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A GENERALIZATION OF A SUBSET-SUM-DISTINCT SEQUENCE

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Abstract. In 1967, as an answer to the question of P. Erdös on a set of integers having distinct subset sums, J. Conway and R. Guy constructed an interesting sequence of sets of integers. They conjectured that these sets have distinct subset sums and that they are close to the best possible with respect to the largest element. About 30 years later (in 1996), T. Bohman could prove that sets from the Conway-Guy sequence actually have distinct subset sums. In this paper, we generalize the concept of subset-sum-distinctness to k-SSD, the k-fold version. The classical subset-sum-distinct sets would be 1-SSD in our definition. We prove that similarly derived sequences as the Conway-Guy sequence are k-SSD.

1. Introduction

We begin the paper with an interesting puzzle: suppose there are six piles of money, each consisting of 100 silver coins. All the coins in one pile are counterfeit, but we do not know which pile it is. We know the correct weight of a legal coin, and we know that the counterfeit coin weighs one gram less than a genuine one. Using a spring scale, identify the pile of counterfeit coins by only one weighing.

The answer for this puzzle is easy: after numbering each pile from 1 to 6, choose i coins from each ith pile, $1 \leq i \leq 6$. Weighing them all together, one can tell the pile of counterfeit coins. For example, if it lacks 5 grams of the expected weight of 21 $(= 1+2+3+4+5+6)$ legal coins, then the fifth pile is the counterfeit. Let us change the puzzle a little: all conditions are the same as above but we do not know how many piles are counterfeit. At this time, one may choose 2^{i-1} coins from ith pile

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for $1 \leq i \leq 6$. Weighing them together, if it lacks $5\ (= 1 + 2^2)$ grams of the expected weight, the first and the third piles are the counterfeit.

When we consider the changed puzzle with the constraint that every pile has only 24 coins, we arrive at the concept of "distinct subset sums". A set of real numbers is said to have distinct subset sums if no two finite subsets have the same sum. To be precise, we define

DEFINITION 1.1. (i) Let A be a set of real numbers. We say that A has the subset-sum-distinct property (briefly SSD-property) if for any two finite subsets X, Y of A,

$$\sum_{x \in X} x = \sum_{y \in Y} y \qquad \Longrightarrow \qquad X = Y.$$

Also, we say that A is SSD or A is an SSD-set if it has the SSD-property.

(ii) A sequence of positive integers $\{a_n\}_{n=1}^{\infty}$ is called a subset-sum-distinct sequence (or briefly, an SSD-sequence) if it has the SSD-property.

With this terminology, to answer for the final puzzle, we just need an SSD-set of six positive integers whose greatest element is less than or equal to 24. In other words, we are very interested in a "dense" SSD-set. In fact, problems related to dense SSD-sets have been considered by many mathematicians in various contexts (see [1, pp.47–48], [2]-[10], [11, pp.59–60], [12, p.114, problem C8], [13]-[16]). The greedy algorithm generates one of the most natural SSD-sequence $\{1, 2, 2^2, 2^3, \dots\}$ which is quite sparse. In 1967, on the request of "dense" SSD-sets, John Conway and Richard Guy constructed so called "Conway-Guy sequence" as following ([13]). First, define an auxiliary sequence u_n by

$$u_0 = 0, u_1 = 1 \ \text{ and } \ u_{n+1} = 2u_n - u_{n-r}, \qquad n \geq 1$$

where $r = \langle\sqrt{2n}\rangle$, the nearest integer to $\sqrt{2n}$. Now, for a given positive integer n, we define

$$a_i = u_n - u_{n-i}, \ 1 \leq i \leq n.$$

The well known Conway-Guy conjecture is that $\{a_i : 1 \leq i \leq n\}$ is SSD for any positive integer n. F. Lunnon showed numerically that they are SSD-sets for $n < 80$ (see [16, p.307, Theorem 4.6]) and the conjecture was completely resolved affirmatively by T. Bohman in 1996 (see [6]).

Note that, in the last modified puzzle, the Conway-Guy sequence gives the unique answer $\{11, 17, 20, 22, 23, 24\}$ when $n = 6$.

2. A generalization

Consider the puzzle of counterfeit coin in the introduction again. This time, we assume that each pile has 169 coins and we know that a counterfeit coin weighs one or two grams less than a genuine one. Also we know that any two coins in the same pile have the same weight. But we do not know how many piles are counterfeit. In this case, we need a set $S = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ of positive integers such that $1 \le a_1 \le a_2 \le a_3 \le a_4 \le a_5 \le a_6 \le 169$ and all sums $\sum_{i=1}^{6} \epsilon_i a_i$ are different when each integer ϵ_i varies from 0 to 2. This means that S is not only subset-sum-distinct but also 2-fold subset-sum-distinct. We define these concepts in full generalization.

DEFINITION 2.1. (i) For a set A of real numbers, we say that A has the k-fold subset-sum-distinct property (briefly k-SSD-property) if for any two finite subsets X, Y of A,

$$\sum_{x\in X} \epsilon_x \cdot x = \sum_{y\in Y} \epsilon_y \cdot y \text{ for some } \epsilon_x,\ \epsilon_y \in \{1,2,\dots,k\} \quad \text{implies} \quad X = Y.$$

Also, we say that A is k-SSD or A is a k-SSD-set if it has the k-SSD-property.

(ii) A sequence of positive integers $\{a_n\}_{n=1}^{\infty}$ is called a k-fold subset-sum-distinct sequence (briefly, k-SSD-sequence) if it has the k-SSD-property.

Note that a classical SSD-set is just a 1-SSD-set. Note also that the greedy algorithm produces the k-SSD-sequence $1, k+1, (k+1)^2, (k+1)^3, \dots$.

For the answer of above puzzle, we need a 2-SSD-set of six elements with minimal height. Lots of calculations shows that $\{109, 147, 161, 166, 168, 169\}$ is the unique answer. That is, after choosing 109, 147, 161, 166, 168, 169 coins from the piles, respectively, one weighs them together. If the scale indicates, for example, $722\ (= 2\cdot 109 + 166 + 2\cdot 169)$ grams less than the expected weight of $920\ (= 109 + 147 + 161 + 166 + 168 + 169)$ genuine coins, one can conclude that every coin in the fourth pile lacks one gram and every coin in the first and the sixth piles lacks two grams.

To obtain dense k-SSD-sets, we immitate the construction of the Conway-Guy sequence. Following Bohman ([6]), we use the difference sequence $d(n)$ instead of using the auxiliary sequence $u(n)$. For $n \geq 1$, define $b(n) = \langle\sqrt{2(n-1)}\rangle$, the nearest integer to $\sqrt{2(n-1)}$. Also, define $d(1) = 1$ and

$$d(n) = \sum_{i=n-b(n)}^{n-1} d(i).$$

Then the sets in the Conway-Guy sequence are

$$S_n = \left\{ \sum_{i=j}^{n} d(i) \mid j = 1, 2, \dots, n \right\}.$$

DEFINITION 2.2. (A generalization of the Conway-Guy sequence) For an integer $k \geq 1$, we define $d_k(1) = 1$, $S_k^1 = \{d_k(1) = 1\}$ and for $n > 1$,

$$d_k(n) = \sum_{i=n-b(n)}^{n-1} k \cdot d_k(i), \quad S_k^n = \left\{ \sum_{i=j}^{n} d_k(i) \mid j = 1, 2, \dots, n \right\}.$$

Note that previously defined $d(n)$, S_n is simply $d_1(n)$, S_1^n, respectively.

Here we enumerate the first few terms of these sequences:
$b(n) : 0,\ 1,\ 2,\ 2,\ 3,\ 3,\ 3,\ 4,\ 4,\ 4,\ 4,\ 5,\ 5,\ \dots$
$d_1(n) : 1,\ 1,\ 2,\ 3,\ 6,\ 11,\ 20,\ 40,\ 77,\ 148,\ 285,\ 570,\ \dots$
$S_1^1 = \{1\}$, $S_1^2 = \{2,\ 1\}$, $S_1^3 = \{4,\ 3,\ 2\}$, $S_1^4 = \{7,\ 6,\ 5,\ 3\}$,
$S_1^5 = \{13,\ 12,\ 11,\ 9,\ 6\}$, $S_1^6 = \{24,\ 23,\ 22,\ 20,\ 17,\ 11\}, \dots$

$d_2(n) : 1,\ 2,\ 6,\ 16,\ 48,\ 140,\ 408,\ 1224, \dots$
$S_2^1 = \{1\}$, $S_2^2 = \{3,\ 2\}$, $S_2^3 = \{9,\ 8,\ 6\}$, $S_2^4 = \{25,\ 24,\ 22,\ 16\}, \dots$

....

$d_k(n) : 1,\ k,\ k(k+1),\ k^2(k+2), \dots$
$S_k^1 = \{1\}$, $S_k^2 = \{k+1,\ k\}$, $S_k^3 = \{(k+1)^2,\ k(k+2),\ k(k+1)\}$,
$S_k^4 = \{k^3+3k^2+2k+1,\ k^3+3k^2+2k,\ k^3+3k^2+k,\ k^3+2k^2\}, \dots$.

One can ask about the size of the largest element of the set S_k^n. For $k = 1$, Erdös and Moser showed that the largest element of S_1^n had a lower bound $C \cdot 2^n/\sqrt{n}$ for some positive constant C (see [2, p.36, Theorem 3.2], [13, p.142]). We note that, using a modified idea of Moser, one can show the largest element of S_k^n has a lower bound

$$C \cdot \frac{(k'+1)^n}{k'\sqrt{n}} \quad \text{where} \quad k' = \begin{cases} k & \text{if } k \text{ is odd} \\ k-1 & \text{if } k \text{ is even} \end{cases}$$

and C is a positive constant which is absolutely independent of k and n. For an acute upper bound of $f(n) = \min\{\max S : |S| = n \text{ and } S \text{ is 1-SSD}\}$, T. Bohman introduced many variations of the Conway-Guy construction (see [7]). We believe that his method can be validly applied on the k-SSD setting after appropriate defining of b-sequences and d-sequences.

Lastly, as we mentioned in the introduction, T. Bohman showed S_1^n is 1-SSD for any positive integer n (see [6, p.3630, Theorem 1]). In the next section, by modifying Bohman's idea a little, we will prove that S_k^n is k-SSD for any positive integers k and n.

3. Main theorem

We prove our main theorem that S_k^n is k-SSD for any positive integers k and n. First, by a *vector*, we mean an *element of* $\mathbf{N}^\infty$ where $\mathbf{N}$ is the set of all positive integres.

DEFINITION 3.1. Let v be a vector.

(i) We say that v is an n-dimensional if n is the largest integer such that $v(n) \neq 0$.

(ii) An n-dimensional vector v is called k-smooth if $|v(1)| \leq k$ and $|v(i) - v(i+1)| \leq k$ for all $i \in \{1, 2, \dots, n-1\}$.

(iii) For an arbitrary set $S = \{a_n > a_{n-1} > \cdots > a_2 > a_1\}$ of positive integers, we define the n-dimensional difference vector d_S by $d_S = (a_n - a_{n-1}, a_{n-1} - a_{n-2}, \dots, a_2 - a_1, a_1, 0, 0, 0, \dots)$.

The following lemma gives an equivalent condition for a set to be k-SSD.

LEMMA 3.2. *Let S be a set of n positive integers. Then S is k-SSD if and only if there exists no nonzero, k-smooth, n-dimensional vector v such that the inner product $v \cdot d_S = 0$.*

Proof. ($\Rightarrow$) Let $S = \{a_n > a_{n-1} > \cdots > a_2 > a_1\}$. Suppose there exists a nonzero, k-smooth, n-dimensional vector v such that the inner product $v \cdot d_S = 0$. Then we have

$$\begin{aligned} 0 &= v \cdot d_S \\ &= v(1)(a_n - a_{n-1}) + v(2)(a_{n-1} - a_{n-2}) + \cdots \\ &\quad + v(n-1)(a_2 - a_1) + v(n)\, a_1 \\ &= v(1)\, a_n + (v(2) - v(1))\, a_{n-1} + \cdots + (v(n) - v(n-1))\, a_1. \end{aligned}$$

Since v is a nonzero n-dimensional vector, not all the coefficients of the a_i's in the last expression are zero. This implies S is not k-SSD.
($\Leftarrow$) Suppose S is not k-SSD. Then we can find two disjoint subsets X, Y of S, such that

$$\sum_{x \in X} \epsilon_x \cdot x - \sum_{y \in Y} \epsilon_y \cdot y = 0 \quad \text{for some} \quad \epsilon_x,\, \epsilon_y \in \{1, 2, \dots, k\}.$$

For appropriate choices of α_x and β_y, we have

$$x = \sum_{i=\alpha_x}^{n} d_S(i), \quad y = \sum_{i=\beta_y}^{n} d_S(i).$$

Notice that the α_x's and β_y's are all distinct. We can write

$$\sum_{x \in X} \epsilon_x \sum_{i=\alpha_x}^{n} d_S(i) - \sum_{y \in Y} \epsilon_y \sum_{i=\beta_y}^{n} d_S(i) = 0.$$

In order to reverse the order of summation, we must count how many times each $d_S(i)$ appears in this summation. This is achieved by setting

$$v(j) = \sum_{\alpha_x \le j} \epsilon_x - \sum_{\beta_y \le j} \epsilon_y \quad \text{for} \quad 1 \le j \le n.$$

Also, set $v(j) = 0$ for all $j > n$. Then $v \cdot d_S$ is the sum on the left hand side of the above equation, and v is k-smooth by the distictness of the α_x's and β_y's. □

LEMMA 3.3. *The sequence* $\{n - b(n)\}_{n=1}^{\infty}$ *is non-decreasing for* $n = 1, 2, 3, \dots$.

Proof. By using simple induction, we see immediately that $\{n - b(n)\}_{n=1}^{\infty}$ is non-decreasing. □

LEMMA 3.4. *For* $k \geq 1$, $d_k(n) + k \cdot \sum_{i=1}^{n} d_k(i) \leq d_k(n+2)$ *for all* $n \geq 1$.

Proof. We use an induction on n. When $n = 1$, we have $d_k(1) + k\, d_k(1) = 1 + k \leq k(k+1) = d_k(3)$. Assume $d_k(n) + k\sum_{i=1}^{n} d_k(i) < d_k(n+2)$. Since

$$\begin{aligned} d_1(n+1) &= \sum_{i=n+1-b(n+1)}^{n} d_1(i) \\ &= \sum_{i=n+1-b(n+1)}^{n-1} d_1(i) + d_1(n) \\ &\leq \sum_{i=n-b(n)}^{n-1} d_1(i) + d_1(n) = 2\, d_1(n)\,, \end{aligned}$$

we know that $d_k(n+1) \leq 2\, d_k(n) + (k-1)\, d_k(n+2)$ for all $k \geq 1$. Thus

$$\begin{aligned} &d_k(n+1) + k\sum_{i=1}^{n+1} d_k(i) \\ &= d_k(n+1) + k\, d_k(n+1) + k\sum_{i=1}^{n} d_k(i) \\ &\leq d_k(n+1) + k\, d_k(n+1) + d_k(n+2) - d_k(n) \\ &\leq 2\, d_k(n) + (k-1)\, d_k(n+2) + k\, d_k(n+1) + d_k(n+2) - d_k(n) \\ &\leq k\, d_k(n) + k\, d_k(n+1) + k\, d_k(n+2) \\ &\leq \sum_{i=n+3-b(n+3)}^{n+2} k\, d_k(i) \quad = \quad d_k(n+3) \end{aligned}$$

where the last inequality follows from the fact that $b(n+3)$ is at least 3 for $n \geq 2$. □

LEMMA 3.5. *If* v *is an* n*-dimensional* k*-smooth vector such that* $|v(n)| \leq k$, *then* $v \cdot d_k < d_k(n+1) + d_k(n+2)$.

Proof. We go by induction on n. For $n = 1$, $v \cdot d_k = v(1) \cdot d_k(1) = v(1) \leq k < k + k(k+1) = d_k(2) + d_k(3)$. Define $z(i) = \max\{v(i) - k,\ 0\}$ for $i \geq 1$. Note $z(1) = 0$. We show that z is k-smooth. Put $\delta = z(i) - z(i+1) = \max\{v(i) - k,\ 0\} - \max\{v(i+1) - k,\ 0\}$. Suppose at first that $z(i) = 0$ or equivalently that $v(i) \leq k$. Then we have $v(i+1) - k \leq v(i) \leq k$ from the smoothness of v. Hence $|\delta| = \max\{v(i+1) - k,\ 0\} \leq k$. Secondly, suppose $z(i) = v(i) - k$ or $v(i) \geq k$. If $z(i+1) = \max\{v(i+1) - k,\ 0\} = v(i+1) - k$, then $|\delta| = |v(i) - k - (v(i+1) - k)| = |v(i) - v(i+1)| \leq k$. If $z(i+1) = \max\{v(i+1) - k,\ 0\} = 0$, then $k \leq v(i) \leq v(i+1) + k \leq 2k$ which implies $0 \leq v(i) - k \leq k$ or $|\delta| = |v(i) - k| \leq k$. Therefore the k-smoothness of z is shown. Note $z(n) = 0$ and $|z(n-1)| \leq k$. Hence by applying the inductive hypothesis to the $n-1$ components of the vector z, we get $z \cdot d_k < d_k(n) + d_k(n+1)$. Letting $\bar{k} = (k, k, k, k, \ldots, k, 0, 0, 0, \ldots)$ of $n-1$ dimension, we obtain

$$\begin{aligned} v \cdot d_k &= (v - \bar{k}) \cdot d_k + \bar{k} \cdot d_k \\ &\leq z \cdot d_k + v(n)\, d_k(n) + \bar{k} \cdot d_k \leq z \cdot d_k + k\, d_k(n) + \bar{k} \cdot d_k \\ &< d_k(n) + d_k(n+1) + k \sum_{i=1}^{n} d_k(i) \\ &\leq d_k(n+1) + d_k(n+2). \end{aligned}$$

The last inequality follows from Lemma 3.4. □

THEOREM 3.6. *For all $m \geq 1$, S_k^m is k-SSD.*

Proof. By Lemma 3.2, it suffices to prove that

$$v \cdot d_k \neq 0 \tag{3.1}$$

for any nonzero k-smooth vector v. Again we use an induction on the dimension of v. Assume that (3.1) is true for any nonzero k-smooth vector of dimension $m-1$. Let $\dim(v) = m$. We may assume $v(m) > 0$. We will construct $m-1$ vectors $w_m, w_{m-1}, \ldots, w_2$ such that for all $i \in \{2, 3, \ldots, m\}$,

(i) $w_i(j) = v(j)$ for $j \geq i$,
(ii) $w_i \cdot d_k = 0$,
(iii) w_i is not k-smooth.

Once these are constructed, since $w_2 \cdot d_k = 0$ and v differs from w_2 in the first component only, the third property (iii) shows that $v \cdot d_k \neq 0$. First, we define vectors $X_2, X_3, X_4, \dots$ by

$$X_i(j) = \begin{cases} 1 & \text{if } i = j \\ -k & \text{if } i - b(i) \le j \le i-1 \\ 0 & \text{otherwise} \end{cases}$$

where $i \ge 2$. Also we define $\rho_m = v(m)$, $w_m = \rho_m X_m$, $\rho_i = v(i) - w_{(i+1)}(i)$, $w_i = w_{i+1} + \rho_i X_i$ for $i = m-1, m-2, \dots, 2$. Note that the vectors X_i's are defined so that $X_i \cdot d_k = 0$. Since w_i's are just linear combinations of X_i's, one can easily see the properties (i) and (ii). □

CLAIM 1. *For* $i \in \{2, 3, 4, \dots, m\}$, $w_i(j) = \sum_{l=i}^{m} \rho_l X_l(j)$.

Proof. Let's use the reverse induction on $i = m, m-1, m-2, \dots, 2$. For $i = m$, we immediately have $w_m(j) = \rho_m X_m(j)$ from the definition. Assuming $w_{i+1}(j) = \sum_{l=i+1}^{m} \rho_l X_l(j)$, we see that $w_i(j) = w_{i+1}(j) + \rho_i X_i(j) = \sum_{l=i+1}^{m} \rho_l X_l(j) + \rho_i X_i(j) = \sum_{l=i}^{m} \rho_l X_l(j)$. □

CLAIM 2. *For* $m - b(m) \le i \le m-1$, $\rho_i > 0$.

Proof. Suppose inductively that $\rho_{i+1}, \dots, \rho_m > 0$. Then

$$w_{i+1}(i) = \sum_{j=i+1}^{m} \rho_j X_j(i) = -k \sum_{j=i+1}^{m} \rho_j \le -k(m-i).$$

Since $v(m) \ge 1$, we have $\rho_i = v(i) - w_{i+1}(i) \ge v(i) + k(m-i) \ge 1 - k(m-i) + k(m-i) = 1$. □

CLAIM 3. *If* $\rho_t, \dots, \rho_m > 0$, *then* w_t *is not* k*-smooth.*

Proof. If $m - b(m) < t \le m-1$, then $w_t(t-1) = -k \sum_{j=t}^{m} \rho_j$ and $w_t(t) = w_{t+1}(t) + \rho_t X_t(t) = \rho_t - k \sum_{j=t+1}^{m} \rho_j$. Thus $w_t(t) - w_t(t-1) = (k+1)\rho_t > k$ which means that w_t is not k-smooth. Suppose $t \le m - b(m)$.

(1) If $t = 2$, then $m \ge 4$ and $w_2(1) = -k(\rho_2 + \rho_3) < -k$. Hence $w_t (= w_2)$ is not k-smooth.

(2) If $t = 3$, then $m \ge 6$ and $w_3(1) = -k\rho_3$, $w_3(2) = -k(\rho_3 + \rho_4 + \rho_5)$. Thus $w_3(1) - w_3(2) = k(\rho_4 + \rho_5) > k$ and so w_t $(= w_3)$ is not k-smooth.

(3) If $4 \le t \le m - b(m)$, then we define $c(u) = \max\{j | j - b(j) \le u\}$. Note $t - b(t) - 1 \ge 1$ and
(3.2)
$$w_t(i) = \sum_{l=t}^{m} \rho_l X_l(i) = \begin{cases} 0 & \text{if } 1 \le i < t - b(t) \\ -k \sum_{l=t}^{c(i)} \rho_l & \text{if } t - b(t) \le i \le t - 1 \,. \end{cases}$$

Since $c(k)$ is strictly increasing, $w_t(t-b(t)), \dots, w_t(t-1)$ is a strictly decreasing sequence of negative numbers. We will show that there exists s such that
(3.3)
$$t-b(t)-1 \le s \le t-2 \quad \text{and} \quad c(s+1) = c(s)+2 \quad \text{and} \quad c(s+1) \ge t+1.$$

If $s = t - b(t) - 1$, then $w_t(s) = 0$ and $w_t(s+1) = -k\sum_{l=t}^{c(s+1)} \rho_l \le -k\sum_{l=t}^{t+1} \rho_l \le -2k$. Hence $w_t(s+1) \le w_t(s) - 2k$ and so w_t is not k-smooth. If $t - b(t) \le s \le t - 1$, then we obtain

$$\begin{aligned} w_t(s+1) &= -k \sum_{l=t}^{c(s+1)} \rho_l = -k \sum_{l=t}^{c(s)+2} \rho_l \\ &= -k \sum_{l=t}^{c(s)} \rho_l - k\rho_{c(s)+1} - k\rho_{c(s)+2} \le w_t(s) - 2k \end{aligned}$$

whence w_t is not k-smooth again.

Now, it remains to show the existence of such an s. Let $y = c(t-1)$. Note $y < m$. Since $y + 1 - b(y+1) > t - 1$, we have $b(y+1) = b(y)$. Applying the fact that $b(n - b(n)) < b(n)$ for all $n \ge 2$, we get $b(t) = b(y+1-b(y+1)) < b(y+1) = b(y)$. Hence there exists u such that $t \le u < y$ and $b(u+1) = b(u)+1$. Taking $s = u-b(u)-1$, we see $t-b(t)-1 \le s = u-b(u)-1 \le y-b(y)-1 \le t-1-1 = t-2$. Also, note $b(u+1) = b(u)+1 \implies 1-b(u+1) \le -b(u) \implies u+1-b(u+1) \le u - b(u) = s+1 \implies u+1 \le c(s+1) \implies t+1 \le c(s+1)$. Since $c(s+1) = u+1$ and $c(s) = u-1$, we get $c(s+1) = c(s)+2$. □

Claim 4. *If there exists i such that $1 < i < m$, $\rho_i \le 0$ then $v \cdot d_k \ne 0$.*

Proof. Choose t so that $\rho_i > 0$ for $t \le i \le m$ and $\rho_{t-1} \le 0$. By claim 2, $t \le m - b(m)$. If $t = 3$, then $m \ge 6$ and $w_3(2) = -k(\rho_3 + \rho_4 + \rho_5)$. Since v is k-smooth, we have $v(2) \ge -2k > w_3(2)$. But

$0 \geq \rho_2 = v(2) - w_3(2)$ implies $w_3(2) \geq v(2)$, a contradiction. Hence we may assume $4 \leq t \leq m - b(m)$ whence (3.2) and (3.3) hold. In claim 3, we proved that $w_t(i) = 0$ for $1 \leq i < t - b(t)$ and $w_t(t - b(t)), \ldots, w_t(t-1)$ is a strictly decreasing sequence of negative numbers and there exists s with $t - b(t) - 1 \leq s \leq t - 2$ and $w_t(s) - 2k \geq w_t(s+1)$.

Now consider the vector $z = v - w_t$. Note that $z \cdot d_k = v \cdot d_k - w_t \cdot d_k = v \cdot d_k$. Also, by the property (i) of w_t, we have $z(i) = 0$ for $i \geq t$. Since $0 \geq \rho_{t-1} = v(t-1) - w_t(t-1)$, we get $v(t-1) \leq w_t(t-1)$. Observing (3.2) for $i = t - b(t) - 1$, we have a strictly decreasing sequence

$$w_t(t - b(t) - 1) > w_t(t - b(t)) > \cdots > w_t(t-2) > w_t(t-1)$$

with the gaps of positive multiples of k. Combining k-smoothness of v with the fact $v(t-1) \leq w_t(t-1)$, we have $v(i) \leq w_t(i)$ for all i such that $t - b(t) - 1 \leq i \leq t - 1$. Furthermore, the existence of the double jump between $w_t(s)$ and $w_t(s+1)$ implies that $w_t(i) > v(i)$ for all $t - b(t) - 1 \leq i \leq s$. Hence we have

$$z(i) \begin{cases} < 0 & \text{if } t - b(t) - 1 \leq i \leq s \\ \leq 0 & \text{if } t - b(t) - 1 \leq i \leq t - 1. \end{cases}$$

For $1 \leq i < t - b(t) - 1$, since $w_t(i) = 0$ and so $z(i) = v(i)$, we see that z is k-smooth on this interval. If $z(i) < 0$ for all $i \leq s$, then clearly $0 > z \cdot d_k = v \cdot d_k$. Suppose there exists some $i < t - b(t) - 1$ with $z(i) \geq 0$. Let $q = \max\{i < t - b(t) - 1 \mid z(i) \geq 0\}$ and let $y = (z(1), z(2), \ldots, z(q), 0, 0, \ldots)$. Then y is a k-smooth q-dimensional vector. We consider two cases seperately:

Case 1. There exists $r > q + 1$ such that $z(r) < 0$.
Applying Lemma 3.5 to the q-dimensional vector y, we have $0 > y \cdot d_k - d_k(q+1) - d_k(q+2) \geq y \cdot d_k - d_k(q+1) - d_k(r) \geq z \cdot d_k = v \cdot d_k$.

Case 2. For all $r > q + 1$, $z(r) = 0$.
In this case, we have $s = q + 1 = t - b(t) - 1$. Hence $z(i) = v(i)$ for $1 \leq i \leq q + 1$ and $z(i) = 0$ for $i > q + 1$ and so z is $(q+1)$-dimensional k-smooth vector. Note $q + 1 = t - b(t) - 1 \leq t - 2 < m$. Thus by the induction (on m), $z \cdot d_k (= v \cdot d_k) \neq 0$. □

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