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The Origin and Recent History for Fuzzy Equations*

Department of Mathematics, Inha University **Q-Heung Choi**
Department of Mathematics, Inha University **Sunyoung Lee**

Abstract

We investigate the origin and recent history for fuzzy equations. And we introduce the existence theorems of solutions for the fuzzy differential equation with infinite delays and fuzzy functional integral equations. We will also add recent researches for controllability of sobolev-type semilinear integro-differential fuzzy system.

0. Introduction and historical background

We investigate a history of researches for the existence of solutions for fuzzy equations, that is, fuzzy differential equations with infinite delays and fuzzy functional integral equations. We also add recent researches. The concept of fuzzy set was first introduced by Zadeh[11] in 1965. The applications of the fuzzy set theory can be found in many branches of mathematical and engineering sciences including artificial intelligence, computer sciences, and operations research. Seikkala (1987)[8] defined the fuzzy derivative which is the generalization of the Hukuhara derivative and the fuzzy integral which is the same as Dubois and Prade (1982)[1]. And by means of the extension principle of Zadeh, Seikkala[8] showed that the fuzzy initial value problem $x'(t)=f(t, x(t))$, $x(0)=x_0$ has a unique fuzzy solution if f satisfies the generalized Lipschitz condition. Park, Kwun and Jeong (1995)[4] studied the following fuzzy integral equation in Banach space.

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$$\varphi(u) = w_0 + \int_{u_0}^{u} F(u, s, \varphi(s))ds, \quad \varphi(u_0) = w_0,$$

where $F: J \times J \times T(X) \to T(X)$ is continuous, $J = [u_0, u_0 + d]$, and $T(X)$ is a regular fuzzy set. They treated the local existence of solutions and approximate solutions for the fuzzy integral equation based on the properties of the α-index of Kuratowski. Park, Lee and Jeong (2000)[5] also studied a fuzzy functional difference equation in Banach space. They also found the approximation solutions of the fuzzy functional integral equation and compare them with the approximate solutions of the functional integral equation. Kaleva (1990)[3] studied the Cauchy problem for fuzzy differential equations, and characterized those subsets of fuzzy sets in which the peano theorem is valid. Subrahmanuam and Subarsanan (1994)[9] proved the existence theorem of fuzzy solutions for fuzzy functional equations of the form $f(x) = h(x, f(s(x)))$. Subrahmanyam and Sudarsanam (1997)[10] studied the following fuzzy functional integral equation.

$$\phi(u) = \int_0^{f_1(t)} h(s, \phi(f_2(s)))ds + g(t, \phi(f_3(t))),$$

where h and g are given fuzzy functions and $f_i : [0, 1] \to [0, 1]$, $i = 1, 2, 3$ are continuous. In [10] using the analytical structure of E^n the space of fuzzy numbers(which are normalized, upper semi- continuous, fuzzy convex, and compactly supported) the existence theorem is proved by invoking Kranoselskii's fixed point theorem.

We will introduce some definitions and properties for fuzzy mapping in section 1. In section 2 we will say that the existence and properties of a unique solution for fuzzy differential equation with infinite delays. In section 3, we reveal the approximate solutions of fuzzy functional integral equations. In section 4, we also add recent researches for controllability of sobolev-type semilinear integro-differential fuzzy system.

1. Definitions and properties for fuzzy mappings

Let X be a real Hilbert space with norm $\|\cdot\|$ and inner product $< , >$. Let K be a closed convex set in X. A fuzzy set in X is a mapping with domain X and values in

[0, 1]. If A is a fuzzy set in X and $x \in X$, the function value $A(x)$ is called the grade of membership of x in A. We denote the collection of all fuzzy sets on X by $F(X)$. Let $A \in F(X)$ and $\alpha \in (0, 1]$. The α-level set of A, denoted $(A)_\alpha$, is defined by $(A)_\alpha = \{x : A(x) \geq \alpha\}$. A mapping T from X into $F(X)$ is called a fuzzy mapping. If $T : X \to F(X)$ is a fuzzy mapping, then $T(u)$, for $u \in X$, is a fuzzy set in $F(H)$ and $T(u, v)$, for $v \in H$, is the degree of membership of v in $T(u)$.

Let $P_K(R^n)$ denote the family of all nonempty compact convex subset of R^n and define the addition and scalar multiplication in $P_K(R^n)$ as usual. Let A and B be two nonempty bounded subsets of R^n. The distance between A and B is defined by the Hausdorff metric

$$d(A, B) = \max\{ \sup_{a \in A} \inf_{b \in B} \|a - b\|, \ \sup_{b \in B} \inf_{a \in A} \|a - b\|\},$$

where $\|\cdot\|$ denote the usual Euclidean norm in R^n. Then it is clear that $(P_k(R^n), d)$ becomes a metric space.

Proposition 1.1. The metric space $(P_k(R^n), d)$ is complete and separable.

We denote

$$E^n = \{u : R^n \to [0, 1] \mid u \text{ satisfies (i)}-\text{(iv) below }\},$$

where

(i) u is normal, i.e., there exists an $x_0 \in R^n$ such that $u(x_0) = 1$,

(ii) u is fuzzy convex, i.e.,

$$u(\lambda x_1 + (1-\lambda)x_2) \geq \min\{u(x_1), u(x_2)\}, \quad x_1, x_2 \in R^n, \quad \lambda \in [0, 1]$$

(iii) u is upper semicontinuous,

(iv) $[u]^0 = \overline{\{x \in R^n \mid u(x) > 0\}}$ is compact.

For $0 < \alpha \leq 1$, denote $[u]^\alpha = \{x \in R^n \mid u(x) \geq \alpha\}$. Then, from (i)-(iv), it follows that the α-level set $[u]^\alpha \in P_K(R^n)$ for all $0 \leq \alpha \leq 1$.

If $g : R^n \times R^n \to R^n$ is a function, then, according to Zadeh's extension principle, we can extend g to $E^n \times E^n \to E^n$ by the equation $g(u, v)(z) = \sup_{z=g(x,y)} \min\{u(x), v(y)\}$. It is well known that

$$[g(u, v)]^\alpha = g([u]^\alpha, [v]^\alpha) \text{ for all } u, v \in E^n,\ 0 \le \alpha \le 1$$

and a continuous function g. Expecially, for addition and scalar multiplication, we have

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha, \qquad [ku]^\alpha = k[u]^\alpha,$$

where $u, v \in E^n$, $0 \le \alpha \le 1$, $k \in R$.

Proposition 1.2. If $u \in E^n$, then

(1) $[u]^\alpha \in P_K(R^n)$ for all $0 \le \alpha \le 1$,

(2) $[u]^{\alpha_2} \subset [u]^{\alpha_1}$ for $0 \le \alpha_1 \le \alpha_2 \le 1$,

(3) If $\{\alpha_k\} \subset [0, 1]$ is a nondecreasing sequence converging to $\alpha > 0$, then

$$[u]^\alpha = \bigcap_{k \ge 1} [u]^{\alpha_k}.$$

Conversely, if $\{A^\alpha \mid 0 \le \alpha \le 1\}$ is a family of subsets of R^n satisfying (1)-(3), then there exists a $u \in E^n$ s.t.

$$[u]^\alpha = A^\alpha \text{ for } 0 < \alpha \le 1 \quad \text{and} \quad [u]^0 = \overline{\bigcup_{0 < \alpha \le 1} A^\alpha} \subset A^0.$$

Define $D : E^n \times E^n \to R^+ \cup \{0\}$ by the equations $D(u, v) = \sup_{0 \le \alpha \le 1} d([u]^\alpha, [v]^\alpha)$, where d is the Hausdorff metric defined in $P_K(R^n)$. Then it is easy to show that D is a metric in E^n.

Using the result in Seikkala[8], we know that

(1) (E^n, D) is a complete metric space,

(2) $D(u+w, v+w) = D(u, v)$ for all $u, v, w \in E^n$,

(3) $D(ku, kv) = |k| D(u, v)$ for all $u, v \in E^n$, $k \in R$.

Definition 1.1. Let $T = [t_0, t_0 + p] \subset R$ be a compact interval. We say that a mapping $F : T \to E^n$ is strongly measurable if, for all $\alpha \in [0, 1]$, the set-valued mapping $F_\alpha : T \to P_K(R^n)$ defined by $F_\alpha(t) = [F(t)]^\alpha$ is (Lebesgue) measurable, where $P_K(R^n)$ is endowed with the topology generated by the Hausdorff metric d.

Definition 1.2. A mapping $F : T \to E^n$ is called integrable bounded if there exists an integrable function h such that $\|x\| \le h(t)$ for all $x \in F_0(t)$.

Definition 1.3. Let $F : T \to E^n$. The integral of F over T denoted by $\int_T F(t)\,dt$ is defined level-wise by the equation

$$\begin{aligned}\left[\int_T F(t)dt\right]^\alpha &= \int_T F_\alpha(t)dt \\ &= \left\{\int_T f(t)dt \mid f: T \to R^n \text{ is a measurabe selection for } F_\alpha\right\}\end{aligned}$$

for all $0 < \alpha \le 1$.

Proposition 1.3.[2] If $F : T \to E^n$ is strongly measurable and integrably bounded, then F is integrable.

The following definitions and propositions are given in reference [2].

Proposition 1.4.[2] Let $F,\ G : T \to E^n$ be integrable and $\lambda \in R$. Then

(1) $\int_T (F(t) + G(t))dt = \int_T F(t)dt + \int_T G(t)dt$

(2) $\int_T \lambda F(t)dt = \lambda \int_T F(t)dt$

(3) $D(F, G)$ is integrable.

(4) $D\left(\int_T F(t)dt, \int_T G(t)dt\right) \le \int_T D(F, G)(t)dt.$

Remark 1.1.[2] Suppose $A \in E^n$ and define $F : [a, b] \to E^n$ by $F(s) = A$ for all $a \le s \le b$. Then, from Example 4.1, we have

$$\int_a^b F(t)dt = (b-a)A.$$

Let $x, y \in E^n$. If there exists a $z \in E^n$ such that $x = y + z$, then we call z the H-difference of x and y, denoted by $x - y$.

Definition 1.4.[7] A mapping $F : T \to E^n$ is differentiable at $t_0 \in T$, if there exists a $F'(t_0) \in E^n$ such that the limits

$$\lim_{h \to 0} \frac{F(t_0 + h) - F(t_0)}{h} \quad \text{and} \quad \lim_{h \to 0} \frac{F(t_0) - F(t_0 - h)}{h}$$

exist and are equal to $F'(t_0)$. Here the limit is taken in the metric space (E^n, D). At the end point of T, we consider only the one-side derivatives.

Proposition 1.5. If $F, G : T \to E^n$ are differentiable and $\lambda \in R$, then

$$(F + G)'(t) = F'(t) + G'(t), \quad (\lambda F)'(t) = \lambda F'(t).$$

Definition 1.5. A mapping $F : T \to E^n$ is called continuous at $t_0 \in T$ if it is continuous at $t = t_0$ in the metric space (E^n, D).

Definition 1.6. A mapping $F : T \to E^n$ is said to be uniformly continuous on T if it is uniformly continuous in the metric space (E^n, D).

Definition 1.7. A mapping $F : T \to E^n$ is said to be bounded if there is a real number M such that $D(F(t), \hat{0}) \leq M$ for all $t \in T$ and $\hat{0} \in E^n$.

Proposition 1.6.[2] Let $F : T \to E^n$ be continuous. Then, for every $t \in T$, the integral $G(t) = \int_{t_0}^{t} F(s)ds$ is differentiable and $G'(t) = F(t)$..

Proposition 1.7.[2] Let $F : T \to E^n$ be differentiable and assume that the derivative $F'(t)$ is integrable over T. Then for each $s \in T$, we have

$$F(s) = F(t_0) + \int_{t_0}^{s} F'(t)dt.$$

2. The existence and properties of solution for fuzzy differential equation with infinite delays

Here we consider the fuzzy functional differenial equations with infinite time delays

$$x'(t)=F(t,x(t),x_t),\quad x_t(s)=x(t+s),\ s\le 0. \tag{2.1}$$

Let $C((-\infty,0];E^n)$ denote the set of all fuzzy continuous and bounded mappings from $(-\infty,0]$ to E^n with $D(\phi(t),\hat{0})\le M$ for some M and for $\hat{0}\in E^n$. We write

$$\Gamma=C((-\infty,0];E^n).$$

Here we denote

$$D(\phi,\hat{0})=\sup\ \{D(\phi(t),\hat{0}) : t\in(-\infty,0]\}\quad \text{for } \phi\in\Gamma,\ \hat{0}\in E^n.$$

For each $\phi\in\Gamma$, set $D(\phi,\hat{0})=D(\phi_{(-\infty,0]},\hat{0})$ for $\hat{0}\in E^n$. If $x(t)$ is a fuzzy mapping on $(-\infty,b)$, $b\le\infty$, we define for each $t\in(-\infty,b)$, $x_t(s)=x(t+s)$, $s\le 0$. Clearly, if $x(t)$ is fuzzy continuous and bounded mapping on each interval $(-\infty,b_1]$, $b_1<b$, then $x_t\in\Gamma$ for $t\in(-\infty,b)$. The E^n-valued fuzzy mapping $F(t,x,\phi)$ on $R\times E^n\times\Gamma$ is said to satisfy following conditions:

(A_1) It is almost periodic fuzzy mapping in t uniformly for (x,ϕ) in closed bounded subsets of $E^n\times\Gamma$, i.e., if $S\subset E^n\times\Gamma$ is closed and bounded, then $\{F(t,x,\phi):(x,\phi)\in S\}$ is almost periodic fuzzy mapping in t and uniformly for (x,ϕ).

(A_2) There exists an $M>0$ such that $D(F(t,0,0),\hat{0})\le M$ for $t\in R$, $\hat{0}\in E^n$.

(A_3) The fuzzy mapping $F(t,x(t),x_t)$ is uniformly continuous on R whenever $x(t)$ is uniformly continuous with $D(x(t),\hat{0})\le M'$ for some constant M', $\hat{0}\in E^n$.

(A_4) There exist positive numbers p, h and r such that $ph<1$, $p\ge M/r$ where M is a constant in (A_2),

$$D(x(t)+hF(t,x(t),x_t),\ y(t)+hF(t,y(t),y_t))\ \le\ (1-ph)\,D(x_t,y_t) \tag{2.2}$$

for $t\in R$ and any fuzzy mappings $x(t)$, $y(t)$ are uniformly continuous mappings on R with $D(x(t),\hat{0})\le r$, $D(y(t),\hat{0})\le r$ for $\hat{0}\in E^n$.

(A_5) For each $r>0$ there exists a $M_1(r)>0$ such that $D(F(t,x(t),\phi),\hat{0})\le M_1(r)$ with $D(x(t),\hat{0})\le r$, $D(\phi,\hat{0})\le r$, for $t\in R$, $\hat{0}\in E^n$ and any $\phi\in \Gamma$

(A_6) In the condition (A_4), (2.2) is also valid for any continuous fuzzy mappings $x(t)$, $y(t)$ on R and there is $p>\frac{M}{r}$.

Remark 2.1. The condition (A_6) implies the condition (A_4)

Definition 2.1. A solution $x(t, t_0, \phi_1)$, which satisfies $x_{t_0}=\phi_1$ with $\phi_1\in\Gamma$ and $D(x(t, t_0, \phi_1),\hat{0})\le M$ for $t\ge t_0$, $\hat{0}\in E^n$, is uniformly stable if, for each $\varepsilon>0$ and each $t_0\ge 0$, there exists a positive number $\delta=\delta(\varepsilon)$ independent of t_0 such that $D(x(t, t_0, \phi_1), y(t, t_0, \phi_2))<\varepsilon$, whenever $D(\phi_1, \phi_2)<\delta$ and $t\ge t_0$, where $y(t, t_0, \phi_2)$, which satisfies $y_{t_0}=\phi_2$ with $\phi_2\in\Gamma$, is any solution of (2.1) for $t\ge t_0$.

Theorem 2.1. Suppose that the fuzzy mapping $F: R\times E^n\times\Gamma\to E^n$ satisfies (A_2)-(A_5). Then (2.1) has only one solution $\bar{x}(t)$ with $D(\bar{x}(t),\hat{0})\le r$ for $t\in R$ and any $\hat{0}\in E^n$.

Theorem 2.2. Suppose that the fuzzy mapping $F: R\times E^n\times\Gamma\to E^n$ has (A_2), (A_3), (A_5) and (A_6). Then (2.1) has only one bounded solution $\bar{x}(t)$ which is uniformly stable and satisfies $D(\bar{x}(t),\hat{0})\le r$ for $t\in R$, $\hat{0}\in E^n$.

Theorem 2.3. If the fuzzy mapping $F: R\times E^n\times\Gamma\to E^n$ is periodic in t with period T independent of (t, ϕ) and satisfies conditions $(A_2)-(A_4)$, then (2.1) has only one T-periodic solution $\bar{x}(t)$ with $D(\bar{x}(t),\hat{0})\le r$, for $t\in R$, $\hat{0}\in E^n$.

Theorem 2.4. Suppose that the fuzzy mapping $F: R\times E^n\times\Gamma\to E^n$ has $(A_1)-(A_4)$. Then (2.1) has a unique almost periodic solution $\bar{x}(t)$ with $D(\bar{x}(t),\hat{0})\le r$ for $t\in R$, $\hat{0}\in E^n$.

3. The approximate solutions for fuzzy functional integral equations

If T is a compact interval of R, then $C(T;E^n)$ denotes the set of all fuzzy continuous mappings from T to E^n. Here we denote

$$D(\phi,\hat{0}) = \sup\{D(\phi(t),\hat{0}) : t \in T\} \quad \text{for } \phi \in C(T;E^n),\ \hat{0} \in E^n.$$

We write $\Gamma_1 = C([-r,0];E^n)$ and for $\phi \in \Gamma_1$, $D(\phi,\hat{0}) = D(\phi_{[-r,0]},\hat{0})$, $\hat{0} \in E^n$. If J is the interval $[\sigma,b]$ or $[\sigma,b)$ $(b>\sigma)$, $I=[\sigma-r,\sigma]\cup J$ and $x \in C(I;E^n)$, then for $t \in J$ we denote by x_t the element of Γ_1 denfined by $x_t(\theta)=x(t+\theta)$, $\theta \in [-r,0]$.

Let U be a closed subset of Γ_1, g be a function from $J\times J\times U \to E^n$ and $f \in C(I;E^n)$ We consider the fuzzy functional integral equation

$$x(t) = \begin{cases} f(t), & t \in [\sigma-r,\sigma] \\ f(t) + \int_\sigma^t g(t,s,x_s)ds, & t \in J \end{cases} \tag{3.1}$$

where $r \geq 0$, $J=[\sigma,b]$ or $[\sigma,b)$.

A function $x: I \to E^n$ will be a solution of equation (3.1) if (i) x is defined and continuous on I and $(x,x_t) \in J\times U$ for all $t \in J$, (ii) for every $t \in I$, x satisfies (3.1).

Definition 3.1. Let Z be a subset of Γ_1 and T_1, T_2 be intervals of R. We say that a mapping $P: T_1 \times T_2 \times Z \to E^n$ satisfies the Caratheodory-type hypotheses if P satisfies the following :

(C_0) $P(t,s,z)=\hat{0}$ if $t<s$, for some $\hat{0} \in E^n$,

(C_1) $P(t,\cdot,z) : T_2 \to E^n$ is strongly measurabe for all $(t,z) \in T_1 \times Z$,

(C_2) $P(t,s,\cdot) : Z \to E^n$ is continuos for $a.e.$ $s \in T_2$, for $t \in T_1$,

(C_3) there exists a function $m(t,s)$ such that for each fixed $t \in T$,

$M(t,\cdot) \in L^1_{loc}(T_2 : R^+)$,

$D(P(t,s,z),\hat{0}) \leq m(t,s)$ for all $(t,s,z) \in T_1 \times T_2 \times Z$, for all $\hat{0} \in E^n$

and $\sup\left\{\int_{T_2} m(t,s)ds : t \in T_1\right\} < \infty$.

If U is a closed subset of Γ_1 and g is a mapping from $J\times J\times U$ to E^n, then we make the following assumption (G):

(G) For each $\tau\in J$ and positive N if $t_0\in J$ then

$$\sup\left\{D\left(\int_\sigma^\tau g(t,s,u_s)ds,\ \int_\sigma^\tau g(t_0,s,u_s)ds\right) : u\in C(I:E^n),\ u_s\in U \text{ for all } s\in[\sigma,\tau],\ D(u(t),\hat{0})\le N \text{ for all } t\in[\sigma,\tau]\ \right\}$$

tends to zero as $t\to t_0$, $\hat{0}\in E^n$.

If $[0,2\varepsilon]$, T_1 and T_2 are intervals of R and w is a function from $T_1\times T_2\times[0,2\varepsilon]$ to R^+, then we make the following assumption (H):

(H) If $t_0\in T_1$, then

$$\sup\left\{\left|\left(\int_{T_2}(w(t,s,r(s))-w(t_0,s,r(s)))ds\right)\right| : r\in C(T_2;[0,2\varepsilon])\right\}$$

tend to zero as $t\to t_0$.

Theorem 3.1. Let f be a fuzzy continuous mapping from I to E^n with $D(f(t),\hat{0})\le M$ for some constant M and for all $\hat{0}\in E^n$. Let U be a closed subset of Γ and let $g: J\times J\times U\to E^n$ be a mapping satisfying Caratheodory-type hypotheses and assumption (G). Suppose for every $(t_0,s_0,\phi_0)\in J\times J\times U$, $t_0<b$, there is a nonempty interval $\overline{J_0}=[t_0,t_0+d]\subseteq J$, a set $B_0=B(\phi_0,\varepsilon)$ in U and a function $w:\overline{J_0}\times J_0\times[0,2\varepsilon]\to R^+$, $(J_0=(t_0,t_0+d])$ satisfying Crathedory-type hypotheses and assumption (H) such that

(i) $D(g(t,s,\phi),g(t,s,\psi))\le w(t,s,D(\phi,\psi))$ for all $(t,s)\in\overline{J_0}\times J_0$ and $\phi,\psi\in B_0$,

(ii) $w(t,s,\cdot)$ is nondecresing for all $(t,s)\in\overline{J_0}\times J_0$,

(iii) for every $e\le d$, $y\equiv 0$ is the unique continuous function u such that

$$u(t)=\begin{cases}0 & \text{for } t\in[t_0-r,t_0],\\ \int_{t_0}^t w(t,s,D(u_s,\hat{0}))ds & \text{for } t\in[t_0,t_0+e],\ \hat{0}\in E^n\end{cases}$$

where for each s, $u_s \in C([-r,0]:R^+)$ is defined by $u_s(\theta)=u(s+\theta)$, $\theta\in[-r,0]$ and $D(u_s,\hat{0})=\sup\{D(u_s(\theta),\hat{0}):\theta\in[-r,0]\}$, $\hat{0}\in E^n$. Let $f(\sigma)\in U$ and $y_s^0\in U$ for each $s\in J$. If the successive approximation

$$y^{n+1}(t)=\begin{cases} f(t), & t\in[\sigma-r,\sigma],\\ f(t)+\int_\sigma^t g(t,s,y_s^n)ds, & t\in J,\end{cases}$$

are well defined on I, $f_\sigma\in U$. Then $(y^n)_n$ converges uniformly on compact subsets of I to a solution of (3.1).

4. Further research

We consider the sobolev type nonlinear integro-differential system of the form

$$\begin{aligned} (Ex(t))'+Ax(t) &= (Bu)(t)+f\left(t,x(t),\int_0^t g(t,s,x(s))ds\right), \qquad t\in J=[0,b], \\ x(0) &= x_0\in E^n, \end{aligned} \tag{4.1}$$

where the state $x(\cdot)$ takes values in the Banach space X and the control function $u(\cdot)$ is given in $L^2(J,U)$, a Banach space of admissible control functions with U as a Banach space. B is a bounded linear operator from U to Y, a Banach space, $g:J\times J\times X\to X$ and $f:J\times X\times X\to Y$. The norm of X is denoted by $\|\cdot\|$ and Y by $|\cdot|$. If the input $u(t)$ are crisp, then it is the classical control system. In this case that $u(t)$ are fuzzy input, we have a different system with fuzzy inputs and fuzzy outputs. However, we still have two problems to be concerned about, *i.e.*,

(1) Is there always a control (or input) $u(t)$ which can transfer the initial state x_0 of the system to any desired range in a finite time?

(2) Can the initial state x_0 of the system be always identified by observing the output $x(t)$ and the input $u(t)$ over a finite time?

Here we want to investigate the second problem. So we consider later for the existence of fuzzy solutions for soboleve-type semilinear integro-differenctial fuzzy systems (4.1).

Reference

1. D. Dubois and H. Prade, "Towards fuzzy differential calculus, Part I : Integration of fuzzy mappings class of second-order," *Fuzzy Sets and Systems* 8(1982), 1-17.
2. O. Kaleva, "Fuzzy differential equations," *Fuzzy Sets and Systems* 24(1987), 301-317.
3. O. Kaleva, "The Cauchy problem for fuzzy differential equations," *Fuzzy Sets and Systems* 35(1990), 389-396.
4. J.Y. Park and Y.C. Kwun and J.U. Jeong, "Existence of solutions of fuzzy integral equations in Banach spaces," *Fuzzy Sets and Systems* 72(1995), 373-378.
5. J.Y. Park, Y.S. Lee and J.U. Jeong, "The approximate solutions of fuzzy functional integro-differential equation," *Fuzzy Sets and Systems* 110(2000), 79-90.
6. J.Y. Park, Y.S. Lee and H.M. Kim, "The existence of solutions for fuzzy differential equations with infinite delays," *Indian J. pure appl. Math.* 31(2) (2000), 137-151.
7. M.L. Puri and D.A. Ralescu, "Differentials of fuzzy functions," *J. Math. Anal. Appl.* 91(1983), 552-558.
8. S. Seikkala, "On the fuzzy initial value problem," *Fuzzy Sets and Systems* 24(1987), 319-330.
9. P.V. Subrahmanyam and S.K. Sudarsanam, "On some fuzzy functional equations," *Fuzzy Sets and Systems* 64(1994), 333-338.
10. P.V. Subrahmanyam and S.K. Sudarsanam, "An existence theorem for a fuzzy functional integral equation," *J. Fuzzy Math.* Vol. 5, No. 3(1997), 723-732.
11. L.A. Zadeh, "Fuzzy sets," *Inform and Control* 8(1965), 338-353.