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ANALYTIC APPROACH TO DEFORMATION OF RESOLUTION OF NORMAL ISOLATED SINGULARITIES: FORMAL DEFORMATIONS

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ABSTRACT. We give an analytic approach to the versal deformation of a resolution of a germ of normal isolated singularities. In this paper, we treat only formal deformation theory and it is applied to complete the CR-description of the simultaneous resolution of a cone over a rational curve of degree n in $\mathbf{P}^n$ $(n \leq 4)$.

Introduction

Let V be a reduced irreducible normal Stein space with a singularity at $o \in V$. We assume that $\dim_{\mathbf{C}} V \geq 2$ throughout the paper. In [7], we made a CR construction of the versal family of deformation of the germ (V, o). The subject of this paper is an analytic approach to deformation of resolution of singularities of V. Let $f : \widetilde{V} \to V$ be a resolution of singularities. We consider it as a holomorphic map $(\widetilde{V}, f^{-1}(o)) \to (V, o)$ between germs of complex spaces. Though a normal Stein space is determined by its resolution, it is well known that all families of deformations of $\widetilde{V}$ cannot be realized as families of resolutions of deformations of V (cf. [9]). Thus we consider the following deformation functor; for $T \in \mathcal{A}n$,

$$\mathcal{R}es_{\widetilde{V}\to V}(T) := \left\{ F : \widetilde{\mathcal{V}} \to \mathcal{V} \;\middle|\; \begin{array}{c} \widetilde{\varpi} : \widetilde{\mathcal{V}} \to T \text{ (resp. } \varpi : \mathcal{V} \to T) \text{ is a family of} \\ \text{deformations of } \widetilde{V} \text{(resp. } V) \text{ and} \\ F : \widetilde{\mathcal{V}} \to \mathcal{V} \text{ is a holomorphic map} \\ \text{satisfying } \varpi \circ F = \widetilde{\varpi} \text{ and } F|_{\widetilde{\mathcal{V}}_0} = f \end{array} \right\} \Big/ \sim$$

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where $\mathcal{A}n$ denotes the category of germs of analytic spaces with the distinguished point 0 and $(F_1 : \widetilde{\mathcal{V}}_1 \to \mathcal{V}_1) \sim (F_2 : \widetilde{\mathcal{V}}_2 \to \mathcal{V}_2)$ if there exist isomorphisms $\widetilde{\chi} : \widetilde{\mathcal{V}}_1 \to \widetilde{\mathcal{V}}_2$ and $\chi : \mathcal{V}_1 \to \mathcal{V}_2$ satisfying $\widetilde{\varpi}_2 \circ \widetilde{\chi} = \widetilde{\varpi}_1$, $\varpi_2 \circ \chi = \varpi_1$, $\chi \circ F_1 = F_2 \circ \widetilde{\chi}$, $\widetilde{\chi}|_{\widetilde{\mathcal{V}}_0} = \mathrm{id}_{\widetilde{V}}$ and $\chi|_{\mathcal{V}_0} = \mathrm{id}_V$.

This type of deformation theory was considered for normal surface singularities in [1] by algebraic method. We approach this deformation via deformation of Cauchy-Riemann structure on $\widetilde{V}$ in order to connect the deformation of resolution to the deformation of CR-structure on a link of the singularities of V.

In this paper, we will consider only formal deformations. For the actual deformation, we need an extra adjustment near the boundary of the inductive construction in Theorem 4 together with an adjustment of the operators in Section 2 and 3. Since it requires a new technique other than the one used in [7], we will discuss it in the next paper.

In Sections 1, we will $\bar{\partial}$-theoretically describe the formal theory of deformation of resolution of normal isolated singularities. The main part of this paper is Sections 2 and 3 where we will construct the homotopy operators for the deformation complex which will be introduced in Section 1. Relying on these operators, we can construct a formal versal family of deformation of resolution of normal isolated singularities by the method in [7]. In Section 4, we will apply our construction to the case of a cone over rational curve of degree $n(\leq 4)$ in $\mathbf{P}^n$ and will give a Cauchy-Riemann theoretic description of so-called the Artin-component.

1. Deformation functor and deformation complex

Let V be embedded in $\mathbf{C}^N$ and denote $f : \widetilde{V} \to V \subset \mathbf{C}^N$ a resolution of singularities. Let $\mathcal{R}es_{\widetilde{V}\to V}$ be the functor as above and let

$$\mathcal{D}ef_{\widetilde{V}\to\mathbf{C}^N}(T) := \left\{ \Phi : \widetilde{\mathcal{V}} \to \mathbf{C}^N \times T \;\middle|\; \begin{array}{c} \widetilde{\varpi} : \widetilde{\mathcal{V}} \to T \text{ is a family of deformations of } \widetilde{V} \\ \text{and } \Phi : \widetilde{\mathcal{V}} \to \mathbf{C}^N \times T \text{ is a holomorphic map} \\ \text{satisfying } pr_2 \circ \Phi = \widetilde{\varpi} \text{ and } \Phi|_{\widetilde{\mathcal{V}}_0} = f \end{array} \right\} / \sim$$

where $(\widetilde{\mathcal{V}}_1, \Phi_1) \sim (\widetilde{\mathcal{V}}_2, \Phi_2)$ if there exists an isomorphism $\widetilde{\chi} : \widetilde{\mathcal{V}}_1 \to \widetilde{\mathcal{V}}_2$ satisfying $\widetilde{\varpi}_2 \circ \widetilde{\chi} = \widetilde{\varpi}_1$ and $\widetilde{\chi}|_{\widetilde{\mathcal{V}}_0} = \mathrm{id}_{\widetilde{V}}$.

PROPOSITION 1.1.

$$\mathcal{R}es_{\widetilde{V}\to V} \simeq \mathcal{D}ef_{\widetilde{V}\to\mathbf{C}^N}.$$

Proof. The only non-trivial part of the proof is to show that, for any family $F : \widetilde{\mathcal{V}} \to \mathbf{C}^N \times T$ in $\mathcal{D}ef_{\widetilde{V}\to\mathbf{C}^N}(T)$, $F(\widetilde{\mathcal{V}}) \to T$ is a flat family. We assume that V is an analytic subset of a neighbourhood of a small ball $B \subset \mathbf{C}^N$ centered at the origin $o \in \mathbf{C}^N$. Let $h_1(w,t),\ldots,h_m(w,t)$ be generators of the ideal sheaf $\mathcal{I}_{F(\widetilde{\mathcal{V}})}$ of $F(\widetilde{\mathcal{V}})$. By Theorem 1.3 of [5] and IV, Theorem 5.6 of [4], it is enough to show the following property: any $(\mu-1)$-th degree polynomials of t, $p_1^{(\mu-1)}(w,t),\ldots,p_m^{(\mu-1)}(w,t) \in H^0(B,\mathcal{O}_B)[t]$ satisfying $\sum_{\sigma=1}^m p_\sigma^{(\mu-1)}(w,t)h_\sigma(w,t) = O(t^\mu)$, can be lifted to a polynomials of degree μ, $p_1^{(\mu)}(w,t),\ldots,p_m^{(\mu)}(w,t) \in H^0(B,\mathcal{O}_B)[t]$ satisfying $p_\sigma^{(\mu)}(w,t) - p_\sigma^{(\mu-1)}(w,t) = O(t^\mu)$ $(\sigma = 1,\ldots,m)$ and

$$\sum_{\sigma=1}^m p_\sigma^{(\mu)}(w,t)h_\sigma(w,t) = O(t^{\mu+1}).$$

It is proved by the argument of the proof of Theorem 5.1 of [2]. Let $r_\mu(w,t) :=$ the μ-th order term of $\sum_{\sigma=1}^m p_\sigma^{(\mu-1)}(w,t)h_\sigma(w,t)$. Then, by the assumption, $r_\mu(F(z,t),t) = 0$. On the other hand, since $r_\mu(w,t)$ is homogeneous of degree μ in t, we have $r_\mu(F(z,t),t) - r_\mu(F(z,0),t) = O(t^{\mu+1})$.

Hence, $f^*r_\mu(t) = 0$. It follows that $r_\mu(t)|_V = 0$ since V is normal. Therefore, there exists homogeneous polynomials of degree μ, $p_{1\mu}(w,t),\ldots,p_{m\mu}(w,t)$ such that

$$\sum_{\sigma=1}^m (p_\sigma^{(\mu)}(w,t) + p_{\sigma\mu}(w,t))h_\sigma(w,t) = O(t^{\mu+1}).$$

□

We shall consider the deformation functor $\mathcal{D}ef_{\widetilde{V}\to\mathbf{C}^N}$. Let $\Theta_{\widetilde{V}}$ and $\Theta_{\mathbf{C}^N}$ denote the sheaves of holomorphic vector fields on $\widetilde{V}$ and $\mathbf{C}^N$ respectively. Then the infinitesimal deformation of this deformation theory is described by the following fundamental exact sequence;

$$0 \to \Theta_{\widetilde{V}} \xrightarrow{df} f^*\Theta_{\mathbf{C}^N} \xrightarrow{r} \mathcal{T}_{\widetilde{V}/\mathbf{C}^N} \to 0 \tag{1.1}$$

where $\mathcal{T}_{\widetilde{V}/\mathbf{C}^N}$ denotes the coherent sheaf $f^*\Theta_{\mathbf{C}^N}/\Theta_{\widetilde{V}}$.

The following proposition is proved by the same argument as the proof of Proposition 1.5 of [7].

PROPOSITION 1.2. (1) *The space of first order deformations is*

$\operatorname{Ker}\{H^1(\widetilde{V},\Theta_{\widetilde{V}}) \xrightarrow{df} H^1(\widetilde{V}, f^*\Theta_{\mathbf{C}^N})\},$

(2) *The space of obstructions to lifting to higher order deformations is* $\mathrm{Ker}\{H^1(\widetilde{V}, \mathcal{T}_{\widetilde{V}/\mathbf{C}^N}) \xrightarrow{H} H^1(\widetilde{V}, \mathcal{O}_{\widetilde{V}}^m)\}$, *where* H *denotes the homomorphism* $\mathcal{T}_{\widetilde{V}/\mathbf{C}^N} \to \mathcal{O}_{\widetilde{V}}^m$ *induced from the bundle homomorphism* $dh : T^{1,0}\mathbf{C}^N \to h^*T^{1,0}\mathbf{C}^m$ *with denoting* $h := (h_1, \dots, h_m)$.

For our treatment of this deformation theory, we shall make a $\bar{\partial}$-theoretic description of the above cohomology spaces.

Let $K_{\widetilde{V}}^{\bullet,\bullet}$ be the following double complex.

(1.2)

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & H^0(\widetilde{V}, f^*\Theta_{\mathbf{C}^N}) & \xrightarrow{H} & H^0(\widetilde{V}, \mathcal{O}_{\widetilde{V}})^m & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow i & & \downarrow i & & \\
0 & \longrightarrow & K_{\widetilde{V}}^{0,0} := A_{\widetilde{V}}^0(\Theta_{\widetilde{V}}) & \xrightarrow{df} & A_{\widetilde{V}}^0(f^*\Theta_{\mathbf{C}^N}) & \xrightarrow{H} & (A_{\widetilde{V}}^0)^{\oplus m} & \longrightarrow & 0 \\
 & & \downarrow \bar{\partial} & & \downarrow \bar{\partial} & & \downarrow \bar{\partial} & & \\
0 & \longrightarrow & A_{\widetilde{V}}^{0,1}(\Theta_{\widetilde{V}}) & \xrightarrow{df} & A_{\widetilde{V}}^{0,1}(f^*\Theta_{\mathbf{C}^N}) & \xrightarrow{H} & (A_{\widetilde{V}}^{0,1})^{\oplus m} & \longrightarrow & 0 \\
 & & \downarrow \bar{\partial} & & \downarrow \bar{\partial} & & \downarrow \bar{\partial} & & \\
0 & \longrightarrow & A_{\widetilde{V}}^{0,2}(\Theta_{\widetilde{V}}) & \xrightarrow{df} & A_{\widetilde{V}}^{0,2}(f^*\Theta_{\mathbf{C}^N}) & \xrightarrow{H} & (A_{\widetilde{V}}^{0,2})^{\oplus m} & \longrightarrow & 0 \\
 & & \downarrow \bar{\partial} & & \downarrow \bar{\partial} & & \downarrow \bar{\partial} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Then the following proposition is direct.

PROPOSITION 1.3. (1) $H^1(K_{\widetilde{V}}^{\bullet,\bullet}) \simeq \mathrm{Ker}\{H^1(\widetilde{V}, \Theta_{\widetilde{V}}) \to H^1(\widetilde{V}, f^*\Theta_{\mathbf{C}^N})\}$.

(2) $H^2(K_{\widetilde{V}}^{\bullet,\bullet}) \simeq \mathrm{Ker}\{H^1(\widetilde{V}, \mathcal{T}_{\widetilde{V}/\mathbf{C}^N}) \to H^1(\widetilde{V}, \mathcal{O}_{\widetilde{V}}^m)\}$.

Let $r : \widetilde{V} \to \mathbf{R}$ be a smooth exhaustion function which is strictly plurisubharmonic on $\widetilde{V} \setminus f^{-1}(0)$; e.g. $r = (\sum_{\beta=1}^N |w_\beta|^2) \circ f$. We assume that $dr \neq 0$ on $\widetilde{V} \setminus r^{-1}(0)$ and denote $\Omega := \{x \in \widetilde{V} | r(x) < c\}$.

We denote $K_{\overline{\Omega}}^{\bullet,\bullet}$ (resp. $(K_{\overline{\Omega}}^{\bullet}, d)$) the double complex (1.2) with $\widetilde{V}$ replaced by $\overline{\Omega}$ (resp. its total simple complex).

PROPOSITION 1.4. $H^q(K_{\widetilde{V}}^{\bullet,\bullet}) \simeq H^q(K_{\overline{\Omega}}^{\bullet,\bullet})$ $(q = 1, 2)$.

Proof. For the proof, we recall the isomorphism

$$H^1(\overline{\Omega}, E) \simeq H^1(\widetilde{V}, E) \tag{1.3}$$

for a holomorphic vector bundle E over $\widetilde{V}$ (cf. [6]).

Let $q = 1$. As Proposition 1.3, we have

$$H^1(K_{\overline{\Omega}}^{\bullet,\bullet}) \simeq \operatorname{Ker}\{H^1(\overline{\Omega}, \Theta_{\widetilde{V}}) \to H^1(\overline{\Omega}, f^*\Theta_{\mathbf{C}^N})\}.$$

Hence our isomorphism follows by the isomorphism (1.3) for $E = \Theta_{\widetilde{V}}$ or $E = f^*\Theta_{\mathbf{C}^N}$.

Let $q = 2$. Let $\mathcal{A}_{\widetilde{V}}^{0,q}(\mathcal{T}_{\widetilde{V}/\mathbf{C}^N}) := \mathcal{T}_{\widetilde{V}/\mathbf{C}^N} \otimes_{\mathcal{O}_{\widetilde{V}}} \mathcal{A}_{\widetilde{V}}^{0,q}$ and define $\bar{\partial}(u \otimes \phi) := u \otimes (\bar{\partial}\phi)$, where we denote $\mathcal{A}_{\widetilde{V}}^{0,q}$ the sheaf of differentiable $(0,q)$-forms on $\widetilde{V}$. We consider the differential complex $(A_{\overline{\Omega}}^{0,q}(\mathcal{T}_{\widetilde{V}/\mathbf{C}^N}), \bar{\partial})$ and denote its q-th cohomology group by $H^q(\overline{\Omega}, \mathcal{T}_{\widetilde{V}/\mathbf{C}^N})$, where we denote $A_{\overline{\Omega}}^{0,q}(\mathcal{T}_{\widetilde{V}/\mathbf{C}^N}) := \{(u \otimes \phi)|_{\overline{\Omega}} \mid u \otimes \phi \in \Gamma(\widetilde{V}, \mathcal{A}_{\widetilde{V}}^{0,q}(\mathcal{T}_{\widetilde{V}/\mathbf{C}^N}))\}$. We remark that the isomorphism

$$H^q(\overline{\Omega}, \mathcal{T}_{\widetilde{V}/\mathbf{C}^N}) \simeq H^q(\widetilde{V}, \mathcal{T}_{\widetilde{V}/\mathbf{C}^N}) \tag{1.4}$$

holds for $q \geq 1$ by the isomorphism (1.3) for $E = \Theta_{\widetilde{V}}$ or $E = f^*\Theta_{\mathbf{C}^N}$ and the following exact sequence together with the five lemma;

$$0 \to A_{\overline{\Omega}}^{0,q}(\Theta_{\widetilde{V}}) \to A_{\overline{\Omega}}^{0,q}(f^*\Theta_{\mathbf{C}^N}) \to A_{\overline{\Omega}}^{0,q}(\mathcal{T}_{\widetilde{V}/\mathbf{C}^N}) \to 0.$$

As Proposition 1.3, we have

$$H^2(K_{\overline{\Omega}}^{\bullet,\bullet}) \simeq \operatorname{Ker}\{H^1(\overline{\Omega}, \mathcal{T}_{\widetilde{V}/\mathbf{C}^N}) \to H^1(\overline{\Omega}, \mathcal{O}_{\widetilde{V}}^m)\}.$$

Hence, the proposition for $q = 2$ follows by the isomorphism (1.4). □

2. Analysis on $K_{\overline{\Omega}}^{\bullet,\bullet}$ ($q = 1$)

Based on the harmonic analysis on $A_{\overline{\Omega}}^{0,1}(\Theta_{\widetilde{V}})$ and $A_{\overline{\Omega}}^{0,1}(f^*\Theta_{\mathbf{C}^N})$, we construct the following operators Z_1 and Q_1 as in [7].

THEOREM 1. *There exist operators* $Z_1 : K_{\overline{\Omega}}^1 \to K_{\overline{\Omega}}^1 \cap \operatorname{Ker} d$ *and* $Q_1 : K_{\overline{\Omega}}^1 \cap \operatorname{Ker} d \to K_{\overline{\Omega}}^0$ *satisfying*

(1) $Z_1|_{\operatorname{Ker} d} = \operatorname{id}|_{\operatorname{Ker} d}$,
(2) $d \circ Q_1 \circ d = d$.

3. Analysis on $K_{\tilde{\Omega}}^{\bullet,\bullet}$ $(q=2)$

In this section, we will prove the existence of the following Z_2 and Q_2.

THEOREM 2. *There exist operators* $Z_2 : K_{\tilde{\Omega}}^2 \to K_{\tilde{\Omega}}^2 \cap \operatorname{Ker} d$ *and* $Q_2 : K_{\tilde{\Omega}}^2 \cap \operatorname{Ker} d \to K_{\tilde{\Omega}}^1$ *satisfying*

(1) $Z_2|_{\operatorname{Ker} d} = \operatorname{id}|_{\operatorname{Ker} d}$,
(2) $d \circ Q_2 \circ d = d$.

For the proof of this theorem, we need the harmonic analysis for $(A_{\tilde{\Omega}}^{0,\bullet}(\mathcal{T}_{\tilde{V}/\mathbf{C}^N}), \bar{\partial})$. Instead of the harmonic theory for the $\bar{\partial}$-complex with values in the coherent sheaf $\mathcal{T}_{\tilde{V}/\mathbf{C}^N}$, we consider the following double complex $L_{\tilde{\Omega}}^{\bullet,\bullet}$.

$$
\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & 0 & \longrightarrow & H^0(\bar{\Omega}, f^*\Theta_{\mathbf{C}^N}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow i & & \\
0 & \longrightarrow & L_{\tilde{\Omega}}^{0,0} := A_{\tilde{\Omega}}^{0}(\Theta_{\tilde{V}}) & \xrightarrow{df} & A_{\tilde{\Omega}}^{0}(f^*\Theta_{\mathbf{C}^N}) & \longrightarrow & 0 \\
 & & \downarrow \bar{\partial} & & \downarrow \bar{\partial} & & \\
0 & \longrightarrow & A_{\tilde{\Omega}}^{0,1}(\Theta_{\tilde{V}}) & \xrightarrow{df} & A_{\tilde{\Omega}}^{0,1}(f^*\Theta_{\mathbf{C}^N}) & \longrightarrow & 0 \\
 & & \downarrow \bar{\partial} & & \downarrow \bar{\partial} & & \\
0 & \longrightarrow & A_{\tilde{\Omega}}^{0,2}(\Theta_{\tilde{V}}) & \xrightarrow{df} & A_{\tilde{\Omega}}^{0,2}(f^*\Theta_{\mathbf{C}^N}) & \longrightarrow & 0 \\
 & & \downarrow \bar{\partial} & & \downarrow \bar{\partial} & & \\
 & & \vdots & & \vdots & &
\end{array}
$$

Then the following proposition is obvious.

PROPOSITION 3.1. (1) $H^1(L_{\tilde{\Omega}}^{\bullet,\bullet}) \simeq H^0(\bar{\Omega}, \mathcal{T}_{\tilde{V}/\mathbf{C}^N})$.
(2) $H^2(L_{\tilde{\Omega}}^{\bullet,\bullet}) \simeq H^1(\bar{\Omega}, \mathcal{T}_{\tilde{V}/\mathbf{C}^N})$.

We consider the harmonic theory on (the total simple complex of) the double complex $L_{\bar{\Omega}}^{\bullet,\bullet}$. Let

$$L_{\bar{\Omega}}^{q} := A_{\bar{\Omega}}^{0,q}(\Theta_{\widetilde{V}}) \oplus A_{\bar{\Omega}}^{0,q-1}(f^*\Theta_{\mathbf{C}^N}),$$
$$D(\phi, g) := (\bar{\partial}\phi, \bar{\partial}g + (-1)^q df\phi).$$

We fix hermitian inner products along the fibres of vector bundles $T^{1,0}\widetilde{V}$ and $T^{1,0}\mathbf{C}^N$. We define an inner product on $L_{\bar{\Omega}}^{q}$ as follows;

$$\big((\phi, g), (\phi', g')\big) := (\phi, \phi') + (g, g').$$

PROPOSITION 3.2. *The boundary condition of D-Neumann problem is*

$$\mathcal{D}_D^q := \big\{(\theta, \xi) \in L_{\bar{\Omega}}^q \,|\, \sigma(\vartheta, dr)\theta|_{\partial\Omega} = 0,\ \sigma(\vartheta, dr)\xi|_{\partial\Omega} = 0\big\}$$

and

$$D^*(\theta, \xi) = (\vartheta\theta + (-1)^{q-1}F^*\xi, \vartheta\xi) \ \ \text{for } (\theta, \xi) \in \mathcal{D}_D^q,$$

where $\vartheta\xi := \xi - \vartheta N\bar{\partial}\xi$ *if* $q = 1$ *and* F^* *is a bundle homomorphism satisfying* $< df\phi, \xi > = < \phi, F^*\xi >$ *for* $\phi \in A_{\bar{\Omega}}^0(\Theta_{\widetilde{V}})$ *and* $\xi \in A_{\bar{\Omega}}^0(f^*\Theta_{\mathbf{C}^N})$.

Proof. For $(\phi, g) \in L_{\bar{\Omega}}^{q-1}$ and $(\theta, \xi) \in L_{\bar{\Omega}}^q$,

$$\begin{aligned}\big(D(\phi, g), (\theta, \xi)\big) &= (\bar{\partial}\phi, \theta) + (\bar{\partial}g + (-1)^{q-1}df\phi, \xi)\\ &= (\phi, \vartheta\theta) - \int_{\partial\Omega} < \phi, \sigma(\vartheta, dr)\theta >\\ &\quad + (g, \vartheta\xi) - \int_{\partial\Omega} < f, \sigma(\vartheta, dr)\xi >\\ &\quad + (-1)^{q-1}(\phi, F^*\xi).\end{aligned}$$

□

We shall construct the D-Neumann operators for $q \geq 2$ following [3]. Denote

$$Q_D(u, v) := (Du, Dv) + (D^*u, D^*v) + (u, v) \ \ \text{for } (u, v) \in \mathcal{D}_D^q$$

and $\widetilde{\mathcal{D}}_D^q$ the completion of $\mathcal{D}_D^q$ with respect to Q_D. Then, since $Q_D(u, u) \geq ||u||^2$ holds for $u \in \mathcal{D}_D^q$, there exists a bounded injective self-adjoint operator $T_D : L_{(2)}^q \to L_{(2)}^q$ with $T_D(L_{(2)}^q) \subset \widetilde{\mathcal{D}}_D^q$ and

$$Q_D(T_D u, v) = (u, v) \ \ \text{holds for all } u,\ v \in \widetilde{\mathcal{D}}_D^q.$$

Let $F_D := T_D^{-1}$. Then $\mathrm{Dom}(F_D) \cap \mathcal{D}_D^q = \big\{u \in \mathcal{D}_D^q \,|\, Du \in \mathcal{D}_D^{q+1}\big\}$ and $F_D u = (\Box_D + I)u$ holds.

Let $H^q_D := \operatorname{Ker}\Box_D$. Then we have the weak orthogonal decomposition

$$L^q_{(2)} = H^q_D + (\operatorname{Range}\Box_D)^c$$

where c denotes closure.

PROPOSITION 3.3.

$$\Box_D(\phi, g)$$
$$= \big(\Box_{\bar\partial}\phi, \Box_{\bar\partial}g\big) + \big((-1)^{q-1}[\bar\partial, F^*]g + F^*df\phi, (-1)^q[\vartheta, df]\phi + dfF^*g\big).$$

Proof. Since

$$D(\phi, g) = \big(\bar\partial\phi, \bar\partial g + (-1)^q df\phi\big) \text{ for } (\phi, g) \in L^q_{\overline{\Omega}}$$
$$D^*(\phi, g) = (\vartheta\phi + (-1)^{q-1}F^*g, \vartheta g) \text{ for } (\phi, g) \in L^q_{\overline{\Omega}}$$

we have

$$D^*D(\phi, g) = (\vartheta\bar\partial\phi + (-1)^q F^*\bar\partial g + F^*df\phi, \vartheta\bar\partial g + (-1)^q\vartheta df\phi)$$
$$DD^*(\phi, g) = (\bar\partial\vartheta\phi + (-1)^{q-1}\bar\partial F^*g, \bar\partial\vartheta g + (-1)^{q-1}df\vartheta\phi + dfF^*g)$$

Hence, we have the proposition. □

By proposition 3.3, we can carry out the standard argument of the estimate in the interior and at the boundary, and the elliptic regularization (cf. [3]). It enables us to conclude the following:

PROPOSITION 3.4. *For* $q \geq 2$,

(1) $H^q_D := \operatorname{Ker}\Box_D$ *is finite dimensional and* $H^q_D \subset L^q_{\overline{\Omega}}$,

(2) $\operatorname{range}\Box_D$ *is closed and we have a strong decomposition* $L^q_{(2)} = H^q_D \oplus \operatorname{Range}\Box_D$.

Let $N_D : L^q_{(2)} \to \operatorname{Dom}(F_D)$ be so-called the Neumann operator defined by the above orthogonal decomposition; that is, $Nu = 0$ for $u \in H^q_D$ and $N_D u$ is the unique solution v of $\Box_D v = u$ $(v \perp H^q_D)$ if $u \perp H^q_D$. Then we have the following

THEOREM 3. *For* $q \geq 2$,

(1) N_D *is compact operator,*

(2) $u = \rho_D u + \Box_D N_D u$ *holds for* $u \in L^q_{(2)}$, *where* ρ_D *denotes the orthogonal projection onto* H^q_D,

(3) $N_D\rho_D = \rho_D N_D = 0$, $\Box_D N_D = N_D\Box_D$, $DN_D = N_D D$ *holds on* $\operatorname{Dom}(D)$ *and* $D^*N_D = N_D D^*$ *holds on* $\operatorname{Dom}(D^*)$,

(4) $N_D(L^q_{\overline{\Omega}}) \subset L^q_{\overline{\Omega}}$,

(5) *if we denote* $||(\phi, g)||_k := ||\phi||_k + ||g||_k$, *then* $||N_D u||_{k+1} \le C||u||_k$ *holds for* $u \in L^q_{\overline{\Omega}}$.

We apply this harmonic analysis to the construction of the operator Z_2, Q_2. We note that

$$K^q_{\overline{\Omega}} = L^q_{\overline{\Omega}} \oplus K^{2,q-2}_{\overline{\Omega}}$$
$$d(u,k) = (Du, \bar{\partial}k + (-1)^{q-1}Hu)$$

where we denote $H(\phi, g) := Hg$.

Proof of Theorem 2. Let $(a_2, b_1, c_{-1}) \in L^2_{\overline{\Omega}} \oplus K^{2,0}_{\overline{\Omega}}$ and denote

$$\begin{cases} (a'_2, b'_1) := (\rho_H \circ \rho + DD^* N_D)(a_2, b_1) \\ c'_0 := \rho c_0 + \bar{\partial}^* N H b'_1. \end{cases}$$

We define

$$Z_2(a_2, b_1, c_{-1}) := (a'_2, b'_1, c'_0).$$

PROPOSITION 3.5. (1) $dZ_2 = 0$,
(2) $Z_2|_{\mathrm{Ker}\ d} = \mathrm{id}|_{\mathrm{Ker}\ d}$.

Proof. (1)

$$\begin{aligned} d(a'_2, b'_1, c'_0) &= (D(a'_2, b'_1), \bar{\partial}c'_0 - Hb'_1) \\ &= (0, 0, \bar{\partial}\bar{\partial}^* NHb'_1 - Hb'_1) \\ &= (0,0,0), \end{aligned}$$

since Hb'_1 is $\bar{\partial}$-exact.

(2) Suppose $d(a_2, b_1, c_{-1}) = (D(a_2, b_1), \bar{\partial}c_0 - Hb_1) = (0,0,0)$. Then

$$(a_2, b_1) = (\rho + DD^* N_D)(a_2, b_1),$$

since (a_2, b_1) is D-closed.

Next, since $H^2(L^\bullet_{\overline{\Omega}}) \simeq \mathrm{Ker}\{H^1(\overline{\Omega}, \mathcal{T}_{\widetilde{V}/\mathbf{C}^N}) \to H^1(\overline{\Omega}, \mathcal{O}^m_{\widetilde{V}})\}$ by Proposition 3.1, we have

$$\rho(a_2, b_1) = \rho_H \circ \rho(a_2, b_1).$$

Hence, it follows that $(a'_2, b'_1) = (a_2, b_1)$ and $c'_0 = \rho c_0 + \bar{\partial}^* NHb_1 = \rho c_0 + \bar{\partial}^* N \bar{\partial} c_0 = c_0$. □

Next, we define Q_2. For $(a'_2, b'_1, c'_0) \in \mathrm{Ker}\ d \cap K^2_{\overline{\Omega}}$, we define

$$\begin{cases} (a''_1, b''_0) := D^* N(a'_2, b'_1) \\ c''_{-1} := c'_0 - Hb''_0 - \bar{\partial}^* NH\rho_D(a'_2, b'_1) \end{cases}$$

and

$$Q_2(a_2', b_1', c_0') := (a_1'', b_0'', c_{-1}'').$$

PROPOSITION 3.6. *$dQ_2d = d$.*

Proof. Let $(a_2', b_1', c_0') := d(a_1, b_0, c_{-1}) = (D(a_1, b_0), c_{-1} + Hb_0)$ and

$$(a_1'', b_0'') = D^*ND(a_1, b_0)$$
$$c_{-1}'' = (c_{-1} + Hb_0) - Hb_0'' - \bar{\partial}^*NH\rho_D(a_2', b_1').$$

Then

$$\begin{aligned} &d(a_1'', b_0'', c_{-1}'') \\ &= (D^*ND(a_1, b_0), (c_{-1} + Hb_0) - Hb_0'' - \bar{\partial}^*NH\rho_D(a_2', b_1' + Hb_0'')) \\ &= (D(a_1, b_0), c_{-1} + Hb_0) \\ &= (a_2', b_1', c_0'). \end{aligned}$$

□

This completes the proof of Theorem 2. □

4. An application

Using the operators Z_2, Q_2, Z_1, Q_1, by the method of [7], we can prove the following

THEOREM 4. *Let $d := \dim_{\mathbf{C}} H^1(K_{\overline{\Omega}}^{\bullet,\bullet})$ and $\ell := \dim_{\mathbf{C}} H^2(K_{\overline{\Omega}}^{\bullet,\bullet})$. Then there exist $b_1(t), \ldots, b_\ell(t) \in \mathbf{C}[[t_1, \ldots, t_d]]$ and $(\phi(t), g(t), k(t)) \in K_{\overline{\Omega}}^1[[t_1, \ldots, t_d]]$ such that*

(1) *$\phi(0) = 0$, $g(0) = 0$, $k(0) = 0$,*
(2) *$\bar{\partial}\phi(t) - \frac{1}{2}[\phi(t), \phi(t)] \equiv 0 \bmod (b_1(t), \ldots, b_\ell(t))$,*
(3) *$(\bar{\partial} - \phi(t))(f + g(t)) \equiv 0 \bmod (b_1(t), \ldots, b_\ell(t))$,*
(4) *$(h + k(t)) \circ (f + g(t)) \equiv 0 \bmod (b_1(t), \ldots, b_\ell(t))$,*
(5) *the linear term of $(\phi(t), g(t), k(t))$ is $\sum_{\gamma=1}^{d} t_\gamma(\phi_\gamma, g_\gamma, k_\gamma)$ where $\{(\phi_\gamma, g_\gamma, k_\gamma)\}_{\gamma=1,\ldots,d}$ is a basis of $H^1(K_{\overline{\Omega}}^{\bullet,\bullet})$,*
(6) *for any family $\left(\Phi : \widetilde{\mathcal{V}} \to \mathbf{C}^N \times \widehat{S}\right) \in \mathcal{D}ef_{\widetilde{V} \to \mathbf{C}^N}(\widehat{S})$ with $\overline{\Omega} \subset \widetilde{\mathcal{V}}_0$, where $\widehat{S}$ is a formal space, there exist a map $\tau : \widehat{S} \to \widehat{T}$, a map $G : \overline{\Omega} \times \widehat{S} \to \widetilde{\mathcal{V}}$ commuting with maps $\overline{\Omega} \times \widehat{S} \to \widehat{S}$ and $\widetilde{\mathcal{V}} \to \widehat{S}$, and a map $\zeta : B \times \widehat{S} \to \mathbf{C}^N \times \widehat{S}$ with $\tau(0) = 0$, $G|_{\overline{\Omega} \times 0} = \mathrm{id}_{\overline{\Omega}}$, $\zeta|_{B \times 0} = \mathrm{id}_B$ and $\Phi \circ G = \zeta \circ (f + g)$ and such that G is holomorphic with respect to the family of complex structures $\phi(\tau(s))$, where we denote by $f + g(t)$ a map $\overline{\Omega} \times \widehat{T} \to \mathbf{C}^N \times \widehat{T}$ defined by $f + g(t)$.*

(7) *If* $H^2(K^{\bullet,\bullet}_{\bar{\Omega}}) = 0$ *then* $b_1(t) = \cdots = b_\ell(t) = 0$.

Relying on this family, we complete the CR-description of the simultaneous resolution of deformations of a cone over rational curve of degree n in $\mathbf{P}^n$ for $n \le 4$.

Let $V_n := \mathbf{C}^2/\mathbf{Z}_n$ and $M_n := S^3/\mathbf{Z}_n$ where $\mathbf{Z}_n$ denotes a cyclic group generated by $g_n \in \mathrm{Aut}(\mathbf{C}^2)$ with $g_n(z_1, z_2) = (\zeta_n z_1, \zeta_n z_2)$ where $\zeta_n := e^{2\sqrt{-1}\pi/n}$. Then V_n is realized as a normal subvariety of $\mathbf{C}^{n+1}$. We denote by $f : V_n \to \mathbf{C}^{n+1}$ the natural inclusion map.

We denote by $\bar{Z}$, Z and T the vector fields on S^3 defined by $\bar{Z} := z_2\frac{\partial}{\partial \bar{z_1}} - z_1\frac{\partial}{\partial \bar{z_2}}$, $Z := \overline{z_2}\frac{\partial}{\partial z_1} - \overline{z_1}\frac{\partial}{\partial z_2}$ and $T := \sqrt{-1}\Big(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} - \overline{z_1}\frac{\partial}{\partial \overline{z_1}} - \overline{z_2}\frac{\partial}{\partial \overline{z_2}}\Big)$.

By applying the algorithm in [7] constructing the versal family of stably embeddable deformation of CR structures on M_n (hence, its image in $\mathbf{C}^{n+1}$ bounds the versal family of deformations of V_n), we have

THEOREM 5. ([8]) *The versal family of stably embeddable deformation of the CR structure on* M_n *is given as follows.*

(1) $(n = 2)$ *The parameter space is* $(\mathbf{C}, 0)$ *and the versal family is given by* $\phi(s) = sZ \otimes \bar{Z}^*$ *together with a family of holomorphic embeddings*

$$f + f(s) : (z_1,\, z_2) \mapsto (z_1^2 - s\bar{z_2}^2,\, z_1 z_2 + s\bar{z_1}\bar{z_2},\, z_2^2 - s\bar{z_1}^2).$$

(2) $(n = 3)$ *The parameter space is* $(\mathbf{C}^2, 0)$ *and the versal family is given by* $\phi(t_0, t_1) = (t_0\overline{z_1} + t_1\overline{z_2})T \otimes \bar{Z}^*$ *together with a family of holomorphic embeddings*

$$\begin{aligned} f + f(t_0, t_1) : (z_1,\, z_2) \mapsto \big(&z_1^3 - \frac{3}{2}\sqrt{-1}z_1\bar{z_2}(1 + |z_1|^2)t_0 - \frac{3}{2}\sqrt{-1}z_1^2\bar{z_2}^2 t_1, \\ &z_1^2 z_2 + \frac{3}{2}\sqrt{-1}|z|^4 t_0 - \frac{3}{2}\sqrt{-1}z_1\bar{z_2}|z_2|^2 t_1, \\ &z_1 z_2^2 + \frac{3}{2}\sqrt{-1}z_2\bar{z_1}|z_1|^2 t_0 - \frac{3}{2}\sqrt{-1}|z_2|^4 t_1, \\ &z_2^3 + \frac{3}{2}\sqrt{-1}z_2^2\bar{z_1}^2 t_0 + \frac{3}{2}\sqrt{-1}z_2\bar{z_1}(1 + |z_2|^2)t_1\big). \end{aligned}$$

(3) $(n = 4)$ *The parameter space is* $(\mathrm{Spec}\big(\mathbf{C}\{s, t_0, t_1, t_2\}/(st_0, st_1, st_2)\big), 0)$ *and the versal family is given by* $\phi(s, t_0, t_1, t_2) = sZ \otimes$

$\bar{Z}^* + (t_0\overline{z_1}^2 + t_1\overline{z_1 z_2} + t_2\overline{z_2}^2)T \otimes \bar{Z}^*$ together with a family of holomorphic embeddings

$$\begin{aligned} f + f(s,0,0,0) : (z_1,\, z_2) \mapsto \big(&z_1^4 - s2z_1^2\bar{z_2}^2 + s^2\bar{z_2}^4, \\ &z_1^3 z_2 - sz_1\bar{z_2}(|z_2|^2 - |z_1|^2) - s^2\bar{z_1}\bar{z_2}^3, \\ &z_1^2 z_2^2 + 2s|z_1|^2|z_2|^2 + s^2\bar{z_1}^2\bar{z_2}^2, \\ &z_1 z_2^3 + sz_2\bar{z_1}(|z_2|^2 - |z_1|^2) - s^2\bar{z_1}^3\bar{z_2}, \\ &z_2^4 - s2z_2^2\bar{z_1}^2 + s^2\bar{z_1}^4\big), \end{aligned}$$

$$\begin{aligned} f + f(0,t_0,t_1,t_2) : (z_1,\, z_2) \mapsto \big(&z_1^4 - t_0 4/3\sqrt{-1}z_1\bar{z_2}(|z_1|^4 + |z_1|^2 + 1) \\ &- t_1 4\sqrt{-1}z_1^2\bar{z_2}^2(|z_1|^2/2 + |z_2|^2/6) - t_2 4/3\sqrt{-1}z_1^3\bar{z_2}^3, \\ &z_1^3 z_2 + t_0 4/3\sqrt{-1}|z_1|^6 - t_1 4\sqrt{-1}z_1\bar{z_2}|z_2|^2(|z_1|^2/2 + |z_2|^2/6) \\ &- t_2 4/3\sqrt{-1}z_1^2\bar{z_2}^2|z_2|^2, \\ &z_1^2 z_2^2 + t_0 4/3\sqrt{-1}z_2\bar{z_1}|z_1|^4 + t_1 4\sqrt{-1}|z_1|^4(|z_1|^2/6 + |z_2|^2/2) \\ &- t_2 4/3\sqrt{-1}z_1\bar{z_2}|z_2|^4, \\ &z_1 z_2^3 + t_0 4/3\sqrt{-1}z_2^2\bar{z_1}^2|z_1|^2 + t_1 4\sqrt{-1}z_2\bar{z_1}|z_1|^2(|z_1|^2/6 + |z_2|^2/2) \\ &- t_2 4/3\sqrt{-1}|z_2|^6, \\ &z_2^4 + t_0 4/3\sqrt{-1}z_2^3\bar{z_1}^3 + t_1 4\sqrt{-1}z_2^2\bar{z_1}^2(|z_1|^2/6 + |z_2|^2/2) \\ &+ t_2 4/3\sqrt{-1}z_2\bar{z_1}(|z_2|^4 + |z_2|^2 + 1)\big). \end{aligned}$$

Where we denote points of M_n by the coordinate (z_1, z_2) of $\mathbf{C}^2$ (the ambient space of its universal covering).

Let $\widetilde{V} = H^{-n}$ and E be the zero-section where H denotes the hyperplane bundle over $\mathbf{P}^1$. Then $(\widetilde{V}, E)$ is the minimal resolution of V_n. We shall compare families of deformations of CR-structures on M_n as above with deformations in $\mathcal{D}ef_{\widetilde{V}\to\mathbf{C}^{n+1}}$.

First, we compare the spaces of first order deformations; $\mathrm{Ker}\{H^1(\overline{\Omega}, \Theta_{\widetilde{V}}) \to H^1(\overline{\Omega}, f^*\Theta_{\mathbf{C}^N})\}$ and $\mathrm{Ker}\{H^1_{\bar{\partial}_b}(M_n, T') \to H^1_{\bar{\partial}_b}(M_n, f^*T^{1,0}\mathbf{C}^N)\}$.

We will do it via the following isomorphisms of holomorphic vector bundles over M_n.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 1_{M_n} & \longrightarrow & T' & \longrightarrow & T^{1,0}M_n & \longrightarrow & 0 \\ & & \downarrow\simeq & & \rho^{1,0}\downarrow\simeq & & d\pi\downarrow\simeq & & \\ 0 & \longrightarrow & \pi^*H^{-n} & \longrightarrow & T^{1,0}\widetilde{V}|_{M_n} & \longrightarrow & \pi^*T^{1,0}\mathbf{P}^1 & \longrightarrow & 0 \end{array} \tag{4.1}$$

where π denotes a holomorphic map $M_n \to \mathbf{P}^1$ induced from the projection of the vector bundle $\widetilde{V} = H^{-n} \to \mathbf{P}^1$ and $\rho^{1,0} : T' \to T^{1,0}\widetilde{V}|_{M_n}$ the projection to the $(1,0)$-part with respect to the complex structure of $\widetilde{V}$.

Let $\xi_n := \frac{\partial}{\partial\theta}$ where θ denotes the angular coordinate of the fibres of H^{-n}. Then, via the n-th covering map $q_n : \mathbf{C}^2 \setminus \{(0,0)\} \to H^{-n} \setminus E$, we have

$$dq_n T = n\xi_n. \tag{4.2}$$

We note that $q_n(S^3) = M_n$ and consider the operation of $N_n := \sqrt{-1}L_{\xi_n}$ on $H^q_{\bar{\partial}_b}(M_n, 1_{M_n})$, $H^q_{\bar{\partial}_b}(M_n, T')$ and $H^q_{\bar{\partial}_b}(M_n, T^{1,0}M_n)$ respectively. We denote $H^q_{(\nu)}(1_{M_n})$, $H^q_{(\nu)}(T')$ and $H^q_{(\nu)}(T^{1,0}M_n)$ the eigenspace belonging to the eigenvalue ν.

PROPOSITION 4.1. $\mathrm{Ker}\{H^1_{\bar{\partial}_b}(M_n, T') \to H^1_{\bar{\partial}_b}(M_n, f^*T^{1,0}\mathbf{C}^N)\} \subset H^1_{(1)}(M_n, T')$.

Proof. Since

$$\sqrt{-1}L_T(\overline{z_1}^s\overline{z_2}^t Z \otimes \bar{Z}^*) = (s+t+4)\overline{z_1}^s\overline{z_2}^t Z \otimes \bar{Z}^*$$
$$\sqrt{-1}L_T(\overline{z_1}^s\overline{z_2}^t T \otimes \bar{Z}^*) = (s+t+2)\overline{z_1}^s\overline{z_2}^t T \otimes \bar{Z}^*,$$

we infer the proposition from (4.2) and Theorem 5. □

On the other hand, $N_n := \sqrt{-1}L_{\xi_n}$ naturally operates on the pullback bundle $\pi^*\mathcal{E}$ of a holomorphic vector bundle $\mathcal{E}$ over $\mathbf{P}^1$ and we have a natural isomorphism

$$\Gamma_{(\nu)}(M_n, \pi^*\mathcal{E}) \simeq \Gamma(\mathbf{P}^1, \mathcal{E} \otimes H^\nu) \ (\nu \in \mathbf{Z}). \tag{4.3}$$

This provides us a way to extend an element of $A^{0,q}_{M_n}(\pi^*\mathcal{E})$ to $A^{0,q}_{\widetilde{V}}(\pi^*\mathcal{E})$ holomorphically along the fibres of H^{-n}.

Since $T^{1,0}\widetilde{V}(-\log E)|_{M_n}$ is a pullback bundle, while $T^{1,0}\widetilde{V}$ is not, we consider the following isomorphisms instead of (4.1);

(4.4)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 1_{M_n} & \longrightarrow & T' & \longrightarrow & T^{1,0}M_n & \longrightarrow & 0 \\ & & \downarrow\simeq & & \rho^{1,0}\downarrow\simeq & & d\pi\downarrow\simeq & & \\ 0 & \longrightarrow & 1_{M_n} & \longrightarrow & T^{1,0}\widetilde{V}(-\log E)|_{M_n} & \longrightarrow & \pi^*T^{1,0}\mathbf{P}^1 & \longrightarrow & 0. \end{array}$$

Since the operations N_n on these bundles commutes with this diagram and with the $\bar{\partial}_b$-operators, we have the following isomorphisms of eigenspaces of cohomology groups;

(4.5)

$$\begin{array}{ccccccc} \longrightarrow & H^q_{(\nu)}(1_{M_n}) & \longrightarrow & H^q_{(\nu)}(T') & \longrightarrow & H^q_{(\nu)}(T^{1,0}M_n) & \longrightarrow \\ & \downarrow\simeq & & \rho^{1,0}\downarrow\simeq & & d\pi\downarrow\simeq & \\ \longrightarrow & H^q_{(\nu)}(1_{M_n}) & \longrightarrow & H^q_{(\nu)}(T^{1,0}(-\log E)|_{M_n}) & \longrightarrow & H^q_{(\nu)}(\pi^*T^{1,0}\mathbf{P}^1) & \longrightarrow. \end{array}$$

Using these isomorphisms, we can compare the spaces of first order deformations.

PROPOSITION 4.2. (1) ($n=2$) *The family of CR structures* $\phi(s) = sZ \otimes \bar{Z}^*$ *is not extendable to a family of deformations of complex structures on* H^{-2}.

(2) ($n=4$) *The family of CR structures* $\phi(s,0,0,0) = sZ \otimes \bar{Z}^*$ *is not extendable to a family of deformations of complex structures on* H^{-4}.

Proof. We note that $Z \otimes \bar{Z}^*$ defines a non-zero cohomology class in $H^1_{(1)}(M_n, T^{1,0}M_n) \simeq H^1(\mathbf{P}^1, T^{1,0}\mathbf{P}^1 \otimes H)$. Hence, the cohomology class cannot be extended to a cohomology class in $H^1(\widetilde{V}, \pi^*T^{1,0}\mathbf{P}^1)$. It implies the impossibility of extending $Z \otimes \bar{Z}^*$ to a cohomology class in $H^1(\widetilde{V}, T^{1,0}\widetilde{V})$.

Hence the first order deformation of CR structure represented by $sZ \otimes \bar{Z}^*$ is not extendable to a first order deformation of complex structure on $\widetilde{V}$. □

Next, we prove the following extendability of families of CR structures to families of complex structures, parametrized by another component.

PROPOSITION 4.3. (1) ($n=3$) *The family of CR structures on* M_3, $\phi(t_0, t_1)$ *is extendable to a family of complex structures on* H^{-3} *which is stably mapped to* $\mathbf{C}^4$.

(2) ($n=4$) *The family of CR structures on* M_4, $\phi(0, t_0, t_1, t_2)$ *is extendable to a family of complex structures on* H^{-4} *which is stably mapped to* $\mathbf{C}^5$.

Proof. First, we prove that the family of CR structures is extendable to a family of complex structures on $\widetilde{V}$. By the commutative diagram (4.5), the cohomology class of $\bar{z}_1^s \bar{z}_2^t \rho^{1,0} T \otimes \bar{Z}^*$ ($s+t=n-2$) is extendable to a cohomology class in $H^1(\widetilde{V}, T^{1,0}\widetilde{V})$. Hence the family of CR structure is extendable to a first order deformation of complex structure on $\widetilde{V}$. Since that extension is holomorphic along the fibres, it preserves the integrability condition. Indeed, the extension to higher order deformation of complex structure is given by

$$\widetilde{\phi}(t_0, t_1) := -3\sqrt{-1}(t_0 + t_1\bar{u})/(1+|u|^2)^3 \frac{\partial}{\partial \zeta} \otimes d\bar{u} \ (n=3),$$

$$\widetilde{\phi}(t_0, t_1, t_2) := -4\sqrt{-1}(t_0 + t_1\bar{u} + t_2\bar{u}^2)/(1+|u|^2)^4 \frac{\partial}{\partial \zeta} \otimes d\bar{u} \ (n=4),$$

where u (resp. ζ) denotes the inhomogeneous coordinate of $\mathbf{P}^1$ (resp. the fibre coordinate of H^{-n}).

Next, we prove the extendability of the family of CR-embeddings to a family of holomorphic mappings. Since $f(t_0,t_1) \in A^0_{(0)}(f^*T^{1,0}\mathbf{C}^4)[t_0,t_1]$ and $f(0,t_0,t_1,t_2) \in A^0_{(0)}(f^*T^{1,0}\mathbf{C}^5)[t_0,t_1,t_2]$, we have their natural extensions $\widetilde{f}(t_0,t_1) \in A^0_{\widetilde{V}}(f^*T^{1,0}\mathbf{C}^4)[t_0,t_1]$ and $\widetilde{f}(t_0,t_1,t_2) \in A^0_{\widetilde{V}}(f^*T^{1,0}\mathbf{C}^5)$ $[t_0,t_1,t_2]$ respectively (cf. (4.3)), which are constant along the fibres of H^{-n} and which respectively satisfy the parametrized Cauchy-Riemann equations

$$\big(\bar\partial - \widetilde{\phi}(t_0,t_1)\big)\big(f+\widetilde{f}(t_0,t_1)\big) = 0,\ \big(\bar\partial - \widetilde{\phi}(t_0,t_1,t_2)\big)\big(f+\widetilde{f}(t_0,t_1,t_2)\big) = 0.$$

In fact, it is given by

$$\begin{aligned} f+\widetilde{f}(t_0,t_1) =&\big(\zeta - \tfrac{3}{2}\sqrt{-1}\overline{u}(2+|u|^2)/(1+|u|^2)^2 t_0 - \tfrac{3}{2}\sqrt{-1}\overline{u}^2/(1+|u|^2)^2 t_1,\\ &\zeta u + \tfrac{3}{2}\sqrt{-1}/(1+|u|^2)^2 t_0 - \tfrac{3}{2}\sqrt{-1}\overline{u}|u|^2/(1+|u|^2)^2 t_1,\\ &\zeta u^2 + \tfrac{3}{2}\sqrt{-1}u/(1+|u|^2)^2 t_0 - \tfrac{3}{2}\sqrt{-1}|u|^4/(1+|u|^2)^2 t_1,\\ &\zeta u^3 + \tfrac{3}{2}\sqrt{-1}u^2/(1+|u|^2)^2 t_0\\ &\quad + \tfrac{3}{2}\sqrt{-1}u(1+2|u|^2)/(1+|u|^2)^2 t_1\big)\ (n=3),\end{aligned}$$

$$\begin{aligned} f+\widetilde{f}(t_0,t_1,t_2) =&\big(\zeta - 4/3\sqrt{-1}\overline{u}(|u|^4+3|u|^2+3)/(1+|u|^2)^3 t_0\\ &- 2/3\sqrt{-1}\overline{u}^2(|u|^2+3)/(1+|u|^2)^3 t_1 - 4/3\sqrt{-1}\overline{u}^3/(1+|u|^2)^3 t_2,\\ &\zeta u + 4/3\sqrt{-1}/(1+|u|^2)^3 t_0 - 2/3\sqrt{-1}|u|^2\overline{u}(3+|u|^2)/(1+|u|^2)^3 t_1\\ &- 4/3\sqrt{-1}|u|^2\overline{u}^2/(1+|u|^2)^3 t_2,\\ &\zeta u^2 + 4/3\sqrt{-1}u/(1+|u|^2)^3 t_0 + 2/3\sqrt{-1}(1+3|u|^2)/(1+|u|^2)^3 t_1\\ &- 4/3\sqrt{-1}|u|^4\overline{u}/(1+|u|^2)^3 t_2,\\ &\zeta u^3 + 4/3\sqrt{-1}u^2/(1+|u|^2)^3 t_0 + 2/3\sqrt{-1}u(1+3|u|^2)/(1+|u|^2)^3 t_1\\ &- 4/3\sqrt{-1}|u|^6/(1+|u|^2)^3 t_2,\\ &\zeta u^4 + 4/3\sqrt{-1}u^3/(1+|u|^2)^3 t_0 + 2/3\sqrt{-1}u^2(1+3|u|^2)/(1+|u|^2)^3 t_1\\ &+ 4/3\sqrt{-1}u(3|u|^4+3|u|^2+1)/(1+|u|^2)^3 t_2\big)\ (n=4).\end{aligned}$$

□

These arguments provide us a Cauchy-Riemann theoretic approach to the Artin-component. By [1], it is known that there is an irreducible component of the parameter space of the versal family of deformations of V_n having the following property; after lifting the component of the parameter space to a finite cover, the family of singularities is simultaneously resolved. It is called the *Artin-component.*

We describe the Artin-component from our viewpoint, in the case as above.

COROLLARY 4.1. *The Artin-components are given as follows;*
(1) $(n=2)$ $(\mathbf{C},0)$,
(2) $(n=3)$ $(\mathbf{C}^2,0)$,
(3) $(n=4)$ $(\mathbf{C}^3,0)$ *defined by* $s=0$.

Proof. The case of $n=3$ is obvious from Proposition 4.3.

The case of $n=4$. First, we show that $H^1(\widetilde{V},\mathcal{T}_{\widetilde{V}/\mathbf{C}^N})=0$. Since V is rational surface singularity, we have $H^1(\widetilde{V},\mathcal{O}_{\widetilde{V}})=H^2(\widetilde{V},\Theta_{\widetilde{V}})=0$. Hence, by (1.1), we have $H^1(\widetilde{V},\mathcal{T}_{\widetilde{V}/\mathbf{C}^N})=0$. By Theorem 4 (7), the parameter space $(S,0)$ of the versal family for $\mathcal{D}ef_{\widetilde{V}\to\mathbf{C}^N}$ is smooth. Let $T=\left(\mathbf{C}\cup\mathbf{C}^3,(0,0)\right)$ be the parameter space of the versal family of deformations of V_4. Then there exists a holomorphic mapping $\tau:S\to T$ in the formal sense. By Propositions 4.2 and 4.3, $\tau(S)$ is contained in $\mathbf{C}^3$. On the other hand, there exists a holomorphic mapping $\sigma:\mathbf{C}^3\to S$ in the formal sense and it holds that $d\sigma\circ d\tau=\mathrm{id}_{T_0\mathbf{C}^3}$ and $d\tau\circ d\sigma=\mathrm{id}_{T_0S}$. Hence, τ (and σ) is a formal isomorphism and therefore $\tau(S)=\mathbf{C}^3$ in the formal sense. It implies that $(\mathbf{C}^3,0)$ defined by $s=0$ is the Artin-component.

The case of $n=2$ follows from the following proposition since Proposition 4.4 implies that if we lift the family of CR structures to a double cover of the parameter space, it bounds a deformation of H^{-2} which is stably mapped in $\mathbf{C}^3$. □

PROPOSITION 4.4. *The family of CR structures* $\psi(u)=uT\otimes\bar{Z}^*$ *together with a family of holomorphic embeddings*

$$f+g(u):\ (z_1,\,z_2)\mapsto(z_1^2-2uz_1\bar{z}_2,\ z_1z_2+u(|z_1|^2-|z_2|^2),\ z_2^2+2u\bar{z}_1z_2)$$

bounds the family of singular varieties defined by $w_0w_2-w_1^2+u^2=0$. *While the family of CR structures* $\phi(s)=sZ\otimes\bar{Z}^*$ *bounds the family defined by* $w_0w_2-w_1^2+s=0$.

The proof of the proposition is a direct computation.

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