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Application of the Chebyshev-Fourier Pseudospectral Method to the Eigenvalue Analysis of Circular Mindlin Plates with Free Boundary Conditions

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An eigenvalue analysis of the circular Mindlin plates with free boundary conditions is presented. The analysis is based on the Chebyshev-Fourier pseudospectral method. Even though the eigenvalues of lower vibration modes tend to convergence more slowly than those of higher vibration modes, the eigenvalues converge for sufficiently fine pseudospectral grid resolutions. The eigenvalues of the axisymmetric modes are computed separately. Numerical results are provided for different grid resolutions and for different thickness-to-radius ratios.

Key Words : Eigenvalue, Chebyshev-Fourier Pseudospectral Method, Circular Mindlin Plate

1. Introduction

Plate vibration is important in many applications in mechanical, civil and aerospace engineering. Real plates may have appreciable thickness so that transverse shear and the rotary inertia are not negligible as assumed in the classical plate theory. As a result, the thick plate model based on the Mindlin theory has gained more popularity.

Research on the plate vibration can be divided into three categories. First, there exist exact solutions only for a very restricted number of simple cases. Second, semi-analytic solutions are available. This includes the Rayleigh-Ritz method and the differential quadrature method. Finally, there are the most widely used discretization methods such as the finite element method, the finite strip method and the finite difference method. As it is more useful to have analytical results than to resort to a numerical method, most efforts focus on developing efficient semi-analytic solutions. Mindlin and Deresiewicz provided analytical solutions of circular and annular plate for the axisymmetric modes (Mindlin, 1951 ; Deresiewicz and Mindlin, 1955; Deresiewicz, 1956). The transfer matrix method was used to compute the eigenvalues of annular Mindlin plate (Irie et al., 1979 ; Irie et al., 1982) and circular Mindlin plates (Irie et al., 1980) including non-axisymmetric modes. With the transfer matrix method, however, one has to provide an initial guess of an eigenvalue and should improve the initial guess by an iterative scheme. The differential quadrature method was also used to compute the eigenvalues of axisymmetric Mindlin plates (Liew et al., 1997).

The algorithms that are easier to implement are more favored these days than those that run faster, because the performances of the computer has been drastically improved during the last decade. Rapid convergence and good accuracy as well as the conceptual simplicity characterize the pseudospectral method. The pseudospectral method, which has stemmed from the spectral method, is more closely related to the differential quadrature method by employing the collocation process. As the formulation is simple and powerful enough

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to produce approximate solutions close to exact solutions, this method has been used extensively in the fluid mechanics research (Pyret and Taylor, 1990).

Even though this method can be used for the solution of structural mechanics problems, it has been largely unnoticed by the structural mechanics community and few articles where the pseudospectral method has been applied are available. The Chebyshev collocation method was applied to the free vibration analyses of axisymmetric circular plate (Soni and Amba-Rao, 1975) and axisymmetric annular plate (Gupta and Lal, 1985), where fourth order differential equations were derived in terms of the bending rotation by eliminating the transverse displacement. The collocation method along with the power series representation of the dependent variables was also used in the free vibration analysis of rectangular plates (Mikami and Yoshimura, 1984). Recently, the pseudospectral method was applied to the eigenvalue problems of the circular Mindlin plates with clamped and simply supported boundary conditions (Lee, 2002) and the rectangular Mindlin plates (Lee, 2003).

In the present study, the pseudospectral method is applied to the free vibration analysis of circular Mindlin plates subject to free boundary conditions.

2. Formulations

2.1 Circular mindlin plate and the pseudospectral method

The equations of motion of a homogeneous and isotropic plate based on the Mindlin theory expressed in the polar coordinate system (Mindlin, 1951) are

$$\begin{aligned}\frac{\partial M_r}{\partial r}+\frac{1}{r}\frac{\partial M_{r\theta}}{\partial\theta}+\frac{M_r-M_\theta}{r}-Q_r=\frac{\rho h^3}{12}\frac{\partial^2\Psi_r}{\partial t^2}\\ \frac{\partial M_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial M_\theta}{\partial\theta}+\frac{2M_{r\theta}}{r}-Q_\theta=\frac{\rho h^3}{12}\frac{\partial^2\Psi_\theta}{\partial t^2}\\ \frac{\partial Q_r}{\partial r}+\frac{1}{r}\frac{\partial Q_\theta}{\partial\theta}+\frac{Q_r}{r}=\rho h\frac{\partial^2 W}{\partial t^2}\end{aligned}\qquad(1)$$

where Ψ_r, Ψ_θ and W represent the bending rotation in the radial direction, the bending rotation in the circumferential direction and the transverse displacement, respectively. Meanwhile, h is the plate thickness, ρ is the density, r, θ and t are the independent variables which represent the distance from the origin, the angle in the circumferential direction and the time. The stress resultants M_r, M_θ, $M_{r\theta}$, Q_r and Q_θ are defined by

$$\begin{aligned}M_r&=D\left[\frac{\partial\Psi_r}{\partial r}+\frac{\nu}{r}\left(\Psi_r+\frac{\partial\Psi_\theta}{\partial\theta}\right)\right]\\ M_\theta&=D\left[\frac{1}{r}\left(\Psi_r+\frac{\partial\Psi_\theta}{\partial\theta}\right)+\nu\frac{\partial\Psi_r}{\partial r}\right]\\ M_{r\theta}&=\frac{D(1-\nu)}{2}\left[\frac{1}{r}\left(\frac{\partial\Psi_r}{\partial\theta}-\Psi_\theta\right)+\frac{\partial\Psi_\theta}{\partial r}\right]\\ Q_r&=\kappa^2Gh\left(\Psi_r+\frac{\partial W}{\partial r}\right)\\ Q_\theta&=\kappa^2Gh\left(\Psi_\theta+\frac{1}{r}\frac{\partial W}{\partial\theta}\right)\end{aligned}\qquad(2)$$

where $D=Eh^3/12(1-\nu^2)$ and G are the flexural rigidity and the shear modulus, E and ν are the Young's modulus and the Poisson ratio, $\kappa^2=\pi^2/12$ is the shear correction factor.

Substitution (2) into (1), while assuming simple harmonic motions in time

$$\begin{aligned}\Psi_r(r,\theta,t)&=\psi_r(r,\theta)\cos\omega t\\ \Psi_\theta(r,\theta,t)&=\psi_\theta(r,\theta)\cos\omega t\\ W(r,\theta,t)&=w(r,\theta)\cos\omega t\end{aligned}\qquad(3)$$

where is ω the angular velocity in [radian/sec], yields

$$\begin{aligned}&\frac{\partial^2\psi_r}{\partial r^2}+\frac{1}{r}\frac{\partial\psi_r}{\partial r}+\frac{1-\nu}{2r^2}\frac{\partial^2\psi_r}{\partial\theta^2}-\left(\frac{1}{r^2}+\frac{\kappa^2Gh}{D}\right)\psi_r+\frac{1+\nu}{2r}\frac{\partial^2\psi_\theta}{\partial\theta\partial r}\\ &\quad-\frac{3-\nu}{2r^2}\frac{\partial\psi_\theta}{\partial\theta}-\frac{\kappa^2Gh}{D}\frac{\partial w}{\partial r}=-\omega^2\frac{\rho h^3}{12D}\psi_r,\\ &\frac{1+\nu}{2r}\frac{\partial^2\psi_r}{\partial\theta\partial r}+\frac{3-\nu}{2r^2}\frac{\partial\psi_r}{\partial\theta}+\frac{1-\nu}{2}\frac{\partial^2\psi_\theta}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2\psi_\theta}{\partial\theta^2}+\frac{1-\nu}{2r}\frac{\partial\psi_\theta}{\partial r}\\ &\quad-\left(\frac{1-\nu}{2r^2}+\frac{\kappa^2Gh}{D}\right)\psi_\theta-\frac{\kappa^2Gh}{D}\frac{1}{r}\frac{\partial w}{\partial\theta}=-\omega^2\frac{\rho h^3}{12D}\psi_\theta,\\ &\frac{\partial\psi_r}{\partial r}+\frac{\psi_r}{r}+\frac{1}{r}\frac{\partial\psi_\theta}{\partial\theta}+\frac{\partial^2w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}=-\omega^2\frac{\rho}{\kappa^2G}w\end{aligned}\qquad(4)$$

Boyd (1989) suggested several options for the pseudospectral representation of functions in the polar coordinates, e.g., the shifted Chebyshev polynomials of quadratic argument, the unshifted Chebyshev polynomials of appropriate parity, and the one-sided Jacobi polynomials. In addition, the unshifted Chebyshev polynomials-Fourier series are readily applicable when the ranges of the independent variables are chosen as $-R \leq r \leq R$ and $0 \leq \theta \leq \pi$ (Fornberg, 1996) because the geometric identity $f(-r, \theta) = f(r, \theta + \pi)$ exists in the polar coordinate system.

Because the governing equations (4) contain $1/r$ and $1/r^2$ terms it is important to avoid the apparent singularities by excluding the collocations at $r=0$. It is convenient to use the normalized variable

$$\xi = \frac{r}{R} \in [-1, 1] \tag{5}$$

where R is the radius of the plate. Then, Eg. (4) can be rewritten as follows

$$\begin{aligned}
&\frac{1}{R^2}\left(\frac{\partial^2\psi_r}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial\psi_r}{\partial\xi}+\frac{1-\nu}{2\xi^2}\frac{\partial^2\psi_r}{\partial\theta^2}\right)-\left(\frac{1}{R^2\xi^2}+\frac{\kappa^2Gh}{D}\right)\psi_r+\frac{1+\nu}{2R^2\xi}\frac{\partial^2\psi_\theta}{\partial\theta\partial\xi} \\
&\qquad -\frac{3-\nu}{2R^2\xi^2}\frac{\partial\psi_\theta}{\partial\theta}-\frac{\kappa^2Gh}{DR}\frac{\partial w}{\partial\xi}=-\omega^2\frac{\rho h^3}{12D}\psi_r, \\
&\frac{1}{R^2}\left(\frac{1+\nu}{2\xi}\frac{\partial^2\psi_r}{\partial\theta\partial\xi}+\frac{3-\nu}{2\xi^2}\frac{\partial\psi_r}{\partial\theta}+\frac{1-\nu}{2}\frac{\partial^2\psi_\theta}{\partial\xi^2}+\frac{1}{\xi^2}\frac{\partial^2\psi_\theta}{\partial\theta^2}+\frac{1-\nu}{2\xi}\frac{\partial\psi_\theta}{\partial\xi}\right) \\
&\qquad -\left(\frac{1-\nu}{2R^2\xi^2}+\frac{\kappa^2Gh}{D}\right)\psi_\theta-\frac{\kappa^2Gh}{DR\xi}\frac{\partial w}{\partial\theta}=-\omega^2\frac{\rho h^3}{12D}\psi_\theta, \\
&\frac{1}{R}\left(\frac{\partial\psi_r}{\partial\xi}+\frac{\psi_r}{\xi}+\frac{1}{\xi}\frac{\partial\psi_\theta}{\partial\theta}\right)+\frac{1}{R^2}\left(\frac{\partial^2w}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial w}{\partial\xi}+\frac{1}{\xi^2}\frac{\partial^2w}{\partial\theta^2}\right)=-\omega^2\frac{\rho}{\kappa^2G}w
\end{aligned} \tag{6}$$

The series expansions of the exact solutions ψ_r, ψ_θ and w have infinite number of terms. In this study, however, they are approximated by the partial sums. The eigenvalue problem of the circular Mindlin plates with clamped and simply supported boundary conditions was solved by assuming the dependent variables ψ_r, ψ_θ and w as follows:

$$\begin{aligned}
\psi_r(\xi, \theta) &\approx \tilde{\psi}_r(\xi, \theta) = \sum_{k=1}^{K}\sum_{l=1}^{L} a_{kl}A_k(\xi)\cos(l-1)\theta \\
\psi_\theta(\xi, \theta) &\approx \tilde{\psi}_\theta(\xi, \theta) = \sum_{k=1}^{K}\sum_{l=1}^{L} b_{kl}B_k(\xi)\sin l\theta \\
w(\xi, \theta) &\approx \tilde{w}(\xi, \theta) = \sum_{k=1}^{K}\sum_{l=1}^{L} c_{kl}C_k(\xi)\cos(l-1)\theta
\end{aligned} \tag{7}$$

where a_{kl}, b_{kl}, c_{kl} are the pseudospectral coefficients. Basis functions $A_k(\xi)$, $B_k(\xi)$ and $C_k(\xi)$ are selected such that each and every of them satisfy the boundary conditions (Lee, 2002). It was possible, for example, to satisfy the clamped boundary conditions:

$$w=0,\ \psi_r=0,\ \psi_\theta=0 \text{ at } \xi=\pm 1 \tag{8}$$

by choosing

$$\begin{aligned}
&A_{2n-1}(\xi) = T_{2n}(\xi) - T_0(\xi),\ A_{2n}(\xi) = T_{2n+1}(\xi) - T_1(\xi) \\
&B_{2n-1}(\xi) = T_{2n}(\xi) - T_0(\xi),\ B_{2n}(\xi) = T_{2n+1}(\xi) - T_1(\xi) \\
&C_{2n-1}(\xi) = T_{2n}(\xi) - T_0(\xi),\ C_{2n}(\xi) = T_{2n+1}(\xi) - T_1(\xi) \\
&\qquad (n=1, 2, \cdots)
\end{aligned} \tag{9}$$

where $T_k(\xi)$ is the Chebyshev polynomial of the first kind and of order k, as the basis functions in the radial direction.

It is, however, difficult to find proper test functions that satisfy the boundary conditions for the free edge given by

$$M_r=0,\ M_{r\theta}=0,\ Q_r=0 \text{ at } \xi=\pm 1 \tag{10}$$

The eigenvalues of the circular Mindlin plate with free boundary conditions are computed by requiring Eg. (10) be satisfied at $2L$ additional points along the domain boundary. The basis function expansions in Eg. (7) are modified as follows:

$$\begin{aligned}
\psi_r(\xi, \theta) &\approx \tilde{\psi}_r(\xi, \theta) = \sum_{k=1}^{K+2}\sum_{l=1}^{L} a_{kl}T_{k-1}(\xi)\cos(l-1)\theta \\
\psi_\theta(\xi, \theta) &\approx \tilde{\psi}_\theta(\xi, \theta) = \sum_{k=1}^{K+2}\sum_{l=1}^{L} b_{kl}T_{k-1}(\xi)\sin l\theta \\
w(\xi, \theta) &\approx \tilde{w}(\xi, \theta) = \sum_{k=1}^{K+2}\sum_{l=1}^{L} c_{kl}T_{k-1}(\xi)\cos(l-1)\theta
\end{aligned} \tag{11}$$

Substituting Eg. (11) into Eg. (6) and setting the residuals be zero at the collocation points (ξ_i, θ_j) given by

$$\begin{aligned}
\xi_i &= -\cos\frac{\pi(2i-1)}{2K}, \qquad (i=1, \cdots, K) \\
\theta_j &= \frac{j\pi}{L}, \qquad (j=1, \cdots, L)
\end{aligned} \tag{12}$$

yields

$$\begin{aligned}
&\sum_{k=1}^{K+2}\sum_{l=1}^{L}\Bigg[a_{kl}\left\{\frac{T''_{k-1}(\xi_i)}{R^2}+\frac{T'_{k-1}(\xi_i)}{R^2\xi_i}-\left(\frac{(1-\nu)(l-1)^2}{2R^2\xi_i^2}+\frac{1}{R^2\xi_i^2}+\frac{\kappa^2Gh}{D}\right)T_{k-1}(\xi_i)\right\}\cos(l-1)\theta_j \\
&\quad +b_{kl}\left\{\frac{(1+\nu)l}{2R^2\xi_j}T'_{k-1}(\xi_i)-\frac{(3-\nu)l}{2R^2\xi_i^2}T_{k-1}(\xi_i)\right\}\cos l\theta_j-c_{kl}\frac{\kappa^2Gh}{DR}T'_{k-1}(\xi_i)\cos(l-1)\theta_j\Bigg] \\
&\qquad =-\omega^2\frac{\rho h^3}{12D}\sum_{k=1}^{K+2}\sum_{l=1}^{L}a_{kl}T_{k-1}(\xi_i)\cos(l-1)\theta_j,
\end{aligned}$$

$$\begin{aligned}&\sum_{k=1}^{K+2}\sum_{l=1}^{L}\Big[-a_{kl}\frac{l-1}{R^2}\Big\{\frac{1+\nu}{2\xi_i}T'_{k-1}(\xi_i)+\frac{3-\nu}{2\xi_i^2}T_{k-1}(\xi_i)\Big\}\sin(l-1)\theta_j\\&+b_{kl}\Big\{\frac{1-\nu}{2R^2}T''_{k-1}(\xi_i)+\frac{1-\nu}{2R^2\xi_i}T'_{k-1}(\xi_i)-\Big(\frac{l^2}{R^2\xi_i^2}+\frac{1-\nu}{2R^2\xi_i^2}+\frac{\kappa^2Gh}{D}\Big)T_{k-1}(\xi_i)\Big\}\sin l\theta_j\\&+c_{kl}\frac{\kappa^2Gh(l-1)}{DR\xi_i}T_{k-1}(\xi_i)\sin(l-1)\theta_j\Big]=-\omega^2\frac{\rho h^3}{12D}\sum_{k=1}^{K+2}\sum_{l=1}^{L}b_{kl}T_{k-1}(\xi_i)\sin l\theta_j,\\&\sum_{k=1}^{K+2}\sum_{l=1}^{L}\Big[\frac{a_{kl}}{R}\Big\{T'_{k-1}(\xi_i)+\frac{T_{k-1}(\xi_i)}{\xi_i}\Big\}\cos(l-1)\theta_j+b_{kl}\frac{l}{R\xi_i}T_{k-1}(\xi_i)\cos l\theta_j\\&+\frac{c_{kl}}{R^2}\Big\{T''_{k-1}(\xi_i)+\frac{T'_{k-1}(\xi_i)}{\xi_i}-\frac{(l-1)^2}{\xi_i^2}T_{k-1}(\xi_i)\Big\}\cos(l-1)\theta_j\Big]\\&=-\omega^2\frac{\rho}{\kappa^2G}\sum_{k=1}^{K+2}\sum_{l=1}^{L}c_{kl}T_{k-1}(\xi_i)\cos(l-1)\theta_j,\\&(i=1,\cdots,K),\quad j=1,\cdots,L\end{aligned}\qquad(13)$$

where ( )′ stands for the differentiation with respect to ξ. This can be rearranged in the matrix form

$$[H]\{d\}+[H^*]\{d^*\}=\omega^2([S]\{d\}+[S^*]\{d^*\})\qquad(14)$$

where the vectors $\{d\}$ and $\{d^*\}$ are defined by

$$\begin{aligned}\{d\}&=\{a_{11}\ a_{12}\ \cdots\ a_{KL}\ b_{11}\ b_{12}\ \cdots\ b_{KL}\ c_{11}\ c_{12}\ \cdots\ c_{KL}\}^T\\\{d^*\}&=\{a_{(K+1)1}\ a_{(K+1)2}\ \cdots\ a_{(K+2)L}\ b_{(K+1)1}\ b_{(K+1)2}\ \cdots\\&\quad b_{(K+2)L}\ c_{(K+1)1}\ c_{(K+1)2}\ \cdots\ c_{(K+2)L}\}^T\end{aligned}\qquad(15)$$

The size of matrices $[H]$ and $[S]$ is $3KL\times3KL$, and that of $[H^*]$ and $[S^*]$ is $3KL\times6L$. The total number of unknowns in $\{d\}$ and $\{d^*\}$ is $3KL+6L$, whereas the number of equations in Eg. (13) is $3KL$. The remaining $6L$ equations are obtained from the boundary conditions. When Eg. (11) is substituted into Eg. (10), the boundary conditions at $\xi_b=\pm1$ are expressed as

$$\begin{aligned}&\sum_{k=1}^{K+2}\sum_{l=1}^{L}[a_{kl}\{T'_{k-1}(\xi_b)+\nu T_{k-1}(\xi_b)\}\cos(l-1)\theta_j\\&\qquad+b_{kl}\nu lT_{k-1}(\xi_b)\cos l\theta_j]=0\\&\sum_{k=1}^{K+2}\sum_{l=1}^{L}[-a_{kl}(l-1)T_{k-1}(\xi_b)\sin(l-1)\theta_j\\&\qquad+b_{kl}\{-T_{k-1}(\xi_b)+T'_{k-1}(\xi_b)\}\sin l\theta_j]=0\\&\sum_{k=1}^{K+2}\sum_{l=1}^{L}\Big\{a_{kl}T_{k-1}(\xi_b)+\frac{c_{kl}}{R}T'_{k-1}(\xi_b)\Big\}\cos(l-1)\theta_j=0\\&\qquad(j=1,\cdots,L)\end{aligned}\qquad(16)$$

The boundary conditions in Eg. (16) can be rearranged in the matrix form as

$$[U]\{d\}+[V]\{d^*\}=\{0\}\qquad(17)$$

where $\{0\}$ is a zero vector. The size of matrices $[U]$ and $[V]$ is $6L\times3KL$ and $6L\times6L$, respectively. Since $\{d^*\}$ can be expressed as

$$\{d^*\}=-[V]^{-1}[U]\{d\}\qquad(18)$$

Eg. (14) can also be reformulated as

$$([H]-[H^*][V]^{-1}[U])\{d\}=\omega^2([S]-[S^*][V]^{-1}[U])\qquad(19)$$

The eigenvalue problem (19) is solved for the estimates of the eigenvalues, and the results are given in the Table 1 and Table 3. The numbers given in Tables 1∼3 are the nondimensional parameters λ_{pq}^2 defined by

$$\lambda_{pq}^2=\omega_{pq}\frac{R^2}{\sqrt{D/\rho h}}\qquad(20)$$

2.2 Axisymmetric vibration of Mindlin plates

The equations of motion of the radially symmetric vibration of Mindlin plates are obtained by setting $M_{r\theta}=0$, $Q_\theta=0$, and $\Psi_\theta=0$ and by letting the differentiations with respect to θ be zero. The equations of motion (1) are then reduced to

$$\begin{aligned}\frac{\partial M_r}{\partial r}+\frac{M_r-M_\theta}{r}-Q_r&=\frac{\rho h^3}{12}\frac{\partial^2\Psi_r}{\partial t^2}\\\frac{\partial Q_r}{\partial r}+\frac{Q_r}{r}&=\rho h\frac{\partial^2W}{\partial t^2}\end{aligned}\qquad(21)$$

where the stress resultants M_r, M_θ and Q_r are redefined by

$$\begin{aligned}M_r&=D\Big(\frac{\partial\Psi_r}{\partial r}+\nu\frac{\Psi_r}{r}\Big)\\M_\theta&=D\Big(\nu\frac{\partial\Psi_r}{\partial r}+\frac{\Psi_r}{r}\Big)\\Q_r&=\kappa^2Gh\Big(\Psi_r+\frac{\partial W}{\partial r}\Big)\end{aligned}\qquad(22)$$

Assuming simple harmonic motions in time

$$\begin{aligned}\Psi_r(r,\ t)&=\psi_r(r)\cos\omega t\\W(r,\ t)&=w(r)\cos\omega t\end{aligned}\qquad(23)$$

and substitution Eg. (22) into Eg. (21) yields

$$\begin{aligned}\frac{d^2\psi_r}{dr^2}+\frac{1}{r}\frac{d\psi_r}{dr}-\Big(\frac{1}{r^2}+\frac{\kappa^2Gh}{D}\Big)\psi_r-\frac{\kappa^2Gh}{D}\frac{dw}{dr}&=-\omega^2\frac{\rho h^3}{12D}\psi_r\\\frac{d\psi_r}{dr}+\frac{\psi_r}{r}+\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}&=-\omega^2\frac{\rho}{\kappa^2G}w\end{aligned}\qquad(24)$$

The boundary conditions for free edge are given by

$$M_r=0,\ Q_r=0 \text{ at } r=R \tag{25}$$

whereas the conditions at the center of the plate for the axisymmetric modes are described as

$$\psi_r=0,\ Q_r=0 \text{ at } r=0 \tag{26}$$

Fornberg (1996) recommended that it is advantageous for the axisymmetric analysis using the pseudospectral method to extend the range of the independent variable to $[-1, 1]$ and then to use the symmetry and antisymmetry to reduce the actual calculations to within $[0, 1]$. For axisymmetric vibration modes, r is normalized as

Table 1 Convergence test of the nondimensionalized frequency parameters λ_{pq}^2 of the non-axisymmetri c vibration modes ($\nu=0.3$, $h/R=0.05$)

p	q	$K\times L$						
		12×6	18×12	18×18	24×12	30×12	36×12	40×12
1	1	20.086	20.399	20.399	20.176	20.328	20.215	20.220
	2	57.287	58.111	58.111	58.292	58.166	58.252	58.245
	3	111.01	113.20	113.20	112.92	113.03	112.95	112.96
	4	200.49	184.22	184.22	182.35	182.23	182.30	182.29
2	0	5.3827	5.2825	5.2825	5.3535	5.3132	5.3401	5.3380
	1	34.460	34.652	34.652	34.571	34.616	34.587	34.590
	2	80.459	81.108	81.108	81.229	81.162	81.201	81.197
	3	158.55	143.92	143.92	143.50	143.59	143.54	143.54
	4	—	224.36	224.36	219.42	219.26	219.33	219.32
3	0	12.238	12.213	12.213	12.308	12.312	12.310	12.310
	1	50.750	51.521	51.521	51.541	51.535	51.538	51.537
	2	105.79	106.45	106.45	106.40	106.41	106.40	106.40
	3	179.94	176.33	176.33	175.94	175.92	175.92	175.92
	4	—	256.05	256.05	257.67	257.66	257.65	257.65
4	0	21.390	21.485	21.485	21.491	21.492	21.492	21.492
	1	70.874	70.790	70.790	70.800	70.799	70.800	70.800
	2	134.87	132.98	132.98	133.62	133.63	133.63	133.63
	3	209.69	210.52	210.52	209.88	209.86	209.86	209.86
	4	—	308.92	308.92	297.17	297.17	297.16	297.16
5	0	32.620	32.753	32.753	32.766	32.766	32.766	32.766
	1	92.147	92.171	92.171	92.171	92.172	92.172	92.172
	2	167.35	162.66	162.66	162.64	162.64	162.64	162.64
	3	246.41	245.03	245.03	245.19	245.20	245.20	245.20
	4	—	347.94	347.94	337.89	337.67	337.67	337.67

Table 2 Convergence test of the nondimensionalized frequency parameters ξ_{pq}^2 of the axisymmetric vibration modes ($\nu=0.3$, $h/R=0.05$)

p	q	$K=3$	$K=6$	$K=9$	$K=12$	$K=15$	$K=18$	$K=30$
0	1	8.9721	8.9686	8.9686	8.9686	8.9686	8.9686	8.9686
	2	33.523	37.788	37.787	37.787	37.787	37.787	37.787
	3	167.06	84.382	84.443	84.443	84.443	84.443	84.443
	4	—	135.15	146.77	146.76	146.76	146.76	146.76
	5	—	249.99	222.03	222.38	222.38	222.38	222.38
	6	—	—	288.69	309.04	308.98	308.98	308.98
	7	—	—	424.34	402.77	404.45	404.44	404.44
	8	—	—	884.33	480.68	507.07	506.96	506.96
	9	—	—	—	628.81	611.67	615.02	615.01
	10	—	—	—	991.06	695.31	727.58	727.37

$$\xi=\frac{r}{R}\in[0,\ 1] \tag{27}$$

and the equations of motion (24) can be rewritten as

$$\begin{gathered}\frac{\psi_r''}{R^2}+\frac{\psi_r'}{R^2\xi}-\left(\frac{1}{R^2\xi^2}+\frac{\kappa^2Gh}{D}\right)\psi_r-\frac{\kappa^2Gh}{RD}w'=-\omega^2\frac{\rho h^3}{12D}\psi_r\\ \frac{\psi_r'}{R}+\frac{\psi_r}{R\xi}+\frac{w''}{R^2}+\frac{w'}{R^2\xi}=-\omega^2\frac{\rho}{\kappa^2G}w\end{gathered} \tag{28}$$

Also the boundary conditions (25) ~ (26) can be rewritten as

$$\begin{cases}\psi_r=0, \qquad w'=0 \quad \text{at } \xi=0\\ \psi_r'+\nu\psi_r=0,\ \psi_r+\dfrac{\omega'}{R}=0 \ \text{at } \xi=1\end{cases} \tag{29}$$

The dependent variables $\psi_r(\xi)$ and $w(\xi)$ can be rearranged and approximated as follows:

$$\begin{gathered}\psi_r(\xi)\approx\tilde{\psi}_r(\xi)=\sum_{k=1}^{K}a_k\left\{T_{2k+1}(\xi)-T_1(\xi)-\frac{4k(k+1)}{1+\nu}\right\}\\ \psi_r(\xi)+\frac{w'(\xi)}{R}\approx\tilde{\psi}_r(\xi)+\frac{\tilde{w}'(\xi)}{R}=\sum_{k=1}^{K}b_k\{T_{2k+1}(\xi)-T_1(\xi)\}\end{gathered} \tag{30}$$

so that the boundary conditions (29) are satisfied. Also the governing equations (28) can be reformulated as

$$\begin{gathered}\frac{\psi_r''}{R^2}+\frac{\psi_r'}{R^2\xi}-\frac{\psi_r}{R^2\xi^2}-\frac{\kappa^2Gh}{D}\left(\psi_r+\frac{\omega'}{R}\right)=-\omega^2\frac{\rho h^3}{12D}\psi_r\\ \frac{1}{R^2}\left(\psi_r+\frac{w'}{R}\right)''+\frac{1}{R^2\xi}\left(\psi_r+\frac{w'}{R}\right)'-\frac{1}{R^2\xi^2}\left(\psi_r+\frac{w'}{R}\right)\\ =-\omega^2\frac{\rho}{\kappa^2G}\left\{\left(\psi_r+\frac{w'}{R}\right)-\psi_r\right\}\end{gathered} \tag{31}$$

Table 3 Nondimensionalized frequency parameters λ_{pq}^2 of the circular Mindlin plates ($\nu=0.3$, $K=12$ for $p=0$ and $K\times L=40\times12$ for $p\geq1$)

p	q	classical theory	h/R 0.005	0.01	0.02	0.05	0.1	0.15	0.2
0	1	9.084	9.0028	9.0017	8.9976	8.9686	8.8679	8.7095	8.5051
	2	38.55	38.436	38.416	38.335	37.787	36.041	33.674	31.111
	3	87.80	87.715	87.609	87.189	84.443	76.676	67.827	59.645
	4		156.70	156.37	155.04	146.76	126.27	106.40	90.059
1	1	20.52	20.33	20.33	20.32	20.22	19.70	18.92	17.98
	2	59.86	59.93	59.87	59.63	58.25	54.26	49.34	44.43
	3	119.0	118.8	118.6	117.8	113.0	99.94	86.24	74.34
	4		197.8	197.3	195.1	182.3	152.7	126.0	105.0
2	0	5.253	5.376	5.374	5.367	5.340	5.281	5.206	5.115
	1	35.25	35.22	35.19	35.11	34.59	33.03	30.94	28.67
	2	83.9	84.35	84.24	83.82	81.20	73.88	65.51	57.72
	3	154.0	153.1	152.8	151.5	143.5	123.8	104.5	88.53
	4		241.8	241.0	237.8	219.3	179.3	145.2	119.3
3	0	12.23	12.427	12.421	12.402	12.310	12.064	11.722	11.314
	1	52.91	52.969	52.906	52.701	51.537	48.227	44.116	39.955
	2	111.3	111.85	111.65	110.91	106.40	94.531	81.930	70.863
	3	192.1	190.50	189.97	187.95	175.92	147.99	122.49	102.27
	4		288.80	287.61	283.14	257.65	205.72	163.95	132.87
4	0		21.813	21.796	21.745	21.492	20.801	19.872	18.816
	1		73.468	73.350	72.969	70.800	64.790	58.044	51.545
	2		142.28	141.95	140.75	133.63	115.96	98.446	83.801
	3		230.73	229.94	226.98	209.86	172.41	140.26	115.57
	4		338.81	337.18	331.03	297.16	232.09	182.25	145.66
5	0		33.456	33.420	33.313	32.766	31.270	29.334	27.255
	1		96.636	96.441	95.800	92.172	82.722	72.464	63.253
	2		175.50	175.00	173.19	162.64	137.95	114.96	96.513
	3		273.81	272.69	268.55	245.20	197.06	157.77	128.42
	4		391.70	389.51	381.33	337.67	258.33	200.14	157.54

The expansions (30) are substituted into Eg. (31) and are collocated at the internal collocation points ξ_i redefined by

$$\xi_i=\cos\frac{\pi(2i-1)}{4K}, \qquad (i=1, \cdots, K) \tag{32}$$

Then, we finally have

$$\begin{aligned}&\sum_{k=1}^{K}\left[a_k\left\{\frac{T''_{2k+1}(\xi_i)}{R^2}+\frac{T'_{2k+1}(\xi_i)-1-\frac{4k(k+1)}{1+\nu}}{R^2\xi_i}-\frac{T_{2k+1}(\xi_i)-T_1(\xi_i)-\frac{4k(k+1)}{1+\nu}\xi_i}{R^2\xi^2}\right\}\right.\\&\left.-b_k\frac{k^2Gh}{D}\{T_{2k+1}(\xi_i)-T_1(\xi_i)\}\right]=-\omega^2\frac{\rho h^3}{12D}\sum_{k=1}^{K}a_k\left\{T_{2k+1}(\xi_i)-T_1(\xi_i)-\frac{4k(k+1)}{1+\nu}\xi_i\right\},\\&\sum_{k=1}^{K}b_k\left\{\frac{T''_{2k+1}(\xi_i)}{R^2}+\frac{T'_{2k+1}(\xi_i)-1}{R^2\xi}-\frac{T_{2k+1}(\xi_i)-T_1(\xi_i)}{R^2\xi^2}\right\}\\&=-\omega^2\frac{\rho}{k^2G}\sum_{k=1}^{K}\left[b_k\{T_{2k+1}(\xi_i)-T_1(\xi_i)\}-a_k\left\{T_{2k+1}(\xi_i)-T_1(\xi_i)-\frac{4k(k+1)}{1+\nu}\xi_i\right\}\right],\\&(i=1, \cdots, K)\end{aligned} \tag{33}$$

The total number of equations matches the number of constants $a_1, \cdots, a_K$, $b_1, \cdots, b_K$ in Eg. (33), and the eigenvalue problem (33) is solved for the estimates of the eigenvalues for the axisymmetric vibration modes.

3. Numerical Results and Discussions

A preliminary run for the convergence check of Eg. (19) is carried out for the thickness-to-radius ratio $h/R=0.05$. Because the eigenvalues for the axisymmetric vibration modes ($p=0$) converge too slowly the computed frequency parameters λ^2_{pq} for $p\geq1$ are given in Table 1, and the convergence of eigenvalues corresponding to the axisymmetric modes is treated separately. The Poisson ratio ν is 0.3 throughout the paper. Table 1 shows that the eigenvalues for the grid resolution $K\times L=18\times12$ are identical to those for $K\times L=18\times18$, which indicates that $L=12$ is sufficient for the eigenvalues for $1\leq p\leq5$ to converge to five significant digits in the circumferential direction. It shows that the convergence of the eigenvalues is slower in general when compared with that of clamped boundary conditions (Lee, 2002). It is also worthwhile to note that the convergence of eigenvalues of the vibration modes for $p\geq4$ is achieved to five digits for $K\times L=30\times12$, while the grid resolution of $K\times L=36\times12$ is required for the convergence of eigenvalues for $p=3$ to five digits. With the spatial resolution of $K\times L=40\times12$, the convergence is achieved to four digits for the vibration modes for $p=1$ and $p=2$. It turns out that the eigenvalues of lower vibration modes converge more slowly than those of higher vibration modes within the range of $p\leq5$ and $q\leq4$, which is opposed to the results of former study (Lee, 2002) where the eigenvalues of lower modes converged faster for the clamped boundary conditions. It is conjectured that this odd behavior is attributable to the way the boundary condition is treated as given in Egs. (16) ~ (19).

Another convergence test of the eigenvalues for the axisymmetric vibration modes is carried out, which is based on the axisymmetric model of Eg. (33), and the results are given in Table 2. Table 2 shows that the eigenvalues of the first four modes ($p=0$, $1\leq q\leq4$) for the thickness-to-radius ratio $h/R=0.05$ converge for the grid resolution $K=9$ to five significant digits and the first eight modes ($p=0$, $1\leq q\leq8$) for $K=18$. It also clearly demonstrates that the eigenvalues of the lower vibration modes converge faster than those of higher vibration modes. Fast convergence of eigenvalues is achieved because the test functions in Eg. (30) satisfying the boundary conditions are used in the computation.

The nondimensionalized frequency parameters λ^2_{pq} for the free vibration of Mindlin plates with free edge are given in Table 3 for different thickness-to-radius ratios. The eigenvalues of the axisymmetric modes ($p=0$) are computed from Eg. (33). The grid resolutions used in Table 3 are $K=12$ for the axisymmetric modes and $K\times L=40\times12$ for the non-axisymmetric modes ($p\geq1$), respectively. Eigenvalues based on the classical theory (Blebins, 1979) are given in Table 3 for comparison. Table 3 shows that the computed eigenvalues are in good agreement with those of the classical theory when h/R is very small, however, they deviate considerably as h/R becomes larger.

4. Conclusions

A Chebyshev-Fourier pseudospectral method has been applied to the free vibration analysis of the circular Mindlin plate with free boundary conditions. The formulation was straightforward and efficient for writing a code for computation. Numerical examples were provided for various thickness-to-radius ratios. It is shown, however, that the eigenvalues of lower vibration modes tend to convergence more slowly than those of higher vibration modes, even though the eigenvalues converge for sufficiently fine pseudospectral grid resolutions. The eigenvalues of the axisymmetric modes were computed separately. The results from this study agree with those of the classical plate theory when the thickness-to-radius ratio is small but present some quantitative differences of natural frequencies for thicker plates.

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