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ESTIMATION OF SCALE PARAMETER AND $P(Y < X)$ FROM RAYLEIGH DISTRIBUTION

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Abstract

We consider the estimation problem for the scale parameter of the Rayleigh distribution using weighted balanced loss function (WBLF) which reflects both goodness of fit and precision. Under WBLF, we obtain the optimal estimator which creates a kind of balance between Bayesian and non-Bayesian estimation. We also deal with the estimation of $R = P(Y < X)$ when Y and X are two independent but not identically distributed Rayleigh distribution under squared error loss function.

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1. Introduction

The Rayleigh distribution is well known as an important model in reliability model. Then the probability density function, conditional on parameter θ, is given by

$$f(x|\theta) = \frac{2}{\theta^2} x e^{-x^2/\theta^2}. \tag{1.1}$$

For a situation where failure rate is linear and an increasing function of time, the Rayleigh distribution which is a special case of Weibull family would be an ideal choice.

In this paper, estimation procedure for the scale parameter θ using a weighted balanced loss function (WBLF) is considered. The WBLF is an extension of the balanced loss function (BLF) for a scalar mean introduced by Zellner (1994) which reflects both goodness of fit and precision of the estimator of θ. Use of a goodness

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of fit criterion such as the sum of squared residuals in a regression problems leads to an estimator which gives good fit and is unbiased. However, as commented by Zellner (1994), it may not be as precise as an unbiased estimator. Thus, there is a need to provide a framework which combines the goodness of fit and precision of estimation formally. For the problem of estimating the mean vector of a multivariate normal distribution, Chung and Kim (1997) showed that the usual estimator $\bar{X}$ is admissible when $p \leq 2$ and otherwise it is inadmissible. Chung *et al.* (1999) also obtained a new class of minimax estimators of multivariate normal means under BLF.

Rodrigues and Zellner (1994) discussed a WBLF for estimation of exponential mean time to failure θ. The WBLF was defined as follows:

$$L_B(\hat{\theta}, \theta) = w\frac{\sum_{i=1}^{n}(X_i - E_{\hat{\theta}}(X))^2}{n\mathrm{Var}_{\hat{\theta}}(X)} + (1-w)\frac{(\hat{\theta}-\theta)^2}{\mathrm{Var}_{\hat{\theta}}(X)} \tag{1.2}$$

for some $0 \leq w \leq 1$, where $\hat{\theta}$ is an estimator of θ. If $\mathrm{Var}_{\hat{\theta}}(X) = 1$ and $E_{\hat{\theta}}(X) = \hat{\theta}$, we obtain the BLF introduced by Zellner (1994). The first term on the right hand side of (1.2) assesses goodness of fit while the second term represents precision of estimation.

We also deal with the problem of estimating $R = P(Y < X)$ using Lindley's approximation when Y and X are independent but not identically distributed according to a Rayleigh distribution. Assume that X is the strength of a component which is subject to a stress Y. This problem arises in the context of mechanical reliability of a system and $P(Y < X)$ is a chosen measure of system performance. The system fails if and only if at any time the applied stress is greater than its strength. Related problems have been widely presented in the literature. Church and Harris (1970) derived the maximum likelihood estimate for $R = P(Y < X)$ in the normal case. Enis and Geisser (1971) suggested a Bayesian approach for estimating R. Weerahandi and Johnson (1992) considered a Bayesian analysis in a stress strength model when X and Y are normally distributed.

Let $\mathbf{X}=(X_1, \ldots, X_n)$ be a random sample from the Rayleigh distribution with the probability density function (1.1). Then the likelihood function of θ on $\mathbf{X}$ is given by

$$L(\theta|\mathbf{X}) = \frac{2^n}{\theta^{2n}}\left(\prod_{i=1}^{n} x_i\right)\exp\left(-\frac{\sum_{i=1}^{n} x_i^2}{\theta^2}\right), \tag{1.3}$$

where $E(X) = \theta\sqrt{\pi}/2$ and $\mathrm{Var}(X) = \theta^2(1 - \pi/4)$. A non-informative prior for θ is used in our estimation problem.

In Section 2, an optimal estimator relative to WBLF is derived and it will be seen that Bayesian and non-Bayesian estimators are dominated in terms of WBLF by the optimal estimator relative to WBLF.

In Section 3, we consider the problem of estimating $R = P(Y < X)$ under squared error loss function (SEL). The maximum likelihood and Bayes estimators are derived. In Monte Carlo simulation, we compare the MLE and Bayes estimators in terms of the estimated risks.

2. ESTIMATION OF SCALE PARAMETER RELATIVE TO WBLF

Let $\mathbf{X}=(X_1, \ldots, X_n)$ be a random sample from a Rayleigh distribution with *pdf* $f(x|\theta)$ given (1.1). From (1.1), we have $E_{\hat{\theta}}(X) = \hat{\theta}\sqrt{\pi}/2$ and $\mathrm{Var}_{\hat{\theta}}(X) = \hat{\theta}^2(1 - \pi/4)$. The WBLF in (1.2) can be re-expressed as

$$L_B(\hat{\theta}, \theta) = \frac{w}{n} \frac{\sum_{i=1}^{n}(x_i - \hat{\theta}\sqrt{\pi}/2)^2}{(1 - \pi/4)\hat{\theta}^2} + (1 - w)\frac{(\hat{\theta} - \theta)^2}{(1 - \pi/4)\hat{\theta}^2}. \tag{2.1}$$

We will show how sensitive estimation results are to the choice of the value of w. Usually, $w = 0$ or $w = 1$ is employed.

Let $\hat{\theta}^*$ be the optimal Bayes estimator of θ. It minimizes the posterior expected weighted balance loss, that is

$$E\{L_B(\hat{\theta}^*, \theta)|\mathbf{X}\} = \min_{\hat{\theta}} E\{L_B(\hat{\theta}, \theta)|\mathbf{X}\}. \tag{2.2}$$

The following theorem gives the optimal estimator and the minimal posterior expected loss.

THEOREM 2.1. *Under the model (1.1) and for any posterior, $\pi(\theta|x)$, we have*

$$\text{(a)} \quad \hat{\theta}^* = \frac{w\bar{X}\hat{\theta}_1 + (1 - w)\hat{\theta}_2\bar{\theta}}{w\bar{X}\sqrt{\pi}/2 + (1 - w)\bar{\theta}}, \tag{2.3}$$

$$\text{(b)} \quad E\{L_B(\hat{\theta}^*, \theta)|\mathbf{X}\} = \frac{1 - w(1 - \pi/4)}{1 - \pi/4} - \frac{1}{1 - \pi/4}\left\{\frac{w\bar{x}\sqrt{\pi}/2 + (1 - w)\bar{\theta}}{\hat{\theta}^*}\right\}, \tag{2.4}$$

where $\hat{\theta}_1 = S^2/\bar{X} + \bar{X}$, $S^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2/n$, $\hat{\theta}_2 = E(\theta^2|X)/\bar{\theta}$ and $\bar{\theta}$ denotes a posterior mean.

PROOF. Using the posterior probability function for θ, the posterior expected loss is

$$L_B(\hat{\theta},\theta) = \frac{wS^2 + w(\hat{\theta}\sqrt{\pi}/2 - \bar{X})^2 + (1-w)(\hat{\theta}-\theta)^2}{(1-\pi/4)\hat{\theta}^2}. \tag{2.5}$$

Solving the equation $\partial\{EL_B(\hat{\theta},\theta)\}/\partial\hat{\theta} = 0$ for $\hat{\theta}$, the result (2.3) will be obtained. The second part of theorem is followed trivially from (2.3) and (2.5). □

Since the usual non-informative prior for θ is proportional to $1/\theta$, the posterior distribution, its mean and $\hat{\theta}_2$ are given by respectively

$$\pi(\theta|\mathbf{x}) = \frac{2(S_1^2)^n}{\Gamma(n)}\left(\frac{1}{\theta}\right)^{(2n+1)} \exp\left(-\frac{S_1^2}{\theta^2}\right),$$

$$\bar{\theta} = \frac{\Gamma(n-1/2)}{\Gamma(n)}S_1,$$

and

$$\hat{\theta}_2 = \frac{\Gamma(n-1)}{\Gamma(n-1/2)}S_1,$$

where $S_1^2 = \sum_{i=1}^n X_i^2$.

Using Theorem 2.1, we have the optimal estimator $\hat{\theta}^*$ and the minimal posterior expected loss as follows:

$$\hat{\theta}^* = \Sigma\hat{\theta}_1^* + (1-\Sigma)\hat{\theta}_2^*, \tag{2.6}$$

and

$$\begin{aligned} E\{L_B(\hat{\theta}^*,\theta)|\mathbf{X}\} &= \frac{1-w(1-\pi/4)}{1-\pi/4} \\ &\quad -\frac{1}{1-\pi/4}\left\{\frac{w\sqrt{\pi}}{2} + (1-w)\frac{\Gamma(n-1/2)}{\Gamma(n)}\frac{S_1}{\bar{X}}\right\}^2 \\ &\quad \times\left\{\frac{w}{1-\lambda} + (1-w)\frac{\Gamma(n-1)}{\Gamma(n)}\frac{S_1^2}{\bar{X}^2}\right\}^{-1}, \end{aligned} \tag{2.7}$$

where

$$\hat{\theta}_1^* = \frac{2\hat{\theta}_1}{\sqrt{\pi}},\quad \hat{\theta}_2^* = \hat{\theta}_2,\quad \Sigma = \frac{w\sqrt{\pi}}{2}\left\{\frac{w\sqrt{\pi}}{2} + (1-w)\frac{\Gamma(n-1/2)}{\Gamma(n)}\frac{S_1}{\bar{X}}\right\}^{-1},$$

$$\lambda = \frac{\sum_{i=1}^n (X_i-\bar{X})^2}{\sum_{i=1}^n X_i^2}.$$

It is obvious from (2.6) that $\hat{\theta}^*$ is a combined estimator of non-Bayesian and Bayesian estimators when $w = 1$ and $w = 0$ respectively. If $w = 1$ in (2.6), $\hat{\theta}^* = \hat{\theta}_1^* = 2\hat{\theta}_1/\sqrt{\pi}$ while if $w = 0$, $\hat{\theta}^* = \hat{\theta}_2^* = \hat{\theta}_2$.

It is of interest to compare (2.7) with the posterior expected losses associated with $\hat{\theta}_1^*$ and $\hat{\theta}_2^*$. The difference between the posterior expected losses corresponding to $\hat{\theta}_1^*$ and $\hat{\theta}^*$ is

$$\begin{aligned}\Delta &= E\{L_B(\hat{\theta}_1^*,\theta)|\mathbf{X}\} - E\{L_B(\hat{\theta}^*,\theta)|\mathbf{X}\}\\ &= \frac{1}{1-\pi/4}\left\{\frac{w\sqrt{\pi}}{2} + (1-w)\frac{\Gamma(n-1/2)}{\Gamma(n)}\frac{S_1}{\bar{X}}\right\}^2\\ &\quad\times\left\{\frac{w}{1-\lambda} + (1-w)\frac{\Gamma(n-1)}{\Gamma(n)}\frac{S_1^2}{\bar{X}^2}\right\}^{-1}\left(1-\frac{\sqrt{\pi}\hat{\theta}^*}{2\hat{\theta}_1}\right)^2\\ &= \frac{1}{u}(1-\Sigma)^2\left(1-\frac{\sqrt{\pi}\hat{\theta}_2}{2\hat{\theta}_1}\right)^2\left(\frac{w\sqrt{\pi}}{2\Sigma}\right)^2, \end{aligned} \tag{2.8}$$

where

$$u = \left(1-\frac{\pi}{4}\right)\left\{\frac{w}{1-\lambda} + (1-w)\frac{\Gamma(n-1)S_1^2}{\Gamma(n)\bar{X}^2}\right\}.$$

From (2.8), it is seen that the inflation of the posterior expected loss associated with use of $\hat{\theta}_1^*$ depends on n, w and $\hat{\theta}_1/\hat{\theta}_2$. It is similar to the case of $\hat{\theta}_2^*$. Further,

$$\frac{\Delta}{E\{L_B(\hat{\theta}^*,\theta)|\mathbf{X}\}} = (1-\Sigma)^2 Z^2\left(\frac{w\sqrt{\pi}}{2\Sigma}\right)^2, \tag{2.9}$$

where

$$Z^2 = \left(1-\frac{\sqrt{\pi}\hat{\theta}_2}{2\hat{\theta}_1}\right)^2\left\{\frac{1-w(1-\pi/4)}{1-\pi/4}u - \left(\frac{w\sqrt{\pi}}{2\Sigma}\right)^2\right\}^{-1}.$$

For the given data $\mathbf{X} = (4.2995, 2.0688, 0.9556, 1.6097, 1.2221)$, (2.9) is equal to 2.75% when $w = 1/2$. That is, under these conditions, the expected loss is inflated by 2.75% if one uses $\hat{\theta}_1^*$ instead of the optimal estimate $\hat{\theta}^*$ given in (2.6). Also, when $Z^2 = 0$, $\hat{\theta}_1^* = \hat{\theta}_2^* = \hat{\theta}^*$, and if $w = 1$, $\hat{\theta}_1^* = \hat{\theta}^*$. Thus the relative loss is equal to zero. It is also the case with $\hat{\theta}_2^*$.

3. STRESS-STRENGTH MODEL UNDER NONINFORMATIVE PRIOR

Let X be a random variable whose *pdf* is a Rayleigh distribution with parameter θ and Y has another Rayleigh distribution with parameter ϕ where X and

Y are independent. We can see that

$$R = P(Y < X) = \frac{\theta^2}{\theta^2 + \phi^2}. \tag{3.1}$$

3.1. Maximum likelihood estimation of R

Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a random sample of size of n drawn from a Rayleigh distribution with parameter θ. From (1.3), the log-likelihood function is

$$l(\theta|\mathbf{x}) \propto -2n \log\theta - \frac{S_1^2}{\theta^2},$$

where $S_1^2 = \sum_{i=1}^n X_i^2$. The maximum likelihood estimate (MLE) of θ is obtained as in the following form:

$$\hat{\theta}_M = \sqrt{\frac{S_1^2}{n}}.$$

If $\mathbf{Y} = (Y_1, \ldots, Y_m)$ is a random sample of size of m from a Rayleigh distribution with parameter ϕ, the MLE of ϕ is

$$\hat{\phi}_M = \sqrt{\frac{S_2^2}{m}},$$

where $S_2^2 = \sum_{i=1}^m Y_i^2$. Therefore, the MLE of R can be obtained as in the following form:

$$\hat{R}_M = \frac{\hat{\theta}_M^2}{\hat{\theta}_M^2 + \hat{\phi}_M^2}. \tag{3.2}$$

3.2. Bayes estimation of R

To obtain the joint posterior distribution of (θ, ϕ) given data, we need prior distributions of (θ, ϕ). Here, we use a Jeffreys's prior, *i.e.* $\pi(\theta, \phi) \propto 1/\theta\phi$. A squared error loss (SEL) function is used. By using the likelihood function and Jeffreys's prior, we can obtain the joint posterior density of (θ, ϕ) given $\mathbf{x}$ and $\mathbf{y}$ in the form of

$$\pi(\theta, \phi|\mathbf{x}, \mathbf{y}) = \frac{4 S_1^{2n} S_2^{2m}}{\Gamma(n)\Gamma(m)} \left(\frac{1}{\theta}\right)^{2n+1} \left(\frac{1}{\phi}\right)^{2m+1} \exp\left\{-\left(\frac{S_1^2}{\theta^2} + \frac{S_2^2}{\phi^2}\right)\right\}, \tag{3.3}$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_m)$.

After the usual transformations of random variables, the posterior density function of R is

$$\pi(r|\mathbf{x},\mathbf{y}) = \frac{\Gamma(n+m)S_1^{2n}S_2^{2m}}{\Gamma(n)\Gamma(m)\left\{S_1^2/r + S_2^2/(1-r)\right\}^{n+m}}\left(\frac{1}{r}\right)^{n+1}\left(\frac{1}{1-r}\right)^{m+1}, \quad 0 < r < 1. \tag{3.4}$$

Under the SEL, the Bayes estimate of R is the posterior expected value of R. This expected value contains an integral which is usually not obtainable in a simple form. We use Lindley's (1980) approximation to compute the Bayes estimate of R.

Lindley (1980) developed approximation procedures for the evaluation of the ratio of the two integrals in the form

$$\frac{\int_\Theta W(\theta)e^{\log L(\theta)}d\theta}{\int_\Theta \pi(\theta)e^{\log L(\theta)}d\theta}, \tag{3.5}$$

where $L(\theta)$ is the likelihood function, $W(\theta)$ and $\pi(\theta)$ are arbitrary functions of θ. Suppose that $\pi(\theta)$ is the prior density of θ and $W(\theta) = G(\theta)\pi(\theta)$. From (3.5),

$$E(G(\theta)|\mathbf{x}) = \frac{\int G(\theta)e^{Q(\theta)}d\theta}{\int e^{Q(\theta)}d\theta}, \tag{3.6}$$

which is the Bayes estimator of $G(\theta)$ under SEL where $Q(\theta) = \log L(\theta) + \log \pi(\theta)$ is the logarithm of the posterior distribution of θ except the normalizing constant. Lindley (1980) estimated both integrals by expanding terms about $\tilde{\theta}$, posterior mode, obtaining an asymptotically second-order approximation. This methods requires third derivatives or derivative-free techniques.

Let $\theta_1 = \theta$ and $\theta_2 = \phi$. For the two parameter case $\theta = (\theta_1, \theta_2)$, Lindley's approximation leads to

$$\begin{aligned}
\hat{G}_B &= E\{G(\theta_1,\theta_2)|\mathbf{x},\mathbf{y}\} \\
&= G(\theta_1,\theta_2) + \frac{1}{2}\left\{\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\partial^2 G}{\partial\theta_i\partial\theta_j}\tau_{ij} + \frac{\partial^3 Q}{\partial^3\theta_1}\left(\frac{\partial G}{\partial\theta_1}\tau_{11} + \frac{\partial G}{\partial\theta_2}\tau_{12}\right)\tau_{11}\right. \\
&\quad + \frac{\partial^3 Q}{\partial^2\theta_1\partial\theta_2}\left(3\frac{\partial G}{\partial\theta_1}\tau_{11}\tau_{12} + \frac{\partial G}{\partial\theta_2}(\tau_{11}\tau_{22} + 2\tau_{12}^2)\right) \\
&\quad + \frac{\partial^3 Q}{\partial\theta_1\partial^2\theta_2}\left(3\frac{\partial G}{\partial\theta_2}\tau_{22}\tau_{21} + \frac{\partial G}{\partial\theta_1}(\tau_{11}\tau_{22} + 2\tau_{21}^2)\right) \\
&\quad \left. + \frac{\partial^3 Q}{\partial^3\theta_2}\left(\frac{\partial G}{\partial\theta_2}\tau_{22} + \frac{\partial G}{\partial\theta_1}\tau_{21}\right)\tau_{22}\right\},
\end{aligned} \tag{3.7}$$

where τ_{ij} is the $(i,j)^{th}$ element in the inverse of the matrix $Q^* = -\partial^2 Q/\partial\theta_i\partial\theta_j$, $i,j = 1,2$.

In our case, Q is given by

$$Q \propto -(2n+1)\log\theta - (2m+1)\log\phi - \frac{S_1^2}{\theta^2} - \frac{S_2^2}{\phi^2}.$$

The joint posterior mode, denoted by $(\tilde{\theta},\tilde{\phi})$, is given by

$$\tilde{\theta} = \sqrt{\frac{S_1^2}{n+1/2}} \quad \text{and} \quad \tilde{\phi} = \sqrt{\frac{S_2^2}{m+1/2}}.$$

For $G(\theta_1,\theta_2) = G(\theta,\phi) = \theta^2/(\theta^2+\phi^2)$ and after some calculations,

$$\tau_{11} = -\frac{\theta^4}{(2n+1)\theta^2 - 6S_1^2}, \quad \tau_{22} = -\frac{\phi^4}{(2m+1)\phi^2 - 6S_2^2}, \quad \tau_{12} = \tau_{21} = 0,$$

$$\frac{\partial^3 Q}{\partial\theta^3} = \frac{-2\theta^2(2n+1) + 24S_1^2}{\theta^5}, \quad \frac{\partial^3 Q}{\partial\phi^3} = \frac{-2\phi^2(2m+1) + 24S_2^2}{\phi^5},$$

$$\frac{\partial^3 Q}{\partial\theta^2\partial\phi} = \frac{\partial^3 Q}{\partial\theta\partial\phi^2} = 0,$$

$$\frac{\partial G}{\partial\theta} = \frac{2\phi^2\theta}{(\theta^2+\phi^2)^2}, \quad \frac{\partial^2 G}{\partial\theta\partial\theta} = \frac{2\phi^2(\phi^2 - 3\theta^2)}{(\theta^2+\phi^2)^3},$$

$$\frac{\partial G}{\partial\phi} = \frac{-2\phi\theta^2}{(\theta^2+\phi^2)^2}, \quad \frac{\partial^2 G}{\partial\phi\partial\phi} = \frac{2\theta^2(3\phi^2 - \theta^2)}{(\theta^2+\phi^2)^3}.$$

Therefore, substituting the above values in (3.7), the Bayes estimate of R is given by

$$\hat{R}_B^* = \tilde{R}\left[1 + \frac{\tilde{R}}{4n+2}(6-4\tilde{R})\left(\frac{1}{\tilde{R}} - 1\right) + \frac{\tilde{R}}{4m+2}\left\{(1-\tilde{R})\left(\frac{3}{\tilde{R}} - 4\right) - 5\left(\frac{1}{\tilde{R}} - 1\right)\right\}\right], \tag{3.8}$$

where $\tilde{R}$ is evaluated at $(\tilde{\theta},\tilde{\phi})$, the posterior mode.

To compare the MLE and the Bayes estimator of R, we compute the risk functions using Monte Carlo simulation. Let W be an exponential distribution with parameter θ and $X = \sqrt{\theta W}$. Then, X has a Rayleigh distribution with

TABLE 1 *Estimated risks of the MLE and the Bayes estimators of R for different sample sizes (n, m) based on 1,000 repetitions*

(n, m)	$\widehat{\text{Risk}}_{ML}$	$\widehat{\text{Risk}}_{B}$
(10,10)	1.9664778E-03	1.5179471E-03
(20,20)	9.8955561E-04	8.5120852E-04
(40,40)	4.3916699E-04	4.0648167E-04
(60,60)	2.8177138E-04	2.6663390E-04

parameter θ. We can generate X and Y from Rayleigh distributions with parameter θ and ϕ, respectively. The risks under SEL of the two estimators were estimated by

$$\widehat{\text{Risk}} = \frac{1}{K}\sum_{k=1}^{K}(\hat{R} - R)^2,$$

where K is a number of replication.

The estimated risks of these estimators are given in Table 1 for different sample sizes n and m. Generally, the estimated risk of the Bayes estimator of R is the smallest. Also, it is seen that the estimated risks of the two estimators are decreasing when (n, m) are increasing.

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