KJHM Vol. 16, No. 1, 2003, 73-78

Minimal basically disconnected covers of countably locally weakly Lindelöf spaces*

단국대학교 수학교육과 **김창일**

Abstract

Observing that if $f: Y \to X$ is a covering map and X is a countably locally weakly Lindelöf space, then Y is countably locally weakly Lindelöf and that every dense countably weakly Lindelöf subspace of a basically disconnected space is basically disconnected, we show that for a countably weakly Lindelöf space X, its minimal basically disconnected cover ΛX is given by the filter space of fixed $\sigma Z(X)^{\#}$- ultrafilters.

0. Introduction

It is known that minimal basically disconnected covers of some spaces are given by certain filter spaces. Among other, Vermeer showed that the minimal basically disconnected cover ΛX of a compact space X is given by the Stone-space $S(\sigma Z(X)^{\#})$ of $\sigma Z(X)^{\#}$ ([4]). In ([2]), the minimal basically disconnected cover of a locally weakly Lindelöf space is characterized by the filter space of fixed $\sigma Z(X)^{\#}$ − ultrafilters on X.

The purpose to write this paper is to extend the above result to countably locally weakly Lindelöf spaces. The concept of countably locally weakly Lindelöf spaces is introduced by Comfort, Hindman, and Negrepontis[1]. We first show that countably locally weakly Lindelöf spaces are left fitting with respect to covering maps, that is, if $f: Y \to X$ is a covering map and X is a countably locally weakly Lindelöf space, then

* The present research was conducted by the research fund Dankook University in 2002.

Y is a countably locally weakly Lindelöf space and that a dense countably locally weakly Lindelöf subspace of a basically disconnected space is also basically disconnected. Using this, we then show that the minimal basically disconnected cover ΛX of a countably locally weakly Lindelöf space X is given by the filter space of fixed $\sigma Z(X)^{\#}$ – ultrafilters on X. All spaces in this paper are assumed to be Tychonoff spaces. For a space X, βX denotes the Stone-Čech compactification of X. For the terminology, we refer to [3].

1. Minimal basically diconnected covers of countably locally weakly Lindelöf spaces

We recall that a space X is called *weakly Lindelöf* if for any open cover α of X, there is a countable subfamily β of α such that $\bigcup\beta$ is dense in X and that a space X is called *locally weakly Lindelöf* if every element of X has a weakly Lindelöf neighborhood.

It is clear that for any collection $\{\alpha_i : i \in I\}$ of open covers of a locally weakly Lindelöf space X and for any $x \in X$, there is a neighborhood G of x in X and for any $i \in I$, there is a countable subfamily β_i of α_i such that $G \subseteq \mathrm{cl}_X(\bigcup\beta_i)$. Indeed, the neighborhood G may be chosen independent of the collection $\{\alpha_i : i \in I\}$. The following definition is a generalization of locally weakly Lindelöf spaces.

Definition 1.1. A space X is called *countably locally weakly Lindelöf* (abbreviation: CLWL) if for any countable collection $\{\alpha_n : n \in N\}$ of open covers of X and for any $x \in X$, there is a neighborhood G of x in X and for any $n \in N$, there is a countable subfamily β_n of α_n such that $G \subseteq \mathrm{cl}_X(\bigcup\beta_n)$.

Every locally weakly Lindelöf space is CLWL but the converse need not be true[2]. Recall that a map $f : Y \to X$ is called *covering* if it is a perfect, continuous, irreducible map. It is known that if $f : Y \to X$ is a covering map and X is a (locally, resp.) weakly Lindelöf space, then Y is also (locally, resp.) weakly Lindelöf([3], [4]).

For any space X, the set $R(X)$ of all regular closed sets in X, when partially ordered by inculsion, becomes a complete Boolean algebra, in which the join, meet and complementation operations are defined as follows:

If $A\in R(X)$ and $\{A_i: i\in I\}\subseteq R(X)$, then

$$\vee\{A_i : i\in I\} = \mathrm{cl}_X(\cup A_i : i\in I),$$

$$\wedge\{A_i : i\in I\} = \mathrm{cl}_X(\mathrm{int}_X(\cap\{A_i : i\in I\})), \text{ and } A' = \mathrm{cl}_X(X-A).$$

It is well-known that for any covering map $f: Y\rightarrow X$, the map $\phi: R(Y)\rightarrow R(X)$, defined by $\phi(A)=f(A)$, is a Boolean isomorphism and the inverse map ϕ^{-1} of ϕ is given by $\phi^{-1}(A)=\mathrm{cl}_Y(f^{-1}(\mathrm{int}_X(A)))=\mathrm{cl}_Y(\mathrm{int}_Y(f^{-1}(A)))$.

Theorem 1.2. Let $f: Y\rightarrow X$ be a covering map and X a CLWL space. Then Y is also CLWL.

proof. Let $\{\alpha_n: n\in N\}$ be a countable family of open covers of X and $y\in Y$. For any $n\in N$, let $\gamma_n=\{\cup\alpha'_n: \alpha'_n$ is a finite subfamily of $\alpha_n\}$. Then γ_n is an open cover of Y that is closed under finite unions. Since f is a covering map, $\gamma'_n=\{X-f(Y-B): B\in\gamma_n\}$ is an open cover of X ([3]). Since X is CLWL, there is an open neighborhood G of $f(y)$ in X and for any $n\in N$, there is a countable subfamily β_n of γ'_n such that $G\subseteq \mathrm{cl}_X(\cup\beta_n)$. Let $\beta_n=\{X-f(Y-B_{n_k}): k\in N$ and $B_{n_k}\in\gamma_n\}$. Then $f^{-1}(G)\subseteq f^{-1}(\mathrm{cl}_X(\cup\{X-f(Y-B_{n_k}): k\in N\}))$. Since $f^{-1}(G)$ is open in Y and f is a covering map,

$$f^{-1}(G)\subseteq \mathrm{cl}_Y(\mathrm{int}_Y(f^{-1}(\mathrm{cl}_X(\cup\{X-f(Y-B_{n_k}): k\in N\}))))$$

$$= \mathrm{cl}_Y(f^{-1}(\cup\{X-f(Y-B_{n_k}): k\in N\}))$$

Note that for any $B\subseteq Y$, $f^{-1}(X-f(Y-B))\subseteq B$. So $f^{-1}(G)\subseteq \mathrm{cl}_Y(f^{-1}(\cup\{X-f(Y-B_{n_k}): k\in N\}))\subseteq \mathrm{cl}_Y(\cup\{B_{n_k}: k\in N\})$. Since each B_{n_k} is a union of a finite subfamily of α_n and hence $\cup\{B_{n_k}: k\in N\}$ is a union of a countable subfamily of α_n. Thus Y is CLWL.

Definition 1.3 A space X is said to be a *basically disconnected space* if for any zero-set Z in X, $\mathrm{int}_X(Z)$ is closed in X.

Let Y be an extension of a space X, then the map $\psi : R(Y) \rightarrow R(X)$, defined by $\psi(A) = A \cap X$, is a Boolean isomorphism and hence for a σ-complete Boolean subalgebra L of $R(Y)$, $L_X = \{A \cap X : A \in L\}$ is a σ-complete Boolean subalgebra of $R(X)$.

Theorem 1.4. Let X be a basically disconnected space and Y a CLWL dense subspace of X. Then Y is a basically disconnected space.

proof. Since X is a basically disconnected space, the set $B(X)$ of clopen sets in X is a σ-complete Boolean subalgebra of $R(X)$ ([4]) and hence $B(X)_Y = \{A \cap Y : A \in B(X)\}$ is also a σ-complete Boolean subalgebra of $R(Y)$. Moreover, $B(X)_Y$ is a base for Y. Take any zero-set Z in Y. Suppose that $\mathrm{cl}_Y(Y-Z) \cap \mathrm{cl}_Y(\mathrm{int}_Y(Z)) \neq \emptyset$. Pick $y \in \mathrm{cl}_Y(Y-Z) \cap \mathrm{cl}_Y(\mathrm{int}_Y(Z))$. Since $Y-Z$ is a cozero-set in Y, there is a sequence (A_n) of closed sets in Y such that $Y-Z = \bigcup\{A_n : n \in N\} = \bigcup\{\mathrm{int}_Y(A_n) : n \in N\}$. Since $(Y-Z) \cap \mathrm{cl}_Y(\mathrm{int}_Y(Z)) = \emptyset$, for any $n \in N$, $A_n \cap \mathrm{cl}_Y(\mathrm{int}_Y(Z)) = \emptyset$ and hence $(Y-A_n) \cup (Y - \mathrm{cl}_Y(\mathrm{int}_Y(Z))) = Y$. Since $B(X)_Y$ is a base for Y, for any $n \in N$, there is a subfamily α_n of $B(X)_Y$ such that $Y - A_n = \bigcup \alpha_n$ and hence $\gamma_n = \alpha_n \cup \{Y - \mathrm{cl}_Y(\mathrm{int}_Y(Z))\}$ is an open cover of Y. Since Y is CLWL, there is a clopen neighborhood G of y in Y and for any $n \in N$, there is a countable subfamily β_n of α_n such that $G \subseteq \mathrm{cl}_Y(\bigcup \beta_n) \cup cl_Y(int_Y(Y - \mathrm{cl}_Y(\mathrm{int}_Y(Z)))) = \mathrm{cl}_Y(\bigcup \beta_n) \cup \mathrm{cl}_Y(Y-Z)$. Since β_n is a countable subfamily of $B(X)_Y$, $\mathrm{cl}_Y(\bigcup \beta_n) \in B(X)_Y$. For any $n \in N$, let $D_n = \mathrm{cl}_Y(\bigcup \beta_n)$. Then for any $n \in N$, $G \subseteq D_n \cup \mathrm{cl}_Y(Y-Z)$ and so $G \subseteq (\bigcap\{D_n : n \in N\}) \cup \mathrm{cl}_Y(Y-Z)$. Thus $G \cap (\bigcup\{Y - D_n : n \in N\}) \cap \mathrm{int}_Y(Z) = \emptyset$. Note that for any $n \in N$, $D_n \subseteq (Y - A_n)$ and hence $\mathrm{int}_Y(A_n) \subseteq (Y - D_n)$. So $Y - Z \subseteq \bigcup\{Y - D_n : n \in N\}$. Since $y \in \mathrm{cl}_Y(Y-Z)$, $y \in \mathrm{cl}_Y(\bigcup\{Y - D_n : n \in N\})$ and since for any $n \in N$, $Y - D_n \in B(X)_Y$,

$cl_Y(\bigcup\{Y-D_n : n\in N\})$ is clopen in Y. Since $G\cap(\bigcup\{Y-D_n : n\in N\})\cap int_Y(Z) = \emptyset$ and G, $int_Y(Z)$ are open in Y, $G\cap cl_Y(\bigcup\{Y-D_n : n\in N\})\cap int_Y(Z)=\emptyset$ and since $cl_Y(\bigcup\{Y-D_n : n\in N\})$ is clopen in Y, $G\cap cl_Y(\bigcup\{Y-D_n : n\in N\})) \cap cl_Y(int_Y(Z))=\emptyset$. This is a contradiction. So $cl_Y(Y-Z)\cap cl_Y(int_Y(Z))=\emptyset$ and hence $cl_Y(int_Y(Z))\subseteq Y-cl_Y(Y-Z) = int_Y(Z)$. Thus Y is a basically disconnected space.

Definition 1.5. Let X be a space.

(a) A pair (Y,f) is called a *cover of* X if $f: Y\to X$ is a covering map.

(b) A cover (Y,f) of X is called a *basically disconnected cover of* X if Y is a basically disconnected space.

(c) A basically disconnected cover (Y,f) of X is called a *minimal basically disconnected cover of* X if for any basically disconnected cover (Z,g) of X, there is a covering map $h: Z\to Y$ with $f\circ h=g$.

For any space X, let $Z(X)^{\#}=\{cl_X(int_X(Z)) : Z \text{ is a zero-set in } X\}$, then $Z(X)^{\#}$ is a sublattice of $R(X)$. Let L be a complete Boolean algebra and M a sublattice of L. Then there is the smallest σ-complete Boolean subalgebra of L containing M, denoted by σM. Vermeer ([4]) showed that every space X has a minimal basically disconnected cover $(\Lambda X,\Lambda_X)$ and that if X is a compact space, then ΛX is the Stone-space of $\sigma Z(X)^{\#}$ and $\Lambda_X(\alpha)=\bigcap\alpha$ $(\alpha\in\Lambda X)$. In [2], for any locally weakly Lindelöf space X, the minimal basically disconnected cover $(\Lambda X,\Lambda_X)$ of X is characterized by : $\Lambda X=\{\alpha : \alpha \text{ is a fixed } \sigma Z(X)^{\#}\text{-ultrafilter on } X\}$ with the topology generated by $\{\Lambda X-A^* : A\in\sigma Z(X)^{\#}\}$ and $\Lambda_X(\alpha)=\bigcap\alpha$, where $A^*=\{\alpha\in\Lambda X : A\in\alpha\}$. For any space X, let $(\Lambda X,\Lambda_X)$ $((\Lambda\beta X,\Lambda_\beta)$, resp.) denote the minimal basically disconnected cover of X (βX, resp.).

Let X, Y be spaces and $f: Y\to X$ a continuous map. For any $U\subseteq X$, let $f_U: f^{-1}(U)\to U$ denote the restriction and corestriction of f to $f^{-1}(U)$ and U, respectively. We can easily show that if X and Y are compact spaces and $f: Y\to X$ is a covering map, then for any dense subspace U of X, $f_U: f^{-1}(U)\to U$ is a covering map. By Theorem 1.2 and Theorem 1.4, we have the following:

Corollary 1.6. For any CLWL space X, $\Lambda_{\beta}^{-1}(X)$ is a basically disconnected space.

Lemma 1.7. ([2]) Let X be a space. If $\Lambda_{\beta}^{-1}(X)$ is a basically disconnected space, then $(\Lambda_{\beta}^{-1}(X), \Lambda_{\beta_X})$, is the minimal basically disconnected cover of X.

Corollary 1.8. For any CLWL space X, $(\Lambda_{\beta}^{-1}(X), \Lambda_{\beta_X})$ is the minimal basically disconnected cover of X.

For any space X, the isomorphism $\psi: R(\beta X) \to R(X)$ $(\psi(A) = A \cap X)$ induces a Boolean isomorphism $\sigma Z(\beta X)^{\#} \to \sigma Z(X)^{\#}$. Using this, we have the following:

Corollary 1.9. For any CLWL space X,
$\Lambda X = \{\alpha : \alpha$ is a fixed $\sigma Z(X)^{\#}$-ultrafilter on X$\}$ with the topology generated by $\{\Lambda X - A^{*} : A \in \sigma Z(X)^{\#}\}$ and $\Lambda_X(\alpha) = \bigcap \alpha$, where $A^{*} = \{\alpha \in \Lambda X : A \in \alpha\}$.

References

1. W. W. Comport, N. Hindman, and S. Negrepontis, F'-space and their products with P-space, Pasific J. Math. 28(1969) 489 - 502.
2. C. I. Kim, Minimal covers and filter spaces, Topol. and its Appl. 72(1996) 31-37.
3. J. Porter and R. G. Woods, Extensions and Absolutes of Hausdorff Spaces, Springer, Berlin, 1988.
4. J. Vermeer, The smallest basically disconnected preimage of a space, Topol. and its Appl. 17(1984) 217-232.