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# A note on partially conformal geodesic transformation on the Kahler manifolds*

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## Abstract

In this paper, we deal with partially conformal geodesic transformations in Kahler geometry by using Fermi coordinates when the submanifold is a geodesic sphere. We derive the necessary and sufficient condition for the existence of such transformation in terms of the Jacobi operator and its derivative.

## 0. Historical background and introduction

In 1972, S. Tochibana introduced the notion of a geodesic conformal transformations around submanifolds in a Riemannian manifold. These transformations are extensions of geodesic symmetries and local reflections with respect to submanifolds. The notion of a reflections generalize that of reflections with respect to linear subspaces in Euclidean space. Recently, E. Garcia-Rio, L. Vanhecke and B. Y. Chen begun a systematic study of geodesic conformal transformation. They show that conformality is a strong condition and motivated the study of the notion of a partially conformal geodesic transformation.

We focus on partially conformal geodesic transformations in Kahler manifolds when the submanifold is a geodesic sphere. This note are devoted to characterizations of complex space forms by using non-Euclidean inversions as defining partially conformal geodesic transformations.

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## 1. Kahler manifolds and Fermi coordinates

Let $(M, g)$ be a connected smooth Riemannian manifold and $\nabla$ its Levi Civita connection. Denote by $R$ its associated Riemannian curvature tensor defined by

$$R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$$

for all vector fields $X, Y \in \aleph(M)$. We put

$$R_{XYZW} = g(R_{XY}Z, W).$$

Let $M$ be a n-dimensional Kahler manifold with structure $(M, g, J)$:

$$J^2 = -1,$$
$$g(JX, JY) = g(X, Y),$$
$$\nabla_X(J)Y = 0$$

for all vector fields $X, Y \in \aleph(M)$. Then

$$R(X, Y)J = JR(X, Y),$$
$$R(JX, JY) = R(X, Y).$$

A plane section of the tangent space $T_pM$ at a point $pINM$ is called a holomorphic section if it is spanned by vectors $X$ and $JX$ in $T_pM$. The sectional curvature of a holomorphic section is called a holomorphic sectional curvature. A Kahler manifold of constant holomorphic sectional curvature $c$ is called a complex space form and its curvature tensor is given by

$$R_{XY}Z = \frac{c}{4}\{g(X, Z)Y - g(Y, Z)X + g(JX, Z)JY - g(JY, Z)JX + 2g(JX, Y)JZ\}.$$

A Kahler manifold $M$ of dimension $\geq 4$ is a complex space form if and only if, for every vector field $X$ on $M$, $R_{XJX}X$ is collinear with $JX$.

Let $B$ be a embedded submanifold of $M$ with dim $B = q$ and $\exp_\nu$ the exponential map of the normal bundle $\nu = T^{\perp}B$ of $B$ and $m \in B$ and $\{E_1, \ldots, E_n\}$ a local orthonormal frame field of $M$ defined along $B$ in a neighborhood of $m$. We specialize the fields such that $E_1, \ldots, E_q$ are tangent to $B$ and $E_{q+1}, \ldots, E_n$ normal vector fields of $B$. For a system of coordinates $(y^1, \ldots, y^q)$ of $B$ in a neighborhood of $m$ such that $(\partial/\partial y^i)(m) = E_i(m)$, $i = 1, \ldots, q$, the Fermi coordinates $(x^1, \ldots, x^n)$ with respect to $m$, $(y^1, \ldots, y^q)$ and $(E_{q+1}, \ldots, E_n)$ are defined by

$$x^i(\exp_\nu(\sum_{q+1}^{n} t^a E_a)) = y^i, \quad i=1, \ldots, q,$$

$$x^a(\exp_\nu(\sum_{q+1}^{n} t^a E_a)) = t^a, \quad a=q+1, \ldots, n$$

in an open neighborhood $U_m$ of $m \in M$.

Put $s(r) = \rho(r)r$, where $r$ is the normal distance. Then $r^2 = \sum_{a=q+1}^{n} (x^a)^2$.

Let $u \in T_m^\perp \subset T_m M$ and $\gamma(r) = \exp_m(ru)$ the normal geodesic with $\gamma(0) = m$, $\gamma'(0) = u = E_n(m)$. Denote by $\{F_1, \cdots, F_n\}$ the frame field along $\gamma$ obtained by parallel translating $\{E_1(m), \cdots, E_n(m)\}$ along $\gamma$. Consider the $n-1$ Jacobi vector fields $Y_a$, $a=1, \cdots, n-1$ along $\gamma$, determined by the initial conditions

$$Y_i(0) = E_i(m), \quad Y_i'(0) = (\nabla_u \partial/\partial x^i)(m), \quad i=1, \ldots, q,$$
$$Y_a(0) = 0, \quad Y_a'(0) = E_a(m), \quad a=q+1, \ldots, n.$$

Then $Y_i(r) = \frac{\partial}{\partial x^i}(\gamma(r))$, $Y_a(r) = r\frac{\partial}{\partial x^a}(\gamma(r))$.

Put $Y_a(r) = D_u(r)F_a$, $a=1, \cdots, n-1$. Then $D_u$ satisfies the Jacobi equation

$$D_u'' + R \circ D_u = 0$$

where $R(x)X = R_{\gamma(r)X}\gamma'(r)$.

Using the initial conditions for $Y_a$,

$$D_u(0) = \begin{pmatrix} I_q & 0 \\ 0 & 0 \end{pmatrix}, \quad D_u'(0) = \begin{pmatrix} T(u) & 0 \\ -{}^t\perp(u) & I_{n-q-1} \end{pmatrix}$$

where $T(u)_{ij} = g(T(u)E_i, E_j)(m)$ $\perp(u)_{ia} = g(\perp_{E_i}E_a, E_n)(m)$

and $(\perp_X N)(m) = (\triangle_X^\perp N)(m)$. Then

$$g_{ij}(p) = ({}^tD_uD_u)_{ij}(r), \quad g_{ia}(p) = \frac{1}{r}({}^tD_uD_u)_{ia}(r),$$

$$g_{ab}(p) = \frac{1}{r^2}({}^tD_uD_u)_{ab}(r), \quad g_{in} = g_{an} = 0, \quad g_{nn} = 1,$$

for $i, j = 1, \cdots, q$ and $a, b = q+1, \cdots, n-1$

## 2. Main results

We consider the local diffeomorphi

$$\phi_B : p = \exp_\nu(ru) \mapsto \phi_B(p) = \exp_\nu(s(r)u)$$

for $u \in T_m^\perp$, $\|u\| = 1$. $\phi_B$ is called the geodesic transformation with respect to $B$, which is locally given by

$$\phi_B : (x^1, \cdots, x^n) \mapsto (x^1, \cdots, x^q, \rho(r)x^{q+1}, \cdots, \rho(r)x^n)$$

Then we have

$$\phi_{B*}\frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^i}, \qquad i = 1, \cdots, q,$$

$$\phi_{B*}\frac{\partial}{\partial x^a} = \rho\frac{\partial}{\partial x^a} + \rho'\frac{\partial}{\partial} r, \qquad a = q+1, \cdots, n.$$

Let $\eta$ be the one form defined by $\eta(X) = g(X, JN)$.

If $\phi_B^* g = e^{2\sigma} g + f\eta \otimes \eta$ for some function $f$ which is depends only on the normal distance function $r$, $\phi_B$ is said to be partially conformal.

**Lemma 1.** A geodesic transformation $\phi_B$ with respect to $B$ is partially conformal if and only if

$$g_{ij}(\phi_B(p)) = (e^{2\sigma} g + f\eta\otimes\eta)_{ij}(p), \qquad \rho g_{ia}(\phi_B(p)) = (e^{2\sigma} g + f\eta\otimes\eta)_{ia}(p),$$

$$\rho^2 g_{ab}(\phi_B(p)) = (e^{2\sigma} g + f\eta\otimes\eta)_{ab}(p), \qquad e^{2\sigma} = (\rho' r + \rho)^2 = (s')^2$$

where $i, j = 1, \cdots, q$ and $a, b = q+1, \cdots, n-1$.

**proof**. Using Fermi coordinates

$$(\phi_B^* g)_{ij}(p) = g_{ij}(\phi_B(p)), \qquad (\phi_B^* g)_{ia}(p) = \rho g_{ia}(\phi_B(p))$$

$$(\phi_B^* g)_{ab}(p) = \rho^2 g_{ab}(\phi_B(p)), \qquad (\phi_B^* g)_{nn}(p) = (\rho' r + \rho)^2 g_{nn}(\phi_B(p))(\mathrm{p})).$$

By the definition of partially conformal, we have the desired result.

**Lemma 2.** Let $(M, g, J)$ be a Kahler manifold and $B$ a real hypersurface. If $\phi_B$ is a partially conformal geodesic transformation with respect to $B$, then $B$ is a Hopf hypersurface with two constant principal curvatures.

**proof**. By Lemma 1

$$g_{ij}(s(r)) = e^{2\sigma} g_{ij}(r) + f(r)(\eta\otimes\eta)_{ij}(r). \qquad (*)$$

Taking limits for $r = 0$, $\delta_{ij} = s'(0)^2\delta_{ij} + f(0)\delta_{1i}\delta_{1j}$. Thus $s'(0)^2 = 1$ and $f(0) = 0$.

Since $\phi_B$ is non-trivial, $s'(0) = -1$. Using power series expansion for both side of

(*), we get

$$\delta_{ij}-2rT_{ij}+O(r^2)=\delta_{ij}+(2T_{ij}-2s''(0)\delta_{ij}+f'(0)\delta_{1i}\delta_{1j})r+O(r^2).$$

Hence $T_{ij}=\frac{1}{2}(s''(0)\delta_{ij}-\frac{1}{2}f'(0)\delta_{1i}\delta_{1j})$.

Therefore $k_1=\frac{1}{2}s''(0)-\frac{1}{2}f'(0)$ and $k_2=\cdots=k_{n-1}=\frac{1}{2}s''(0)$.

**Theorem 3.** $(M, g, J)$ is an n-dimensional Kahler manifold of complex space form $M_n(c)$, $c\neq 0$ if and only if the non-Euclidean inversion

$$\tan(s+\alpha)\frac{\sqrt{c}}{4}\tan(r+\alpha)\frac{\sqrt{c}}{4}=\tan^2\alpha\frac{\sqrt{c}}{4} \qquad (**)$$

defines a partially conformal geodesic transformation with respect to each geodesic sphere $G(\alpha)$ of small radius $\alpha$.

**proof.** Let (**) be a partially conformal geodesic transformation with respect to $G(\alpha)$. Then $G(\alpha)$ is a hypersurface of $M$. Put

$$s+\alpha=\frac{4}{\sqrt{c}}\tan^{-1}\bar{t},\ \ r+\alpha=\frac{4}{\sqrt{c}}\tan^{-1}t \ \text{ and } \ D=\tan^2\alpha\frac{\sqrt{c}}{4}.$$

Then (**) takes the form $\bar{t}t=D$ and $s=\frac{4}{\sqrt{c}}\tan^{-1}(D/t)-\alpha$.

By the power series expansion and lemma 2,

$$s=-r=\frac{1}{2}r^2\sqrt{c}\ \cot\alpha\frac{\sqrt{c}}{2}+O(r^3),$$

$$f=-\left(\frac{\sin(s+\alpha)\frac{\sqrt{c}}{2}}{\sin(r+\alpha)\frac{\sqrt{c}}{2}}\right)^2+\left(\frac{\sin(s+\alpha)\sqrt{c}}{\sin(r+\alpha)\sqrt{c}}\right)^2.$$

Hence $k_1=\sqrt{c}\cot\alpha\sqrt{c}$ and $k_2=\cdots=k_{n-1}=\frac{\sqrt{c}}{2}\cot\alpha\frac{\sqrt{c}}{2}$.

Thus $(M, g, J)$ is a complex space $M_n(c)$.

Conversely, suppose $M=M_n(c)$ and $c$ to be positive. Then

$$R=\begin{pmatrix} c & 0 \\ 0 & \frac{c}{4}I_{n-2}\end{pmatrix},\qquad A=\begin{pmatrix}\frac{1}{\sqrt{c}}\sin\alpha\sqrt{c} & 0 \\ 0 & \frac{2}{\sqrt{c}}\sin\alpha\frac{\sqrt{c}}{2}I_{n-2}\end{pmatrix}.$$

Hence $T(\exp_\nu(ru)) = \begin{pmatrix} \sqrt{c}\cos a\sqrt{c} & 0 \\ 0 & \frac{\sqrt{c}}{2}\cos a\frac{\sqrt{c}}{2} I_{n-2} \end{pmatrix}$.

Since $D_u(r) = (\cos r\sqrt{c})D_u(0) + \left(\frac{\sin r\sqrt{c}}{\sqrt{c}}\right) D_{u'}(0)$,

$$g_{ij}(\exp\nu(ru)) = ({}^tD_uD_u)_{ij}(r) = \left(\cos r\frac{\sqrt{c}}{2} + \cot a\frac{\sqrt{c}}{2}\sin r\frac{\sqrt{c}}{2}\right)^2\delta_{ij}.$$

From $e^{2\sigma}g_{ij}(p) = g_{ij}(\phi_p(p))$, we get

$$e^{2\sigma}\left(\cos r\frac{\sqrt{c}}{2} + \cot a\frac{\sqrt{c}}{2}\sin r\frac{\sqrt{c}}{2}\right)^2\delta_{ij} = \left(\cos s(r)\frac{\sqrt{c}}{2} + \cot a\frac{\sqrt{c}}{2}\sin s(r)\frac{\sqrt{c}}{2}\right)^2\delta_{ij}.$$

Thus $\frac{ds}{dr}\sin(r+a)\frac{\sqrt{c}}{2} = \pm\sin(s+a)\frac{\sqrt{c}}{2}$. Therefore

$$\frac{ds}{\sin(s+a)\frac{\sqrt{c}}{2}} = \pm\frac{dr}{\sin(r+a)\frac{\sqrt{c}}{2}} \qquad (***)$$

So (**) is the only solution of (***) leaving $G(a)$ invariant.

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