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# ON THE SPECIAL FINSLER METRIC

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ABSTRACT. Given a Riemannian manifold $(M, \alpha)$ with an almost Hermitian structure $f$ and a non-vanishing covariant vector field $b$, consider the generalized Randers metric $L = \alpha + \beta$, where $\beta$ is a special singular Riemannian metric defined by $b$ and $f$. This metric $L$ is called an $(a, b, f)$-metric. We compute the inverse and the determinant of the fundamental tensor $(g_{ij})$ of an $(a, b, f)$-metric. Then we determine the maximal domain $\mathcal{D}$ of $TM \setminus O$ for an $(a, b, f)$-manifold where a $y$-local Finsler structure $L$ is defined. And then we show that any $(a, b, f)$-manifold is quasi-C-reducible and find a condition under which an $(a, b, f)$-manifold is C-reducible.

## 1. Introduction

Let $M$ be a smooth $2m$-dimensional manifold. We will consider a Finsler metric $L = \alpha + \beta$, where $\alpha$ is a Riemannian metric on $M$ and $\beta$ is a singular Riemannian metric on $M$. We call such a Finsler metric a generalized Randers metric. In case where $\beta$ is a 1-form on $M$, $L$ is a usual Randers metric.

We denote a point of $M$ by $x = (x^i)$ and a tangent vector at that point $x$ by $y = (y^i)$. Let $\alpha(x, y) = (a_{ij}(x) y^i y^j)^{1/2}$ be a Riemannian metric on $M$. Given an almost Hermitian structure $f^i_j(x)$ of $(M, \alpha)$ and a non-vanishing covariant vector field $b_i(x)$ on $M$, we have a singular Riemannian metric $\beta(x, y) = (b_{ij}(x) y^i y^j)^{1/2}$, where $b_{ij} = b_i b_j + f_i f_j$ and $f_i = b_r f^r_i$. Such $L = \alpha + \beta$ is an interesting example of a generalized Randers metric, which we call an $(a, b, f)$-metric. For the further study about $(a, b, f)$-metrics, we refer to [3] and [4].

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Note that a manifold with an $(a,b,f)$-metric becomes a Rizza manifold. A Rizza manifold $(M,L,f)$ is by definition a Finsler manifold $(M,L)$ with an almost complex structure $f^i_j(x)$ satisfying the condition

$$L(x,\phi_\theta(y)) = L(x,y),$$

where ${\phi_\theta}^i_j = \cos\theta\cdot\delta^i_j + \sin\theta\cdot f^i_j$. But an $(a,b,f)$-metric is not a $y$-global Finsler metric. And so we have to restrict a domain in the tangent bundle $T(M)$ over $M$, say, $\{y : \beta(y)\neq 0\}$. In section 4, we show that the $n\times n$ Hessian matrix $(g_{ij}) := ((\frac{1}{2}L^2)_{y_i y_j})$ is positive definite on $\{y:\beta(y)\neq 0\}$ by checking the sign of determinant of $(g_{ij})$. For this purpose, we compute the determinant of $(g_{ij})$.

It is interesting and valuable to study Finsler space with some important tensors of special form. For example, M. Matsumoto[6] initiated the study of a Finsler metric whose Cartan tensor $A_{ijk} := \frac{L}{4}(L^2)_{y^i y^j y^k}$ satisfies

$$A_{ijk} = \mathfrak{S}_{(ijk)}\{Q_{ij}R_k\},$$

where $Q_{ij}$ is a symmetric Finsler tensor field satisfying $Q_{ij}y^j = 0$ and $R_k$ is assumed to satisfy $R_k y^k = 0$. Here we use the notation $\mathfrak{S}_{(ijk)}$ to denote the summation of the cyclic permutation of indices $i,j,k$, i.e.,

$$\mathfrak{S}_{(ijk)}\{S_{ijk}\} = S_{ijk} + S_{jki} + S_{kij}.$$

In case $R_k = A_k$ with $A_k := g^{ij}A_{ijk}$, the Finsler manifold is called quasi-C-reducible. Furthermore, if $Q_{ij} = \frac{1}{n+1}h_{ij}$ where $h_{ij}$ is the angular metric $h_{ij} := g_{ij} - L_i L_j$, we call the Finsler manifold to be C-reducible. In section 4, we show that any $(a,b,f)$-manifold is quasi-C-reducible and find a sufficient condition that an $(a,b,f)$-manifold is C-reducible. To get $A_k$, we compute the inverse $(g^{ij})$ of $(g_{ij})$.

## 2. Preliminaries

Let $(M,\alpha)$ be a $2m$-dimensional Riemannian manifold and let $f^i_j(x)$ be an almost Hermitian structure of $(M,\alpha)$. For a non-vanishing covariant vector field $b_i(x)$ on $M$, we have a singular Riemannian metric

$$\beta(x,y) = (b_{ij}(x)y^i y^j)^{1/2},$$

where $b_{ij} = b_i b_j + f_i f_j$, $f_i = b_r f^r_i$ and we consider a generalized Randers metric $L=\alpha+\beta$. Such a generalized Randers metric $L=\alpha+\beta$ is called an $(a,b,f)$-metric and $(M,L)$ an $(a,b,f)$-manifold.

Recall the definition of a $y$-global Finsler metric $F$ on $M$.

DEFINITION 2.1. A $y$-global Finsler metric on $M$ is a function $F: TM \to \mathbb{R}$ such that

(F1) Nonnegativity: $F \geq 0$ on $TM$.
(F2) Regularity: $F$ is smooth on $TM \setminus O$.
(F3) Absolute homogeneity: $F(x, \lambda y) = |\lambda| F(x, y)$ for all $\lambda \in \mathbb{R}$.
(F4) Strong convexity: The $n \times n$ Hessian matrix $(g_{ij}) := ((\frac{1}{2}F^2)_{y_i y_j})$ is positive definite at every point of $TM \setminus O$.

Note that for the most important physical applications, the assumptions are too restrictive. And so we have to consider a $y$-local Finsler structure $F$ defined only on a domain $\mathcal{D}$ of $TM \setminus O$ with $\mathcal{D} \cap T_x M \neq \emptyset$ for every $x \in M$.

Now we find the maximal domain $\mathcal{D}$ of $TM \setminus O$ for $(a, b, f)$-metric. Because $L(y) = \alpha(y) + \beta(y)$ is positive for any $y \in TM \setminus O$ and both $\alpha$ and $\beta$ are regular away from $\{y : \beta(y) = 0\} = \ker B$ with $B = (b_{ij})$, our possible domain $\mathcal{D}$ is the complement $\complement(\ker B)$ of $\ker B$. In section 4, we show that $(g_{ij})$ is positive definite on $\complement(\ker B)$, i.e., all the eigenvalues of $(g_{ij})$ are positive on $\complement(\ker B)$.

We use the following lemma extensively in the next section. For its proof, see [1].

LEMMA 2.1. *Let $(P_{ij})$ be a real symmetric non-singular matrix with the inverse $(P^{ij})$. And let $(Q_{ij}) = (P_{ij} \pm c_i c_j)$ with $1 \pm c^2 \neq 0$ and $c^2 := c_i P^{ij} c_j$. Then the matrix $(Q_{ij})$ is non-singular and its inverse is $(Q^{ij}) = (P^{ij} \mp \frac{1}{1 \pm c^2} c^i c^j)$ where $c^i = P^{ij} c_j$ and $\det(Q_{ij}) = (1 \pm c^2) \det(P_{ij})$.*

## 3. The computation of the determinant and the inverse of $(g_{ij})$

In this section, we compute the inverse and the determinant of the fundamental tensor $(g_{ij})$ of $(a, b, f)$-metric. Here we assume that $y \in \complement \ker B$.

For $L = \alpha + \beta$, we have

$$g_{ij} = \frac{L}{\alpha} a_{ij} + \frac{L}{\beta} b_i b_j + \frac{L}{\beta} f_i f_j + L_i L_j - \frac{L}{\alpha} \alpha_i \alpha_j - \frac{L}{\beta} \beta_i \beta_j \,,$$

where $\alpha_i = \frac{\partial \alpha}{\partial y^i}$, $\beta_i = \frac{\partial \beta}{\partial y^i}$, $L_i = \alpha_i + \beta_i$. We put $\alpha^i = a^{ir} \alpha_r$, $\beta^i = a^{ir} \beta_r$, $b^i = a^{ij} b_j$ and $f^i = a^{ij} f_j$. Then we can apply Lemma 2.1 to $(g_{ij})$ five times.

PROPOSITION 3.1. *For the fundamental tensor* $(g_{ij})$ *of an* $(a,b,f)$*-metric* $L=\alpha+\beta$, *the determinant of* $(g_{ij})$ *is*

$$\det(g_{ij})=\frac{L\gamma}{\alpha\beta}\det A$$

*and the inverse* $(g^{ij})$ *of* $(g_{ij})$ *is given by*

$$(3.1)\quad g^{ij}=\frac{\alpha}{L}a^{ij}-\frac{\alpha^2}{\gamma L}b^{ij}+\frac{\alpha^2\gamma}{L^3}\alpha^i\alpha^j-\frac{\alpha}{L^2\beta}(\alpha^i\beta^j+\alpha^j\beta^i)+\frac{\alpha^2}{L\gamma}\beta^i\beta^j,$$

*where* $A=(\frac{L}{\alpha}a_{ij}),\ \gamma=\beta+b^2\alpha,\ b^{ij}=b^ib^j+f^if^j$.

*Proof.* First, we set

$$P_{ij}=\frac{L}{\alpha}a_{ij},\ c_{1i}=\sqrt{\tfrac{L}{\beta}}b_i \text{ and } (Q_1)_{ij}=\frac{L}{\alpha}a_{ij}+\frac{L}{\beta}b_ib_j.$$

Note that $c_1^2=c_{1i}P^{ij}c_{1j}=\frac{\alpha}{\beta}b^2$, where $b^2=a^{ij}b_ib_j$ and $(a^{ij})$ is the inverse of $(a_{ij})$. And note also that $b^2=a^{ij}b_ib_j$ is positive, because $(a_{ij})$ is positive definite. In particular, the quantity $1+c_1^2=\frac{\gamma}{\beta}>0$, where $\gamma=\beta+b^2\alpha>0$. By Lemma 2.1, we have

$$\det Q_1=\frac{\gamma}{\beta}\det\Big(\frac{L}{\alpha}a_{ij}\Big)=\frac{\gamma}{\beta}\det A,$$
$$(Q_1)^{ij}=\frac{\alpha}{L}a^{ij}-\frac{\alpha^2}{\gamma L}b^ib^j.$$

Secondly, let

$$(Q_2)_{ij}=(Q_1)_{ij}+\frac{L}{\beta}f_if_j,\ c_{2i}=\sqrt{\tfrac{L}{\beta}}f_i$$

and apply Lemma 2.1 in the same way. Then we have $c_2^2=c_{2i}(Q_1)^{ij}c_{2j}=\frac{\alpha}{\beta}b^2, 1+c_2^2=\frac{\gamma}{\beta}>0$. And Lemma 2.1 says that

$$\det Q_2=\frac{\gamma}{\beta}\det Q_1=\frac{\gamma^2}{\beta^2}\det A\,,$$
$$(Q_2)^{ij}=\frac{\alpha}{L}a^{ij}-\frac{\alpha^2}{\gamma L}b^{ij}.$$

Thirdly, let

$$(Q_3)_{ij}=(Q_2)_{ij}+L_iL_j,\ c_{3i}=L_i.$$

Then we have $c_3^2 = c_{3i}(Q_2)^{ij}c_{3j} = 1,\ 1 + c_3^2 = 2$. And by Lemma 2.1,

$$\det Q_3 = \frac{2\gamma^2}{\beta^2}\det A\,,$$
$$(Q_3)^{ij} = \frac{\alpha}{L}a^{ij} - \frac{\alpha^2}{\gamma L}b^{ij} - \frac{1}{2L^2}y^i y^j\,.$$

Fourthly, let

$$(Q_4)_{ij} = (Q_3)_{ij} - \frac{L}{\beta}\beta_i\beta_j,\ c_{4i} = \sqrt{\tfrac{L}{\beta}}\beta_i.$$

Then we have $c_4^2 = c_{4i}(Q_3)^{ij}c_{4j} = \frac{1}{\beta}\Big(b^2\alpha - \frac{b^4\alpha^2}{\gamma} - \frac{\beta^2}{2L}\Big), 1 - c_4^2 = \frac{\beta(2L+\gamma)}{2L\gamma} > 0$. And by Lemma 2.1,

$$\det Q_4 = \frac{(2L+\gamma)\gamma}{L\beta}\det A\,,$$
$$\begin{aligned}(Q_4)^{ij} = {} & \frac{\alpha}{L}a^{ij} - \frac{\alpha^2}{\gamma L}b^{ij} - \frac{1}{L(2L+\gamma)}y^i y^j \\ & - \frac{\alpha}{L(2L+\gamma)\beta}(a^{ik}b_{kl}y^l y^j + y^i y^k b_{kl}a^{lj}) \\ & + \frac{2\alpha^2}{(2L+\gamma)\beta^2\gamma}a^{ik}b_{kl}y^l y^m b_{mn}a^{nj}.\end{aligned}$$

Finally, let

$$g_{ij} = (Q_4)_{ij} - \frac{L}{\alpha}\alpha_i\alpha_j,\ c_{5i} = \sqrt{\tfrac{L}{\alpha}}\alpha_i.$$

Then we get $c_5^2 = c_{5i}(Q_4)^{ij}c_{5j} = \frac{L+\gamma}{2L+\gamma} - \frac{L\beta}{\alpha(2L+\gamma)}, 1 - c_5^2 = \frac{L^2}{\alpha(2L+\gamma)} > 0$. And by Lemma 2.1,

$$\det(g_{ij}) = \frac{L^2}{\alpha(2L+\gamma)}\cdot\frac{(2L+\gamma)\gamma}{L\beta}\det A = \frac{L\gamma}{\alpha\beta}\det A,$$
$$\begin{aligned}g^{ij} = {} & \frac{\alpha}{L}a^{ij} - \frac{\alpha^2}{\gamma L}b^{ij} + \frac{\gamma}{L^3}y^i y^j \\ & - \frac{\alpha}{L^2\beta}(a^{ik}b_{kl}y^l y^j + y^i y^k b_{kl}a^{lj}) + \frac{\alpha^2}{L\beta^2\gamma}a^{ik}b_{kl}y^l y^m b_{mn}a^{nj}.\end{aligned}$$

If we set $\alpha^i = \frac{y^i}{\alpha}$ and $\beta^i = \frac{a^{ir}b_{rs}y^s}{\beta}$, then the last equation yields equation (3.1). □

## 4. Theorems

In this section, with the aid of Proposition 3.1, we show the positivity of $g_{ij}$ and the quasi-C-reducibility of an $(a,b,f)$-metric and find a sufficient condition of being C-reducible.

Now we are ready to prove that $(g_{ij})$ is positive definite on $\mathbb{C}(\ker B)$. This implies that $\mathbb{C}(\ker B)$ is the maximal domain $\mathcal{D}$ of $TM \setminus O$ for an $(a,b,f)$-manifold where a $y$-local Finsler structure $L$ is defined.

THEOREM 4.1. *$(g_{ij})$ is positive definite on $\mathbb{C}(\ker B)$.*

*Proof.* Consider a one-parameter family of the $(a,b,f)$-metric $L^\epsilon = \alpha + \epsilon\beta$ with $0 \le \epsilon \le 1$. Let $g^\epsilon$ be the fundamental tensor of $L^\epsilon$. For $\epsilon > 0$, by Proposition 3.1, we have

$$\det(g^\epsilon_{ij}) = \frac{L^\epsilon \gamma^\epsilon}{\epsilon\alpha\beta}\det A^\epsilon,$$

where $A^\epsilon = (\frac{L^\epsilon}{\alpha}a_{ij})$, $\gamma^\epsilon = \epsilon\beta + \epsilon^2 b^2 \alpha > 0$, and so $\det(g^\epsilon_{ij})$ is positive. In particular, none of the eigenvalues of $(g^\epsilon_{ij})$ can vanish. For $\epsilon = 0$, $L^\epsilon = \alpha$ and all the eigenvalues of $(g^\epsilon_{ij}) = (g_{ij})$ are positive. Since $\det(g^\epsilon_{ij})$ is continuous for $\epsilon$, all the eigenvalues of $(g^\epsilon_{ij})$ are positive by the intermediate value theorem. And so all the eigenvalues of $(g_{ij})$ are positive. This means that $(g_{ij})$ is positive definite. □

Next, we show that $(a,b,f)$-manifolds are quasi-C-reducible and we determine a sufficient condition under which $(a,b,f)$-manifolds are C-reducible. We start with the definitions of quasi-C-reducibility and of C-reducibility.

DEFINITION 4.1. A Finsler manifold of dimension $n$, $n \ge 3$, is quasi-C-reducible if there exists a symmetric Finsler tensor field $Q_{ij}$ satisfying $Q_{ij}y^j = 0$ and $A_{ijk} = \mathfrak{S}_{(ijk)}\{Q_{ij}A_k\}$, where $A_k := g^{ij}A_{ijk}$.

DEFINITION 4.2. A Finsler manifold of dimension $n$, $n \ge 3$, is C-reducible if $A_{ijk}$ is in the form $A_{ijk} = \frac{1}{n+1}\mathfrak{S}_{(ijk)}\{h_{ij}A_k\}$, where $h_{ij} := g_{ij} - L_iL_j$ is the angular metric of $L$.

Note that for $(a,b,f)$-metric, the Cartan tensor is

$$\begin{aligned} A_{ijk} :&= \frac{L}{4}(L^2)_{y^iy^jy^k} = \frac{L}{2}(g_{ij})_{y^k} \\ &= \frac{L}{2}\mathfrak{S}_{(ijk)}\left\{\left(\frac{\alpha_{ij}}{\alpha} - \frac{\beta_{ij}}{\beta}\right)(\alpha\beta_k - \beta\alpha_k)\right\}. \end{aligned}$$

By Proposition 3.1, we get

$$A_k = \frac{\lambda}{2}(\alpha\beta_k - \beta\alpha_k),$$

where $\lambda = \left(\frac{n+1}{\alpha} - \frac{b^2L}{\beta\gamma}\right)$. Since $rank(b_{ij}) = 2$, $\lambda \neq 0$. And if we let

$$Q_{ij} = \frac{L}{\lambda}\left(\frac{\alpha_{ij}}{\alpha} - \frac{\beta_{ij}}{\beta}\right),$$

we have $A_{ijk} = \mathfrak{S}_{(ijk)}\{Q_{ij}A_k\}$. Because $Q_{ij}$ is symmetric and $Q_{ij}y^j = 0$ by Euler's theorem, we have

THEOREM 4.2. *$(a, b, f)$-manifolds are quasi-C-reducible.*

Since the angular metric $h_{ij}$ for $(a, b, f)$-manifold is $L \cdot (\alpha_{ij} + \beta_{ij})$, we can conclude

THEOREM 4.3. *If an $(a, b, f)$-metric $L = \alpha + \beta$ satisfies*

$$\frac{\alpha_{ij}}{\alpha} - \frac{\beta_{ij}}{\beta} = \frac{\lambda}{n+1}(\alpha_{ij} + \beta_{ij}),$$

*or equivalently $b^2\alpha\alpha_{ij} = (n\gamma + \beta)\beta_{ij}$, then the $(a, b, f)$-manifold is C-reducible.*

REMARK. If $A_i = 0$ for a $C$-reducible manifold, then $A_{ijk} = 0$ immediately. And so the manifold is Riemannian. For a C-reducible $(a, b, f)$-manifold with $A_i = 0$, we can show that

$$g_{ij}(x) = g_{pq}(x) f_i^p f_j^q.$$

In other words, such an $(a, b, f)$-manifold is an almost Hermitian manifold. For its proof, we refer the readers to [2].

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## References

[1] D. Bao, S. S. Chern, and Z. Shen, *An introduction to Riemannian–Finsler geometry*, Graduate Texts in Mathematics, Springer-Verlag, Heidelberg, New York, 2000.
[2] Y. Ichiiyō, *Almost Hermitian Finsler manifolds*, Tensor(N.S.) **37** (1982), 279–284.
[3] Y. Ichiiyō and M. Hashiguchi, *On $(a, b, f)$-metrics*, Rep. Fac. Sci. Kagoshima Univ. Math. Phys. Chem **28** (1995), 1–9.
[4] ———, *On $(a, b, f)$-metrics II*, Rep. Fac. Sci. Kagoshima Univ. Math. Phys. Chem **29** (1996), 1–5.

[5] M. Matsumoto, *Foundations of Finsler geometry and special Finsler spaces*, Kaiseisha Press, Shigaken, 1986.
[6] ______, *On C-reducible Finsler spaces*, Tensor(N.S.) **24** (1972), 29–37.

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