt{-2} \\ \mathtt{1.5} & \mathtt{-2.5} \end{pmatrix};\ \mathtt{y} = \begin{pmatrix} \mathtt{Cos[t]} \\ \mathtt{Sin[t]} \end{pmatrix};\ \mathtt{z} = \begin{pmatrix} \mathtt{t} \\ \mathtt{t} \end{pmatrix};$$

We compare the two graphs before transformation with those after transformation.

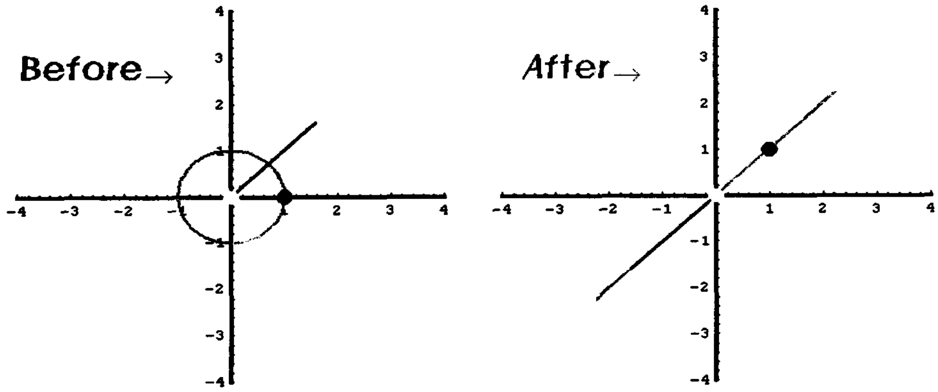
```
trans[x, y, z]
```



With this, and other choices of *z*, we illustrate the fact that linear transformations take circles to ellipses and lines to lines.

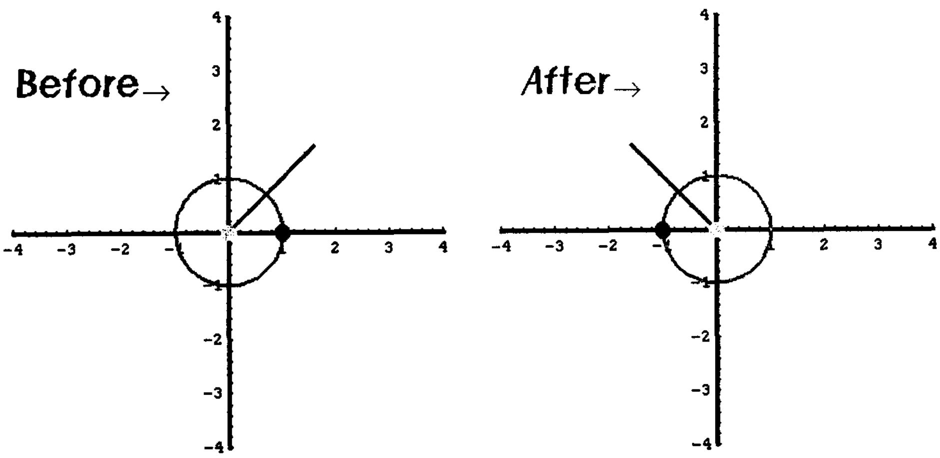
If T is singular (but nonzero), we note that the image of T is 1-dimensional. The circle is collapsed to a line segment.

$$\mathbf{x1} = \begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix}; \ \mathbf{trans[x1,\ y,\ z]}$$



We consider the identity transformation, a homothety (central stretching), a reflection and a rotation.

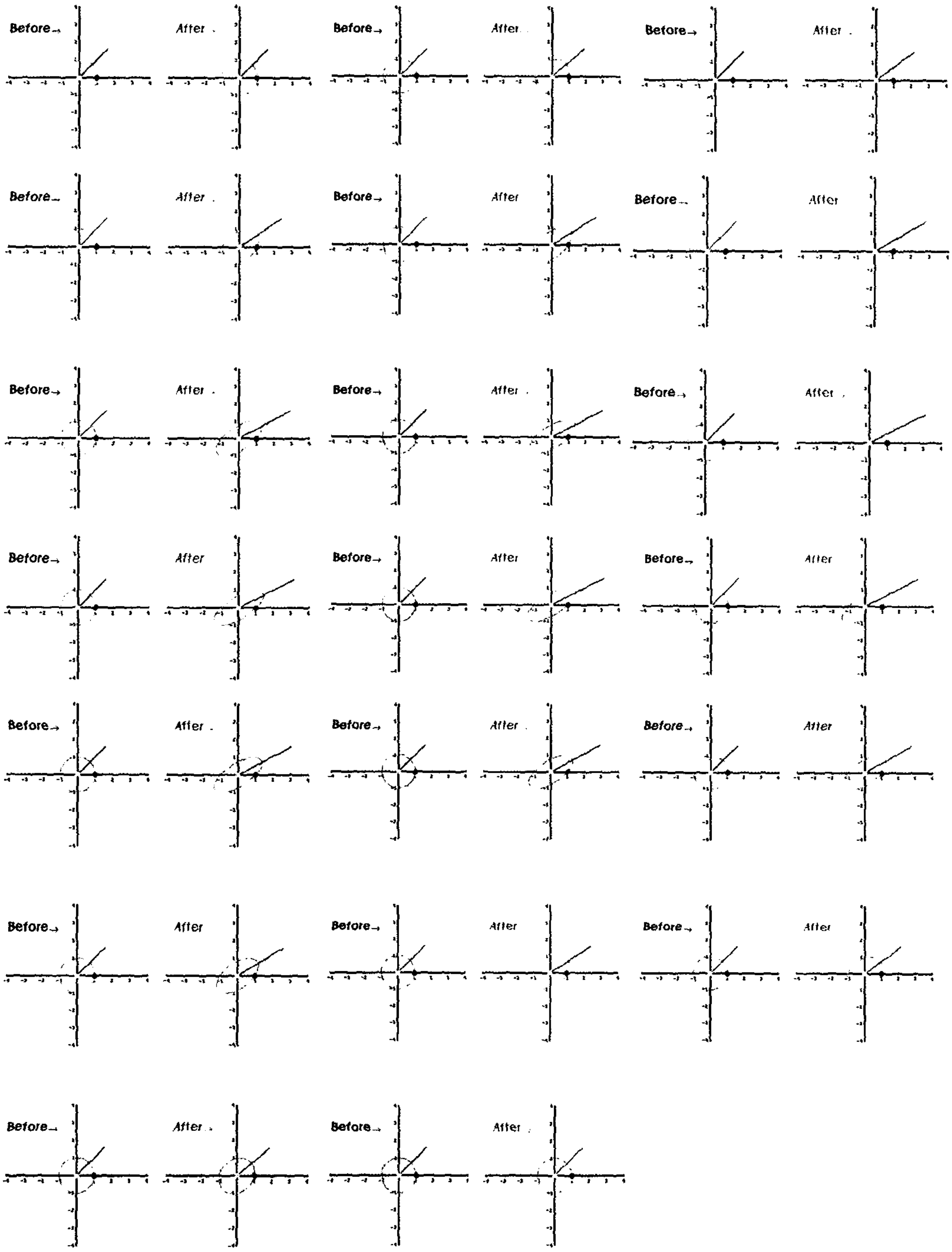
$$\mathbf{trans}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{y,\ z}\right] \ \mathbf{trans}\left[\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \mathbf{y,\ z}\right] \ \mathbf{trans}\left[\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{y,\ z}\right]$$



What can be said about the role of each (i, j) -component of the transformation matrix?

We can animate how the graphs of the two curve when the (1,2)-component and (2,1)-component (in various combinations) range from -1 to 1. To do this we use the Table function in Mathematica.

$$\mathbf{Table}\left[\mathbf{trans}\left[\begin{pmatrix} 1 & -w^2 + 2w \\ 0 & 1 \end{pmatrix}, \mathbf{y,\ z}\right], \mathbf{\{w,\ 0,\ 2,\ 0.1\}}\right];$$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After

$$\texttt{Table}\left[\texttt{trans}\left[\begin{pmatrix} 1 & w^2-2w \\ 0 & 1 \end{pmatrix}, \texttt{y}, \texttt{z}\right], \{\texttt{w}, 0, 2, 0.1\}\right];$$

$$\texttt{Table}\left[\texttt{trans}\left[\begin{pmatrix} 1 & 0 \\ -w^2+2w & 1 \end{pmatrix}, \texttt{y}, \texttt{z}\right], \{\texttt{w}, 0, 2, 0.1\}\right];$$

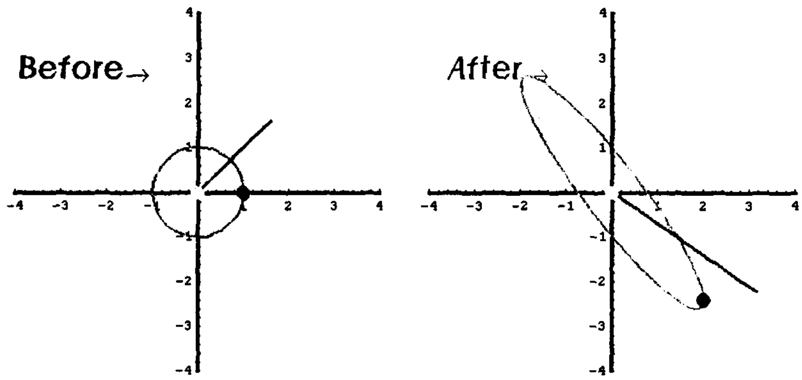
$$\texttt{Table}\left[\texttt{trans}\left[\begin{pmatrix} 1 & 0 \\ w^2-2w & 1 \end{pmatrix}, \texttt{y}, \texttt{z}\right], \{\texttt{w}, 0, 2, 0.1\}\right];$$

$$\texttt{Table}\left[\texttt{trans}\left[\begin{pmatrix} 1 & -w^2+2w \\ -w^2+2w & 1 \end{pmatrix}, \texttt{y}, \texttt{z}\right], \{\texttt{w}, 0, 2, 0.1\}\right];$$

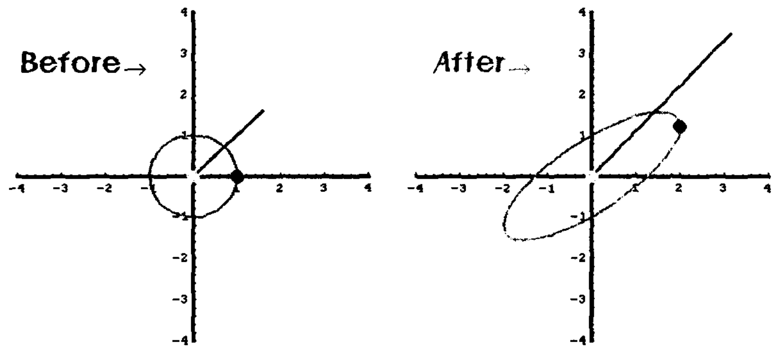
$$\texttt{Table}\left[\texttt{trans}\left[\begin{pmatrix} 1 & w^2-2w \\ w^2-2w & 1 \end{pmatrix}, \texttt{y}, \texttt{z}\right], \{\texttt{w}, 0, 2, 0.1\}\right];$$

The function *trans* can also be used to illustrate the fact that the composition of linear transformations is not commutative.

$$\texttt{trans}\left[\begin{pmatrix} -1 & 0 \\ 1.2 & 1 \end{pmatrix} . \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}, \texttt{y}, \texttt{z}\right]$$



$$\texttt{trans}\left[\begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix} . \begin{pmatrix} -1 & 0 \\ 1.2 & 1 \end{pmatrix}, \texttt{y}, \texttt{z}\right]$$



After this experiment, many students will have a vivid reminder of the fact that matrix multiplication and the composition of functions are not commutative.

Finally, we replace the line by the parabola $y = x^2$ to illustrate the behaviour of transformations in a slightly more complicated situation. Experimenting with this and other examples, students may strengthen their understanding of the plane transformation geometry.

2. Space Linear Transformation

A 3×3 matrix

(1.2) $$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \text{ for any real } a_{ij} (1 \le i, j \le 3),$$

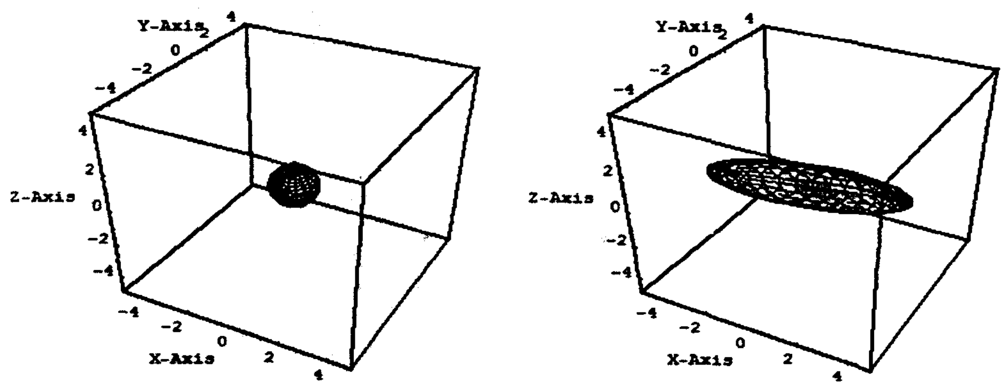
represents a space linear transformation $T: E^3 \to E^3$. We define the function

trans3d[x_, y_],

(see the Program [1-2] in Appendix), where x is a 3×3 matrix and y is a surface. Using the same process as for a plane linear transformation, we can study various aspects of a space linear transformation.

$$\textbf{Clear[y]; x2} = \begin{pmatrix} 1 & -1 & 3 \\ 0 & -1 & 2 \\ -1 & 0 & -1 \end{pmatrix}; \textbf{ y} = \begin{pmatrix} \text{Cos[t] Cos[u]} \\ \text{Sin[t] Cos[u]} \\ \text{Sin[u]} \end{pmatrix};$$

```
trans3d[x2, y]
```



$$\textbf{Table}\left[\textbf{trans3d}\left[\begin{pmatrix} 1 & w & 0 \\ 2-w & 1 & 0 \\ 0 & 1-0.5w & 1 \end{pmatrix}, \textbf{y}\right], \{w, 0, 2, 0.1\}\right];$$

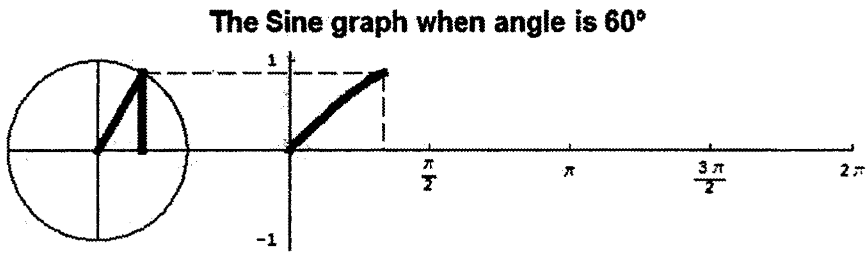
As an example, we show how a sphere is transformed by one particular transformation and also by a parameterized family of transformations. The sphere is transformed into a family of ellipsoids and as the parameter passes through 1, the ellipsoid is collapsed to a plane ellipse. Presentation of these examples to students points the way to many further experiments that are likely to be rewarding.

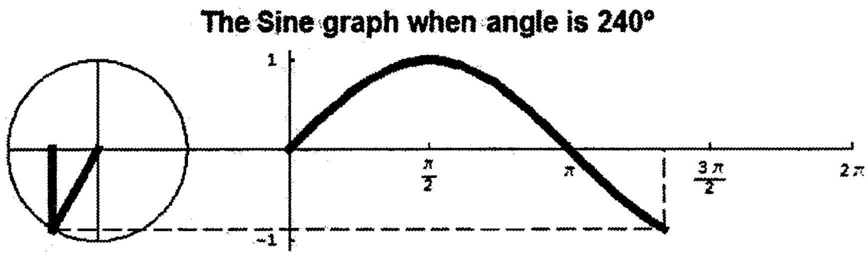
Model 2: Music

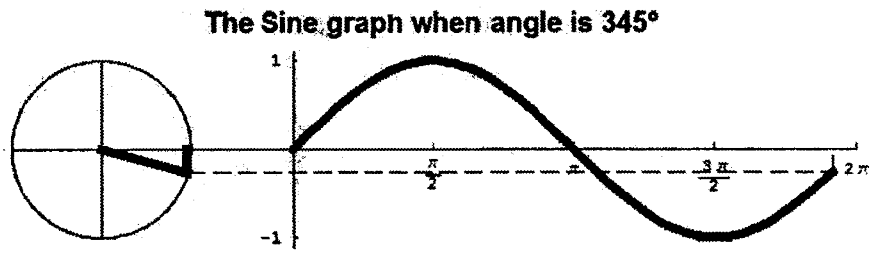
In this section we give various animations of trigonometric functions and show how musical notes can be produced using, for example, the sine and exponential functions (Kim 2003).

First, we define the function SinPlot (see the Program[2-1] in Appendix), which can be used to illustrate the relationship between the sine graph and the unit circle. Then we can animate drawing of the sine graph as its argument runs from 0 to 2π.

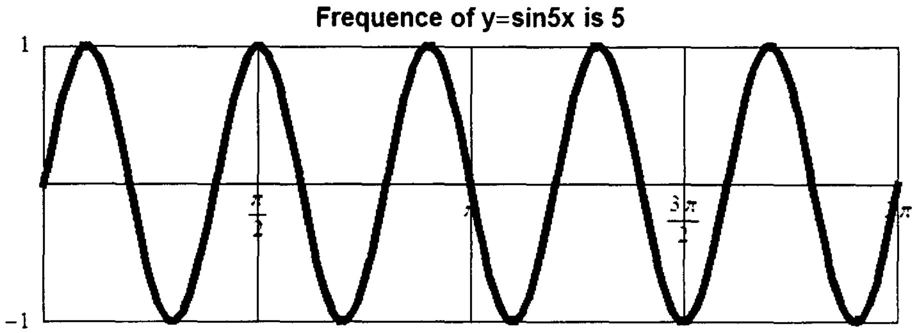
```
Do[SinPlot[θ], {θ, (π / 180) * 15, 2 π, (π / 180) * 15}];
```

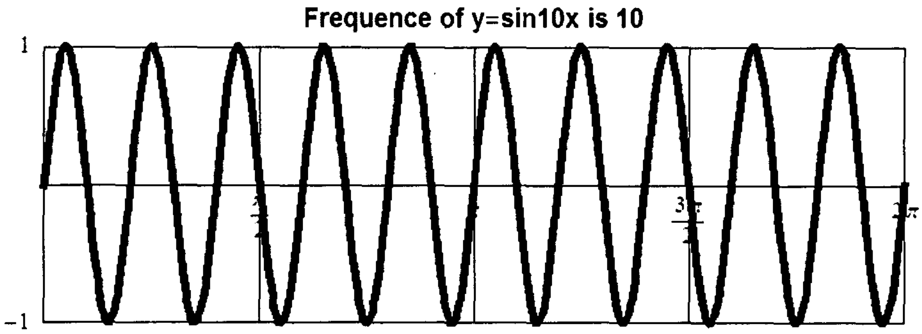




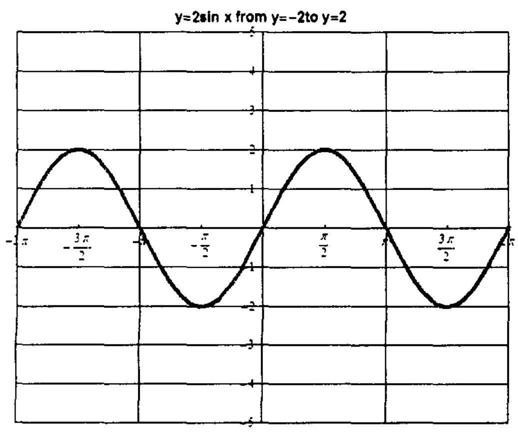


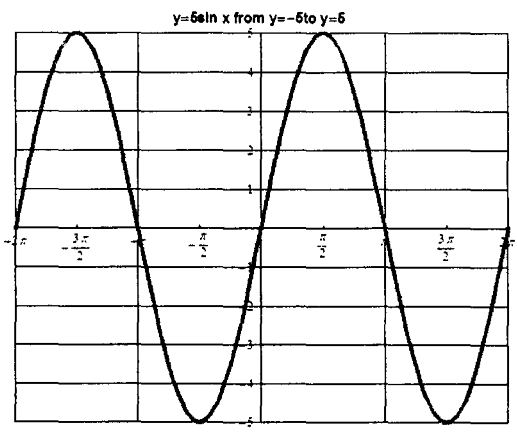
As an aid in explaining the relationship between the sine function and sound production, we plot graphs of the sin(x) and sin(bx) for real a, b. This illustrates the effect of frequency and amplitude on the graph.






```
Table[Plot[Sin[b x], {x, 0, 2π}, PlotStyle → {RGBColor[0, 0, 1], Thickness[0.01]},
  PlotLabel → FontForm[" Frequence of y=sin" <> ToString[b] <> "x is " <> ToString[b] <> " ",
   {"Helvetica-Bold", 16}], Background → GrayLevel[1], DefaultFont → {"Times", 14},
  GridLines → {{0, π/2, π, 3π/2, 2π}, {-1, 0, 1}}, Ticks → {{0, π/2, π, 3π/2, 2π}, {-1, 1}},
  AspectRatio → Automatic, ImageSize → 650, PlotRange → {{0, 2π}, {-1, 1}}];, {b, 1, 10, 1}];
```





```
Table[Plot[a Sin[x], {x, -2π, 2π}, PlotStyle→{RGBColor[0, 1, 0], Thickness[0.01]},
   PlotLabel→FontForm["y=" <> ToString[a] <> "sin x from y=" <> ToString[-a] <>
     "to y=" <> ToString[a] <> " ", {"Helvetica-Bold", 16}], Background→GrayLevel[1],
   DefaultFont→{"Times", 14},
   Ticks→{{-2π, -3π/2, -π, -π/2, 0, π/2, π, 3π/2, 2π}, {-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5}},
   GridLines→{{-2π, -π, 0, π, 2π}, {-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5}},
   AspectRatio→Automatic, ImageSize→600, PlotRange→{{-2π, 2π}, {-5, 5}}];, {a, 1, 5, 1}];
```

Mathematica has a Play function, which is analogous to Plot but produces sound instead of graphs. For example, we can produce a sound by running the following:

$$\text{Play}\left[\frac{20\,\text{Sin}[\,100 + x^3 + \text{Sin}[10 + x^3] + 20\,\text{Cos}[100\,x^3 + \text{Sin}[10\,x^3]]]}{100 + x\text{^}2 + 100\,\text{Sin}[x\text{^}3]},\ \{x,\ 0,\ 5\}\right];$$

```
Play[Sin[5000 x], {x, 0, 3}];
```

In order to represent musical notes, we take account of the familiar 12 tone scale and use a frequency that increases by the same multiplicative factor at each step. The factor is chosen to be

$$2^{\frac{1}{12}}$$

so that when the octave reached (12 steps) the frequency will be doubled. The following is an example that produces 6 notes.

$$\begin{aligned}
&\text{Play}\Big[\text{Which}\Big[0 < x < 1,\ \frac{\text{Sin}[1000\,x\,2^{\frac{-4}{12}}]}{e^{x\,2^{\frac{-4}{12}}}}\ \text{Sin}[1000\,x\,2^{\frac{-4}{12}}],\\
&\qquad 1 < x < 1.5,\ \frac{\text{Sin}[1000\,(x-1)\,2^{\frac{3}{12}}]}{e^{(x-1)\,2^{\frac{3}{12}}}}\ \text{Sin}[1000\,(x-1)\,2^{\frac{3}{12}}],\\
&\qquad 1.5 < x < 2,\ \frac{\text{Sin}[1000\,(x-1.5)\,2^{\frac{0}{12}}]}{e^{(x-1.5)\,2^{\frac{0}{12}}}}\ \text{Sin}[1000\,(x-1.5)\,2^{\frac{0}{12}}],\\
&\qquad 2 < x < 4,\ \frac{\text{Sin}[1000\,(x-2)\,2^{\frac{3}{12}}]}{e^{(x-2)\,2^{\frac{3}{12}}}}\ \text{Sin}[1000\,(x-2)\,2^{\frac{3}{12}}],\\
&\qquad 4 < x < 5,\ \frac{\text{Sin}[1000\,(x-4)\,2^{\frac{8}{12}}]}{e^{(x-4)\,2^{\frac{8}{12}}}}\ \text{Sin}[1000\,(x-4)\,2^{\frac{8}{12}}]\Big],\\
&\{x,\ 0,\ 5\}\Big]
\end{aligned}$$

Using the same method, we have prepared a function that plays the song "songaji" which is popular in Korea.

Model 3: Integral Calculus

1. Riemann sums

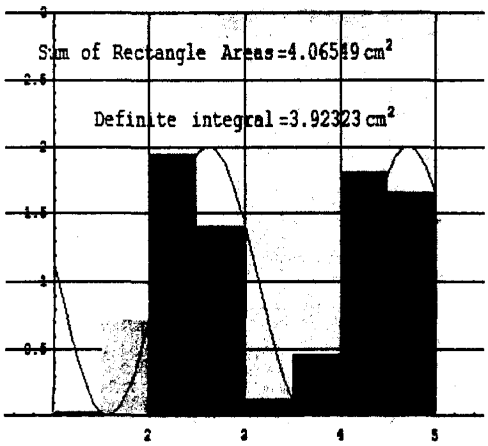
A common use/demonstration of integration is to determine the area under a curve and the volume under a surface (Wellin 2000). In this section we introduce methods for the visualization of these two concepts. For a given function and interval, we plot the graph of the function and numerically calculate the area under it.

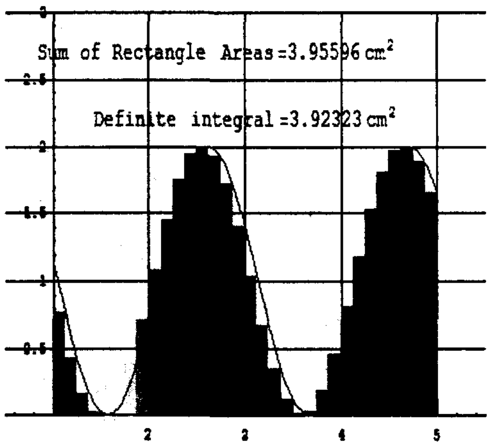
First, we consider the area under the curve $f(x) = 1 + \sin 3x$.

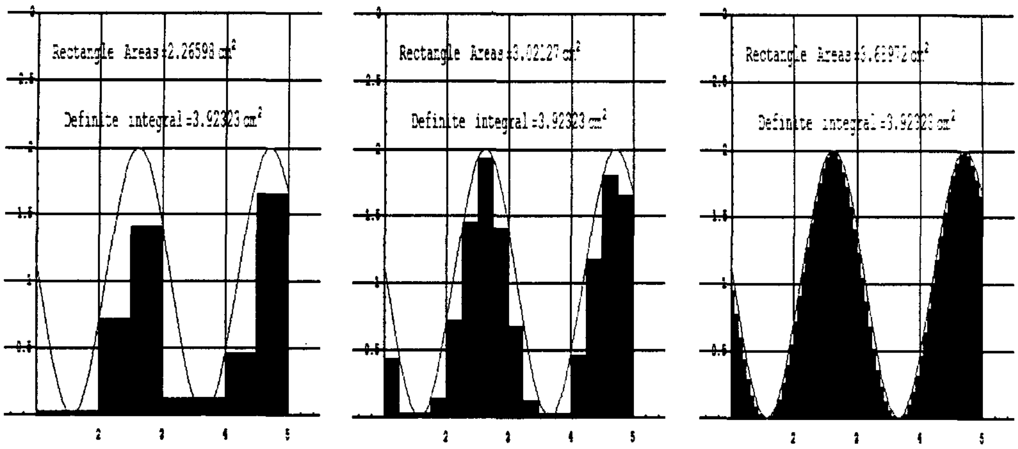
```
f[x_] := Sin[3 x] + 1;
fplot = Plot[f[x], {x, 1, 5}, AspectRatio -> 3/5, PlotRange → {{0.5, 5.5}, {0, 3}},
        ImageSize → 400, GridLines → Automatic, Axes → True, PlotStyle → {Thickness[0.005], Hue[0.8]}];

Clear[h, x]; h = N[Integrate[f[x], {x, 1, 5}]]
```

A natural way of estimating this area is firstly to subdivide the domain into evenly spaced intervals and then add the areas of certain approximating rectangles. Now, in order to see how the sum of the rectangle areas approximates the definite integral, we can compare right-hand, left-hand, maximal and minimal Riemann sums for many different numbers of subdivisions.

This displays the four Riemann sums for $f(x)$ using the four methods for approximating rectangles in the case where we divide the given interval to 2^5 subdivisions (see the Program [3-1] in Appendix).

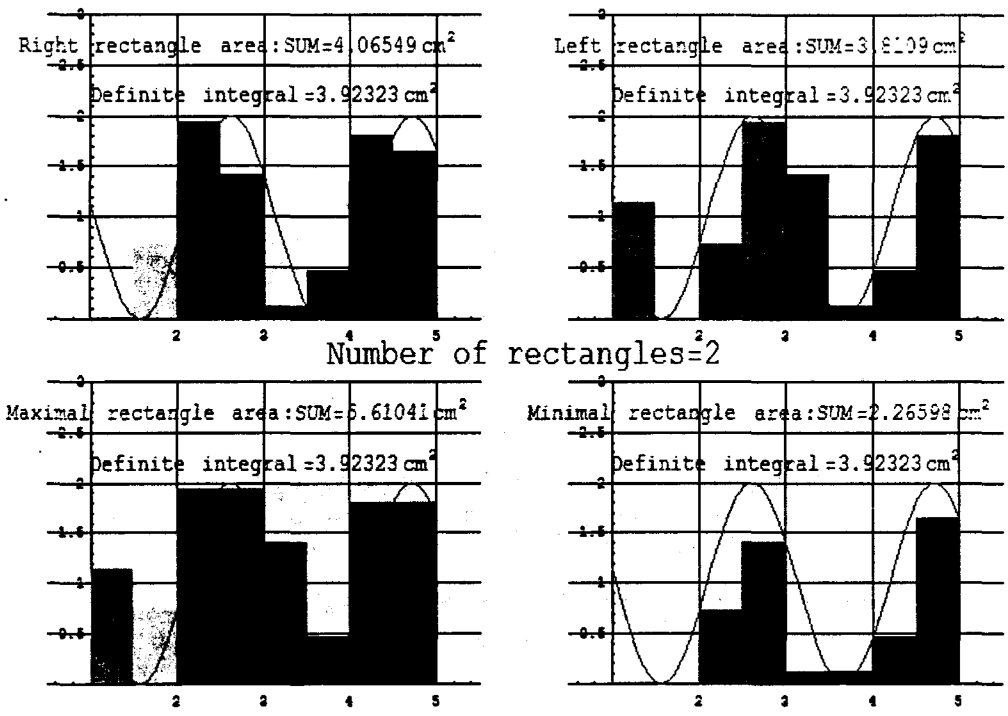


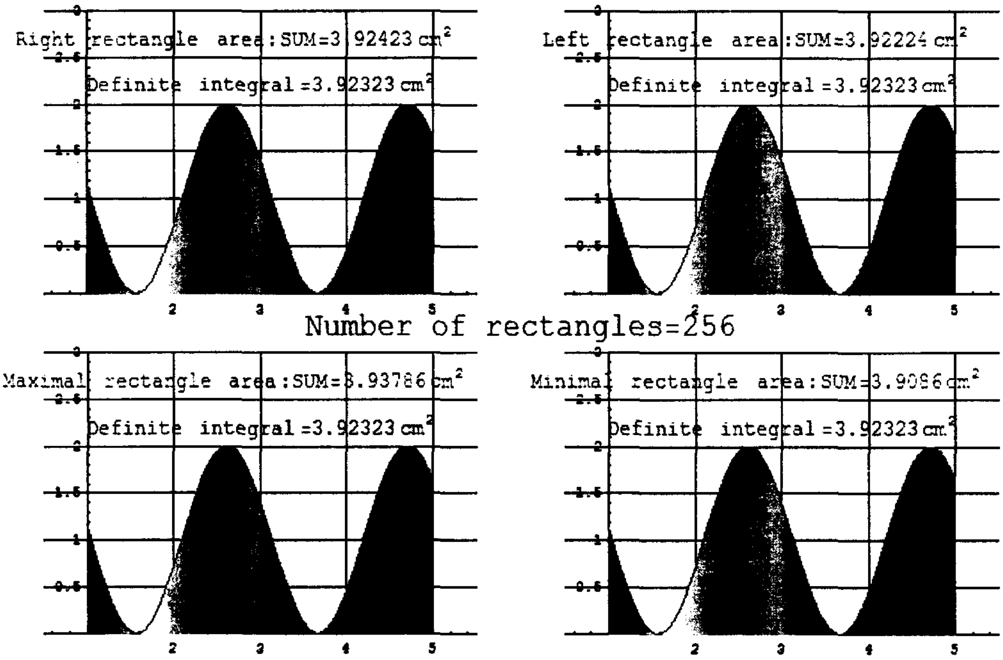




The following animations will give most students extraordinary experience and we believe that this is a meaningful in itself (see the Program [3-2] in Appendix).

Moreover, using the function Block we can allow the user to specify any function, any interval and any number of subdivisions, and then produce various Riemann sums. Here, we compare the numerical value of area with the middle method Riemann sum.





2. *Volume of A Surface of Revolution*

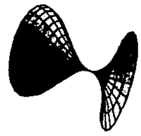
Since the volume of a surface of revolution is the integral of an area of cross sections orthogonal to the axis of revolution, and the cross section is a circle, it is relatively easy to visualize it. As an example, we give a method to visualize the volume of a wine glass.

First, we choose a suitable curve.

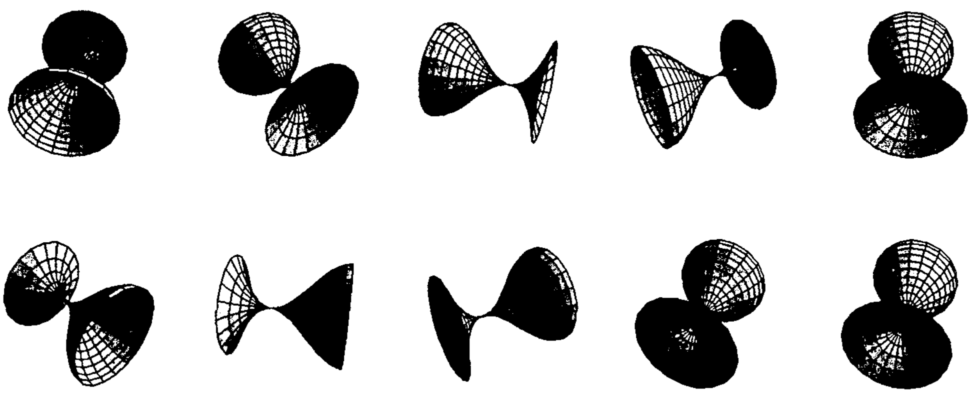
```
Plot[1/3 (x^3 - 6 x^2 + 9 x), {x, 1, 4}, PlotStyle→ {Hue[0.6], Thickness[0.012]}];
```

By revolving it around the x-axis, we obtain a wine glass (Kim 2001).

```
g = ParametricPlot3D[{t, 1/3 (t^3 - 6 t^2 + 9 t) Cos[u], 1/3 (t^3 - 6 t^2 + 9 t) Sin[u]}
        , {t, 1, 4}, {u, 0, 2 π}, Boxed→ False
        , Axes→ False, PlotPoints→ Automatic
        , PlotRange→ {{1, 4}, {-1.5, 1.5}, {-1.7, 1.7}}];
```

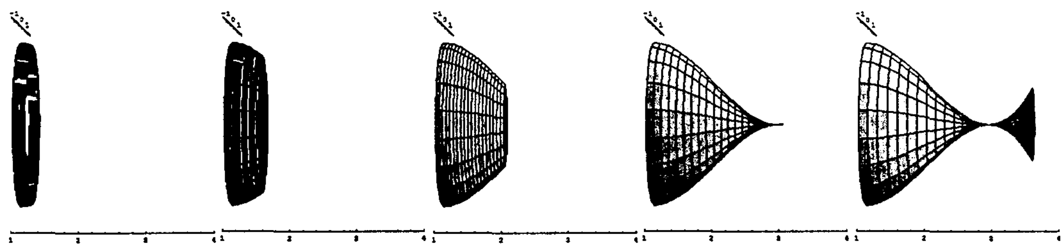


Using the function SpinShow, we can see the wine glass from various viewpoints.

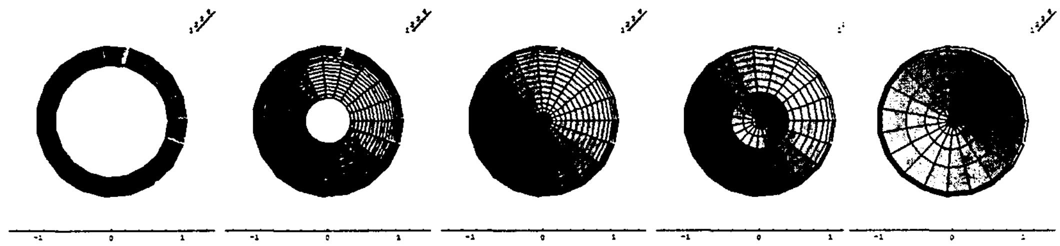


The following animations illustrate computation of the volume of the surface of revolution given by the integration of the area of the circular cross-section.

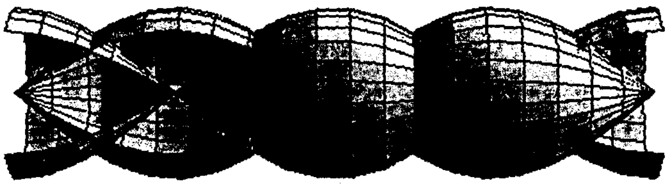
```
Table[ParametricPlot3D[{t, 1/3 (t^3 - 6 t^2 + 9 t) Cos[u], 1/3 (t^3 - 6 t^2 + 9 t) Sin[u]}
    , {t, 1, 1 + n}, {u, 0, 2 π}, Boxed → False, Axes → {True, True, False}, PlotPoints → Automatic
    , PlotRange → {{1, 4}, {-1.5, 1.5}, {-1.6, 1.6}}, ViewPoint -> {0.039, -4.970, 0.000}
    ], {n, 0.1, 3, 0.1}];
```



```
Table[ParametricPlot3D[{t, 1/3 (t^3 - 6 t^2 + 9 t) Cos[u], 1/3 (t^3 - 6 t^2 + 9 t) Sin[u]}
    , {t, 1, 1 + n}, {u, 0, 2 π}, Boxed → False, Axes → {True, True, False}
    , PlotPoints → Automatic, PlotRange → {{1, 4}, {-1.5, 1.5}, {-1.6, 1.6}}
    , ViewPoint -> {5, 0, 0}], {n, 0.1, 3, 0.1}];
```

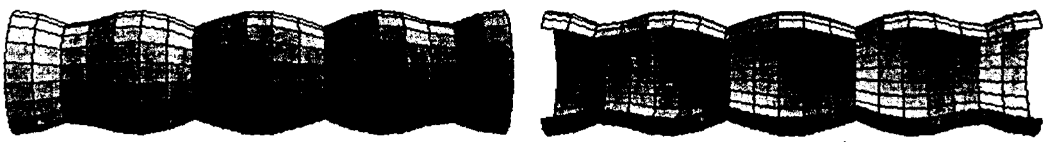


On the other hand, we can attempt to find another example to visualize the surface of revolution obtained by revolving two curves $\sin x$ and $\cos x$ around the x-axis. Of course, the two curves intersect transversely so that the two surfaces interpenetrate. The following "cut-away" view illustrates the "inner" and "outer" surfaces.



This demonstration leads the student to two observations: (i) it is sufficient and simpler to use the absolute value of the function being revolved (here we use $|\sin x|$ and $|\cos x|$); (ii) in order to get a single surface, we can choose for each x, the maximum value of $|\sin x|$ and $|\cos x|$ as our function value. This has the effect of eliminating the inner surface. This is shown below, both with an outer view and a cut-away view.

```
ParametricPlot3D
    [{t, Max[Abs[Sin[t]] , Abs[Cos[t]] ] Cos[u], Max[Abs[Sin[t]], Abs[Cos[t]]] Sin[u]}
    , {t, 0, 2 π}, {u, 0, 2 π}
    , Boxed→ False, Axes → False, PlotPoints →Automatic
    , PlotRange→ {{0, 2 π}, {-1.5, 1.5}, {-1.7, 1.7}}
    , ViewPoint -> {0.028, -4.069, -0.020}, ImageSize→ 400, PlotPoints→ 100    ];
```



III. CONCLUSION

It is clear that computer systems are presenting new challenges and opportunities in the mathematics classroom (Barrodale 1971). Their ubiquity has the potential to affect curriculum and teaching-learning methods both in high schools and universities. As Muller (1998) observes, new educational technologies "will erect new barriers for some people, while [freeing others] to explore the world of Mathematics in a very different environment." In this paper, we have presented six models designed to encourage effective teaching and learning in the mathematics classroom. Since abstract mathematical definitions and concepts need to be represented easily in order to facilitate student's understanding, mathematical software and other technologies may stimulate better mathematics education. The more novel the environment is, the better the chance to arouse the students' curiosity. It may well be that these experiments raise more questions than they answer.

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http://mttc.inje.ac.kr

http://socas.inje.ac.kr/hskim

Appendix (The Mathematica program used in MAIN.)

1. Program [1-1]

```
trans[matrix_, a01_, b02_] :=
 Module[{input01, input02, output01, output02, inplot01, inplot02, outplot01, outplot02},
  input01 = a01;
  input02 = b02;
  output01 = Flatten[matrix.input01];
  initial01 = Flatten[matrix.{1, 0}];
  initial02 = Flatten[matrix.{0, 0}];
  output02 = Flatten[matrix.input02];
  inplot01 = ParametricPlot[Evaluate[Flatten[input01]], {t, 0, 2 π},
           PlotRange → {{-4, 4}, {-4, 4}},
           AspectRatio → 1, PlotStyle -> {Thickness[0.01], Hue[.5]},
           AxesStyle → Thickness[0.01],
           DisplayFunction → Identity];
  inplot02 = ParametricPlot[Evaluate[Flatten[input02]], {t, 0, π/2},
           PlotRange → {{-4, 4}, {-4, 4}},
           AspectRatio → 1, PlotStyle -> {Thickness[0.01], Hue[.9]},
           AxesStyle → Thickness[0.01],
           DisplayFunction → Identity];
  pt01 = Show[Graphics[{PointSize[0.04], Hue[0.7], Point[{1, 0}]}], DisplayFunction → Identity];
  ppo01 = Show[Graphics[{PointSize[0.04], Hue[0.2], Point[{0, 0}]}], DisplayFunction → Identity];
  inplot = Show[{inplot01, inplot02, pt01, ppo01}, PlotRange → {{-4, 4}, {-4, 4}},
           AspectRatio → 1, DisplayFunction → Identity,
           Prolog → {Text[StyleForm["Before→", FontSize → 20], {-2.5, 2.7}]}];
  outplot01 = ParametricPlot[Evaluate[Flatten[output01]], {t, 0, 2 π},
           PlotStyle -> {Thickness[0.01], Hue[.5]},
           PlotRange → {{-4, 4}, {-4, 4}}, AspectRatio → 1,
           AxesStyle → Thickness[0.01],
           DisplayFunction → Identity];
  outplot02 = ParametricPlot[Evaluate[Flatten[output02]], {t, 0, π /2},
           PlotStyle -> {Thickness[0.01], Hue[.9]},
           PlotRange → {{-4, 4}, {-4, 4}}, AspectRatio → 1,
           AxesStyle → Thickness[0.01],
           DisplayFunction → Identity];
  pt02 = Show[Graphics[{PointSize[0.04], Hue[0.7], Point[initial01]}], DisplayFunction → Identity];
  qp02 = Show[Graphics[{PointSize[0.04], Hue[0.2], Point[initial02]}], DisplayFunction → Identity];
  outplot = Show[{outplot01, outplot02, pt02, qp02},
           PlotRange → {{-4, 4}, {-4, 4}},
           AspectRatio → 1, DisplayFunction → Identity,
           Prolog →
    {Text[StyleForm["After→", FontSize → 20], {-2.5, 2.7}]}];
  Show[GraphicsArray[{inplot, outplot}, ImageSize → 600]];
 ]
```

2. Program [1-2]

```
trans3d[matrix_, para_] :=
 Module[{outplot, inplot},
  input01 = para;
  initial01 = Flatten[matrix.{0, 0, 1}];
  output = Flatten[matrix.input01];
  p1 = ParametricPlot3D[Evaluate[Flatten[input01]], {t, 0, 2 π},
    {u, -π/2, π/2},    PlotRange → {{-5, 5}, {-5, 5}, {-5, 5}}, AspectRatio → 1,
    AxesStyle -> Thickness[0.009],
    AxesLabel → {"X-Axis", "Y-Axis", "Z-Axis"},
    BoxStyle →  Thickness[0.009], DisplayFunction → Identity];
  pt01 = Show[Graphics3D[{PointSize[0.04], Hue[0.99], Point[{0, 0, 1}]}],
    DisplayFunction → Identity];
  inplot = Show[{p1, pt01}, DisplayFunction → Identity
          ];
  gr = ParametricPlot3D[Evaluate[Flatten[output]], {t, 0, 2 π},
    {u, -π/2, π/2},    PlotRange → {{-5, 5}, {-5, 5}, {-5, 5}}, AspectRatio → 1,
    AxesStyle -> Thickness[0.009],
    AxesLabel → {"X-Axis", "Y-Axis", "Z-Axis"},
    BoxStyle →  Thickness[0.009], DisplayFunction → Identity];
  pt02 = Show[Graphics3D[{PointSize[0.04], Hue[0.99], Point[initial01]}],
    DisplayFunction → Identity];
  outplot = Show[{gr, pt02}, DisplayFunction → Identity
          ];
  Show[GraphicsArray[{inplot, outplot}, ImageSize → 600]];
 ]
```

3. Program [2-1]

```
SinPlot[θ_] :=
     Module[{},
            lns = {{Thickness[0.001],
                  Line[{{-π + 1, 1}, {-π + 1, -1}}]},
                  {Thickness[0.01],
                   Line[{{-π + 1, 0}, {-π + 1 + Cos[θ], Sin[θ]}}],
                   Line[{{-π + 1 + Cos[θ], 0}, {-π + 1 + Cos[θ], Sin[θ]}}]},
                   {Dashing[{0.02, 0.01}],
                   Thickness[0.001],
                   Line[{{-π + 1 + Cos[θ], Sin[θ]}, {θ, Sin[θ]}}],
                   Line[{{θ, Sin[θ]}, {θ, 0}}]}};
             cls = {Thickness[0.001], Circle[{-π + 1, 0}, 1],
    Circle[{-π + 1, 0}, 1.0, {0, Mod[θ, 2 π]}]};
  Plot[Sin[t], {t, 0, θ},
   PlotStyle → {RGBColor[1, 0, 0], Thickness[0.01]},
   PlotLabel → FontForm["The Sine graph when angle is " <> ToString[θ 180 / π] <> "°",
     {"Helvetica-Bold", 16}],
   Epilog → {cls, lns},
   Ticks → {{0, π / 2, π, 3 π / 2, 2 π}, {-1, 0, 1}},
   AspectRatio → Automatic, ImageSize → 600,
   PlotRange → {{-π, 2 π}, {-1.1, 1.1}}];
 ];
```

4. Program [3-1]

```
m = 5;
Table[n = k;
  plotpoint = Table[{x, f[x]}, {x, 1, 5, 1/2^n}];
  y2 = 1/2^n N[Sum[plotpoint[[i, 2]], {i, 1, Length[plotpoint] - 1}]];
  Show[Graphics[Table[{Hue[i/Length[plotpoint]],
      Rectangle[{plotpoint[[i + 1, 1]], 0}, plotpoint[[i]]]}, {i, 1, Length[plotpoint] - 1}],
    Axes -> True, PlotRange -> {{0.5, 5.5}, {0, 3}},
    AspectRatio -> 3/5, ImageSize -> 400, GridLines -> Automatic, Prolog -> {{Text[StyleForm["Rectangle Areas="
        <> ToString[y2] <> "cm^2",
       FontSize -> 14, FontColor -> Hue[0.999]], {2.7, 2.7}]},
     {Text[StyleForm["Definite integral=" <> ToString[h] <> "cm^2", FontSize -> 14, FontColor -> Hue[0.599]], {3, 2.2}]}}], fplot];,
  {k, 1, m}];

  Prolog ->
   {{Text[StyleForm["Sum of Rectangle Areas="
        <> ToString[y1] <> "cm^2",
       FontSize -> 14, FontColor -> Hue[0.999]], {2.7, 2.7}]},
     {Text[StyleForm["Definite integral=" <> ToString[h] <> "cm^2", FontSize -> 14, FontColor -> Hue[0.599]], {3, 2.2}]}}], fplot];,
  {k, 1, m}];
```

```
m = 5;
Table[n = k;
  plotpoint = Table[{x, f[x]}, {x, 1, 5, 1/2^n}];
  y1 = 1/2^n N[Sum[plotpoint[[i + 1, 2]], {i, 1, Length[plotpoint] - 1}]];
  Show[Graphics[Table[{Hue[i/Length[plotpoint]],
      Rectangle[{plotpoint[[i, 1]], 0}, plotpoint[[i + 1]]]}, {i, 1, Length[plotpoint] - 1}],
    Axes -> True, PlotRange -> {{0.5, 5.5}, {0, 3}},
    AspectRatio -> 3/5, ImageSize -> 400, GridLines -> Automatic,
    Prolog ->
     {{Text[StyleForm["Sum of Rectangle Areas="
          <> ToString[y1] <> "cm^2",
         FontSize -> 14, FontColor -> Hue[0.999]], {2.7, 2.7}]},
      {Text[StyleForm["Definite integral=" <> ToString[h] <> "cm^2", FontSize -> 14, FontColor -> Hue[0.599]], {3, 2.2}]}}], fplot];,
  {k, 1, m}];
```

```
Table[n = k;
  plotpoint = Table[{x, f[x]}, {x, 1, 5, 1/2^n}];
  y2 = 1/2^n N[Sum[plotpoint[[i, 2]], {i, 1, Length[plotpoint] - 1}]];
  Show[Graphics[Table[{Hue[i/Length[plotpoint]],
      Rectangle[{plotpoint[[i + 1, 1]], 0}, plotpoint[[i]]]}, {i, 1, Length[plotpoint] - 1}],
    Axes → True, PlotRange → {{0.5, 5.5}, {0, 3}},
    AspectRatio -> 3/5, ImageSize → 400, GridLines → Automatic, Prolog → {{Text[StyleForm["Rectangle Areas="
        <> ToString[y2] <> "cm²",
       FontSize → 14, FontColor → Hue[0.999]], {2.7, 2.7}]},
     {Text[StyleForm["Definite integral=" <> ToString[h] <> "cm²", FontSize → 14, FontColor → Hue[0.599]], {3, 2.2}]}}], fplot];,
  {k, 1, m}];
Table[n = k;
  plotpoint = Table[{x, f[x]}, {x, 1, 5, 1/2^n}];
  y3 = 1/2^n N[Sum[Max[plotpoint[[i, 2]], plotpoint[[i + 1, 2]]], {i, 1, Length[plotpoint] - 1}]];
  Show[Graphics[Table[{Hue[i/Length[plotpoint]],
      Rectangle[{plotpoint[[i, 1]], 0}, {plotpoint[[i + 1, 1]], Max[plotpoint[[i, 2]], plotpoint[[i + 1, 2]]]}]},
     {i, 1, Length[plotpoint] - 1}],
    Axes → True, PlotRange → {{0.5, 5.5}, {0, 3}},
    AspectRatio -> 3/5, ImageSize → 400, GridLines → Automatic, Prolog → {{Text[StyleForm["Rectangle Areas="
        <> ToString[y3] <> "cm²",
       FontSize → 14, FontColor → Hue[0.999]], {2.7, 2.7}]},
     {Text[StyleForm["Definite integral=" <> ToString[h] <> "cm²", FontSize → 14, FontColor → Hue[0.599]], {3, 2.2}]}}], fplot];,
  {k, 1, m}];
Table[n = k;
  plotpoint = Table[{x, f[x]}, {x, 1, 5, 1/2^n}];
  y4 = 1/2^n N[Sum[Min[plotpoint[[i, 2]], plotpoint[[i + 1, 2]]], {i, 1, Length[plotpoint] - 1}]];
  Show[Graphics[Table[{Hue[iverLength[plotpoint]],
      Rectangle[{plotpoint[[i, 1]], 0}, {plotpoint[[i + 1, 1]], Min[plotpoint[[i, 2]], plotpoint[[i + 1, 2]]]}]},
     {i, 1, Length[plotpoint] - 1}],
    Axes → True, PlotRange → {{0.5, 5.5}, {0, 3}},
    AspectRatio -> 3/5, ImageSize → 400, GridLines → Automatic, Prolog → {{Text[StyleForm["Rectangle Areas="
        <> ToString[y4] <> "cm²",
       FontSize → 14, FontColor → Hue[0.999]], {2.7, 2.7}]},
     {Text[StyleForm["Definite integral=" <> ToString[h] <> "cm²", FontSize → 14, FontColor → Hue[0.599]], {3, 2.2}]}}], fplot];,
  {k, 1, m}];
```

5. Program [3-2]

```
m = 8;
Table[n = k;
  plotpoint = Table[{x, f[x]}, {x, 1, 5, 1/2^n}];
     y1 = 1/2^n N[Sum[plotpoint[[j + 1, 2]], {j, 1, Length[plotpoint] - 1}]];
     y2 = 1/2^n N[Sum[plotpoint[[j, 2]], {j, 1, Length[plotpoint] - 1}]];
     y3 = 1/2^n N[Sum[Max[plotpoint[[j, 2]], plotpoint[[j + 1, 2]]], {j, 1, Length[plotpoint] - 1}]];
     y4 = 1/2^n N[Sum[Min[plotpoint[[j, 2]], plotpoint[[j + 1, 2]]], {j, 1, Length[plotpoint] - 1}]];
     a1 = Show[Graphics[Table[{Hue[j/Length[plotpoint]],
                            Rectangle[{plotpoint[[j, 1]], 0}, plotpoint[[j + 1]]]},
                           {j, 1, Length[plotpoint] - 1}],
          Axes → True, PlotRange → {{0.5, 5.5}, {0, 3}}, AspectRatio -> 3/5,
          ImageSize → 400, GridLines → Automatic,
          Prolog → {{Text[StyleForm["Right rectangle area:SUM=" <> ToString[y1] <> "cm²",
                              FontSize -> 14, FontColor -> Hue[.999]],
                             {2.7, 2.7}]},
                    {Text[StyleForm["Definite integral=" <> ToString[h] <> "cm²",
                              FontSize -> 14, FontColor -> Hue[.599]],
                             {3, 2.2}]}}], fplot, DisplayFunction → Identity];
     a2 = Show[Graphics[Table[{Hue[j/Length[plotpoint]],
                            Rectangle[{plotpoint[[j + 1, 1]], 0}, plotpoint[[j]]]},
                           {j, 1, Length[plotpoint] - 1}],
          Axes → True, PlotRange → {{0.5, 5.5}, {0, 3}}, AspectRatio -> 3/5,
          ImageSize → 400, GridLines → Automatic,
          Prolog → {{Text[StyleForm["Left rectangle area:SUM=" <> ToString[y2] <> "cm²",
                              FontSize -> 14, FontColor -> Hue[.999]],
                             {2.7, 2.7}]},
                    {Text[StyleForm["Definite integral=" <> ToString[h] <> "cm²",
                              FontSize -> 14, FontColor -> Hue[.599]],
                             {3, 2.2}]}}], fplot, DisplayFunction → Identity];
     a3 = Show[Graphics[Table[{Hue[j/Length[plotpoint]],
                        Rectangle[{plotpoint[[j, 1]], 0},
                                 {plotpoint[[j + 1, 1]],
                                  Max[plotpoint[[j, 2]], plotpoint[[j + 1, 2]]]}]},
                               {j, 1, Length[plotpoint] - 1}],
          Axes → True, PlotRange → {{0.5, 5.5}, {0, 3}}, AspectRatio -> 3/5,
          ImageSize → 400, GridLines → Automatic,
          Prolog → {{Text[StyleForm["Maximal rectangle area:SUM=" <> ToString[y3] <> "cm²",
                              FontSize -> 14, FontColor -> Hue[.999]],
                             {2.7, 2.7}]},
                    {Text[StyleForm["Definite integral=" <> ToString[h] <> "cm²",
                              FontSize -> 14, FontColor -> Hue[.599]],
                             {3, 2.2}]}}], fplot, DisplayFunction → Identity];
     a4 = Show[Graphics[Table[{Hue[j/Length[plotpoint]],
                            Rectangle[{plotpoint[[j, 1]], 0},
                                     {plotpoint[[j + 1, 1]],
                                      Min[plotpoint[[j, 2]], plotpoint[[j + 1, 2]]]}]},
                               {j, 1, Length[plotpoint] - 1}],
          Axes → True, PlotRange → {{0.5, 5.5}, {0, 3}}, AspectRatio -> 3/5,
          ImageSize → 400, GridLines → Automatic,
          Prolog → {{Text[StyleForm["Minimal rectangle area:SUM=" <> ToString[y4] <> "cm²",
                              FontSize -> 14, FontColor -> Hue[.999]],
                             {2.7, 2.7}]},
                    {Text[StyleForm["Definite integral=" <> ToString[h] <> "cm²",
                              FontSize -> 14, FontColor -> Hue[.599]],
                             {3, 2.2}]}}], fplot, DisplayFunction → Identity];
  Show[GraphicsArray[{{a1, a2}, {a3, a4}}, ImageSize → 700,
       Prolog → {Text[StyleForm
                 ["Number of rectangles=" <> ToString[2^k],
                            FontSize → 20, FontColor → Hue[0.6]], {1, 0.6}]}]];
 , {k, 1, m}];
```

6. Program [3-3]

```
Block[{}, f[x_] = Input["Enter the function, for example, x+Sin[3x] as functions of x."];
 k1 = Input["Enter the minimum of domain."];
 k2 = Input["Enter the maximum of domain."];
 Print["Given Function : ", f[x]];
 Print["Domain of function : ", {k1, k2}];
 Print["The graph "];
 Plot[f[x], {x, k1, k2}, PlotRange → {{k1, k2}, {Min[{Exponent[f[x], x]*k1, f[k1]}], Max[{Exponent[f[x], x]*k2, f[k2]}]}},
  AspectRatio -> 3/5,
        ImageSize → 500, GridLines → Automatic,
        Axes → True, PlotStyle → {Thickness[0.008], Hue[0.8]}];]
```