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THE CONDITIONS FOR REPELLING THE AUTOMORPHISM ORBIT FROM THE BOUNDARY POINT

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ABSTRACT. In this paper, we first prove that there are no automorphism orbits accumulating at a boundary point of the largest isolated finite type. We also present a generalization of the results of Isaev and Krantz on the structure of the orbit accumulation points.

1. Introduction

For a domain Ω in $\mathbb{C}^n$, we denote by $\mathrm{Aut}(\Omega)$ the group of holomorphic automorphisms of Ω. It is obvious that $\mathrm{Aut}(\Omega)$ is a topological group with respect to the law of composition and the compact-open topology. In particular, it is a theorem of H. Cartan that $\mathrm{Aut}(\Omega)$ is in fact a Lie group, if Ω is bounded.

In light of the outstanding question "*Which domains possess non-compact automorphism group?*" there is much interest focused upon the existence and non-existence of orbits of the automorphism group action accumulating at a given boundary point. The well-known Greene-Krantz conjecture belongs to such a line of research. In this paper, we discuss the *finite* type boundary points that repel automorphism orbits.

Denote by τ the D'Angelo type (see [9]). Let $\mathcal{M}$ be the Catlin multitype (see [7]). Now, we present our main theorem.

THEOREM 1.1. *Let Ω be a domain in $\mathbb{C}^n$. Assume that there exists a point $p \in \partial\Omega$ admitting an open neighborhood U in $\mathbb{C}^n$ satisfying the conditions:*

(1) *the boundary $\partial\Omega$ is C^∞ smooth convex and of finite type in the sense of D'Angelo,*

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(2) $\mathcal{M}(\tilde{p}) \neq \mathcal{M}(p)$, *for every* $\tilde{p} \in U \cap \partial\Omega \setminus \{p\}$.
Then, there are no automorphism orbits in Ω *accumulating at* p.

In $n = 2$, there is a more general result without convexity assumption in [6]. This is the first result in higher dimensional complex space.

Another important theme in the study of pseudoconvex domains concerns the set $S(\Omega)$ of all orbit accumulation boundary points of the given domain Ω. An article by S. Fu, A. Isaev and S. G. Krantz ([10]) have analyzed the structure of $S(\Omega)$ for the case when Ω is Reinhardt, showing that $S(\Omega)$ forms a manifold of odd dimension between 1 and $2n-1$ inclusive. The result obtained by A. Isaev and S. G. Krantz ([14]) is that it is a perfect set if Ω is a bounded pseudoconvex domain with finite type boundary and if $S(\Omega)$ contains at least 3 points. We present in this article a resonant result in a more general situation, without assuming the boundedness or the Rienhardtness condition.

THEOREM 1.2. *Let* Ω *be a domain in* $\mathbb{C}^n$ *with a boundary point* $p \in \partial\Omega$ *admitting an open neighborhood* U *in which* $\partial\Omega$ *is* C^∞ *smooth convex of finite type in the sense of D'Angelo. If* p *is an automorphism orbit accumulation point, then* p *is also an accumulation point of the set* $S(\Omega)$.

In [14], A. Isaev and S. G. Krantz drew several questions. One of them is that for a smoothly bounded domain D, can the set $S(D)$ have uncountably many connected components? For example, can it be a Cantor-type set?

Here is a partial answer for this question.

PROPOSITION 1.3. *Let* Ω *be a smooth bounded pseudoconvex domain of finite type in* $\mathbb{C}^2$. *Then* $S(\Omega)$ *is connected.*

We organize this paper as follows. We give a brief introduction about the scaling theory on convex domains of finite type in Section 2. For smooth exposition, we introduce several boundary invariants for example, the Catlin multitype, and so on. In Section 3, we prove a key lemma and Theorem 1.1 and 1.2. In the last section, we give an introduction to the Hausdorff set convergence and prove Proposition 1.3.

2. The Catlin multitype and scaling theory

Let $(z_1, \ldots, z_n)$ denotes the standard Euclidean coordinate system of $\mathbb{C}^n$.

2.1. The Catlin multitype and linear multitype

Let ρ denote a smooth defining function such that $\Omega = \{(z_1, \dots, z_n) \mid \rho(z_1, \dots, z_n) < 0\}$. Let Γ_n denote the set of all n-tuples of numbers $\Lambda = (\lambda_1, \dots, \lambda_n)$ with $1 \le \lambda_j \le \infty$ such that

$$\lambda_1 \le \cdots \le \lambda_n, \tag{2.1}$$

and for each k, either $\lambda_k = \infty$ or there is a set of nonnegative integers $a_1, \dots, a_k$ with $a_k > 0$, such that

$$\sum_{j=1}^{k} \frac{a_j}{\lambda_j} = 1.$$

An element of Γ_n will be referred to as a *weight.* The set of weights can be ordered lexicographically, i.e. if $\Lambda' = (\lambda'_1, \dots, \lambda'_n)$ and $\Lambda'' = (\lambda''_1, \dots, \lambda''_n)$, then $\Lambda' < \Lambda''$ if for some k, $\lambda'_j = \lambda''_j$ for all $j < k$, but $\lambda'_k < \lambda''_k$.

Now let p be a given point in the boundary of a domain Ω with a defining function ρ. A weight $\Lambda \in \Gamma_n$ is said to be *distinguished* if there exist holomorphic coordinates $(z_1, \dots, z_n)$ about p with p mapped to the origin such that

$$\text{if } \sum_{j=1}^{n} \frac{\alpha_j + \beta_j}{\lambda_j} < 1, \text{ then } D^\alpha \overline{D}^\beta \rho(p) = 0.$$

Here D^α and $\overline{D}^\beta$ denote the partial derivative operators

$$\frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}} \quad \text{and} \quad \frac{\partial^{|\beta|}}{\overline{\partial} z_1^{\beta_1} \cdots \overline{\partial} z_n^{\beta_n}}$$

respectively.

Definition 2.1. The *multitype* $\mathcal{M}(p)$ is defined to be the smallest weight $\mathcal{M} = (m_1, \dots, m_n)$ in Γ_n such that $\mathcal{M} \ge \Lambda$ for every distinguished weight Λ.

A weight $\Lambda \in \Gamma_n$ is said to be *linear distinguished* if there exist linear holomorphic coordinates $(z_1, \dots, z_n)$ about p with p mapped to the origin such that

$$\text{If } \sum_{j=1}^{n} \frac{\alpha_j + \beta_j}{\lambda_j} < 1, \text{ then } D^\alpha \overline{D}^\beta \rho(p) = 0.$$

Definition 2.2. The *linear multitype* $\mathcal{L}(p)$ is defined to be the smallest weight $\mathcal{L} = (l_1, \dots, l_n)$ in Γ_n such that $\mathcal{L} \ge \Lambda$ for every distinguished weight Λ.

2.2. The D'Angelo type and line type

If f is a smooth, complex-valued function, defined near the origin in $\mathbb{C}$, let $\nu(f)$ denotes the order of vanishing of $f - f(0)$ at the origin. For vector-valued $F = (f_1, \dots, f_n)$, let $\nu(F) = \min_j \nu(f_j)$. The following definition was formulated by D'Angelo in [9]. In two dimensional complex space, J. J. Kohn introduced and studied the boundary invariant type in [16].

DEFINITION 2.3. Let Ω be a smooth domain with defining function ρ. A point p is of finite q-dimensional variety type if there exists a constant m such that

$$\Delta_q(p) = \sup_F \frac{\nu(\rho \circ F)}{\nu(F)} \leq m$$

for F a holomorphic parameterization of a q-dimensional complex analytic subvariety of $\mathbb{C}^n$ with $F(0) = p$. $\Delta_q(p)$ is called the *q-dimensional variety type at p*.

A complex line in $\mathbb{C}^n$ is a set of points of the form $\{a\zeta + b \mid \zeta \in \mathbb{C}\}$ for fixed $a, b \in \mathbb{C}^n$. In a manner analogous to Definition 2.3, we will consider the order of contact of $\partial\Omega$ with complex lines.

DEFINITION 2.4. p is a point of finite line type if there exists a constant K such that

$$\sup_l \nu(\rho \circ l) \leq K$$

for l a parameterization of a complex line with $l(0) = p$. The smallest K for which the inequality holds will be called the *line type* of p denoted by $L(p)$.

It follows immediately from the definitions that

$$L(p) \leq \Delta_1(p).$$

2.3. The relation about several types

Let p be a boundary point of Ω. Assume that $\mathcal{M}(p) = (m_1, \dots, m_n)$ is the Catlin multitype at p. By [7], we have the following inequality

$$m_{n+1-1} \leq \Delta_q(p)$$

for each $q = 1, \dots, n$.

According to [5, 18], if Ω is convex, then $L(p) = \Delta_1(p)$. J. Yu ([24]) proved that the Catlin multitype and the linear multitype coincide:

$$\mathcal{L}(p) = \mathcal{M}(p),$$

if Ω is a smooth bounded domain in $\mathbb{C}^n$ and if Ω is convex near p.

EXAMPLE 2.5. Let $H(z_2, \ldots, z_n)$ be a real valued polynomial defined on $\mathbb{C}^{n-1}$ and let $\alpha_2, \ldots, \alpha_n$ be positive real numbers with $\alpha_2 \leq \cdots \leq \alpha_n$. Then H is called *weighted homogeneous* of degree one with respect to $\alpha_2, \ldots, \alpha_n$ if

$$H(t^{1/\alpha_2} z_2, \ldots, t^{1/\alpha_n} z_n) = tH(z_2, \ldots, z_n)$$

for every $t \in \mathbb{R}$.

The polynomial H is *non-degenerate* if the set $\{H = 0\}$ does not contain any analytic set of positive dimension. We denote M_H by

$$M_H = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid \text{Re } z_1 + H(z_2, \ldots, z_n) < 0\},$$

which is usually called a *model domain.*

If H is a weighted homogeneous non-degenerate polynomial of degree one with respect to $\alpha_2, \ldots, \alpha_n$, then $\mathcal{M}(o) = (1, \alpha_2, \ldots, \alpha_n)$, where $o \in \partial M_H$ is the origin of $\mathbb{C}^n$.

2.4. The scaling method and its convergence

Since S. Pinchuk developed the scaling method, the other researcher improved the scaling method for example, E. Bedford, K. T. Kim, S. Krantz, F. Berteloot, J. McNeal, H. Gaussier, and A. Kodama and so on. In order to smooth exposition, we would like to introduce scaling method in convex domain in $\mathbb{C}^n$. The content of this subsection is contained in [11, 12, 19].

Let Ω be a domain in $\mathbb{C}^n$ of finite type and let $\mathcal{M}(p) = (1, \alpha_2, \ldots, \alpha_n)$. Assume that $\partial\Omega$ is convex near a point p of $\partial\Omega$. There exists a neighborhood V of p in $\mathbb{C}^n$ such that $\Omega \cap V$ is convex and is defined by a convex function ρ of the form

$$\rho(z_1, \ldots, z_n) = \text{Re } z_1 + H(z_2, \ldots, z_n) + \cdots,$$

where H is a non-degenerate polynomial of degree one with respect to $\alpha_2, \ldots, \alpha_n$ and the dots denote terms of degree more than one.

From now on, we assume that p is an orbit accumulation point of Ω. Then there are a sequence $\{h_j\}$ of automorphisms of Ω and an interior point $q \in \Omega$ such that

$$\lim_{j \to \infty} h_j(q) = p.$$

We may assume that $h_j(q) \in V \cap \Omega$ for every j and let T_j be the translation defined by $T_j(z) = z - h_j(q)$. We will construct a unitary map A_j such that A_j converges to a unitary map A.

We first compute the distance from $h_j(q)$ to $\partial\Omega \cap V$. For a sufficiently small neighborhood V, there is a unique point $p_1^j \in \partial\Omega \cap V$ such that $\operatorname{dist}(h_j(q), p_1^j) = \operatorname{dist}(h_j(q), \partial\Omega \cap V)$. Let z_1^j be the corresponding complex line satisfying that p_1^j lies in the real positive axis z_1^j. We set $\delta_1^j = \operatorname{dist}(h_j(q), p_1^j)$. We consider the orthogonal complement of the complex line z_1^j through $h_j(q)$ and compute the distance from $h_j(q)$ to $\partial\Omega \cap V$ on each complex line in this complement. Because of the assumption of finite type, the largest such distance is finite and is achieved at a point p_2^j on the real positive axis x_2^j of the complex line z_2^j. We set $\delta_2^j = \operatorname{dist}(h_j(q), p_2^j)$. Repeating this process, we obtain real numbers $\delta_1^j, \ldots, \delta_n^j$. We can define a unitary map A_j such that

1) $A_j \circ T_j$ is a holomorphic coordinate system centered at $h_j(q)$, denoted by $(z_1^j, \ldots, z_n^j)$,
2) for each $k = 1, \ldots, n$, the segment of $h_j(q)$ and p_k^j lies the positive real axis of z_k^j, where a holomorphic coordinate system $(z_1^j, \ldots, z_n^j)$ centered at $h_j(q)$.

Define

$$\Lambda_j(z_1, \ldots, z_n) = \left(\frac{z_1}{\delta_1^j}, \ldots, \frac{z_n}{\delta_n^j}\right).$$

Let

$$\begin{aligned} \varepsilon_j &= -\rho \circ h_j(q) \\ \rho_j &= \frac{1}{\varepsilon_j} \rho \circ (\Lambda_j \circ A_j \circ T_j)^{-1}. \end{aligned}$$

then ρ_j is given in a neighborhood of the origin by

$$\begin{aligned} \rho_j(z) &= -1 + \frac{1}{\varepsilon_j} \mathrm{Re}\left(\sum_{\nu=1}^{n} a_\nu^j \delta_\nu^j z_j\right) \\ &\quad + \frac{1}{\varepsilon_j} \sum_{2 \le |\alpha|+|\beta| \le 2m} C_{\alpha\beta}^j \delta^{|\alpha|+|\beta|} z^\alpha \bar{z}^\beta + \frac{1}{\varepsilon_j} \mathcal{O}(|z|^{2m+1}). \end{aligned}$$

By Proposition 2.1 in [11], ρ_j converges uniformly on compact subset of $\mathbb{C}^n$ to a smooth convex function $\tilde{\rho}$ of the form

$$\tilde{\rho}(z) = -1 + \mathrm{Re} \sum b_j z_j + \tilde{H}(z),$$

where $\tilde{H}$ is a real convex polynomial of degree less than or equal to $2m$. This result depends on the uniform estimates of the coefficients of ρ_j Also Gaussier proved that Ω is taut and that the sequences $\{f_j\}$ and $\left\{f_j^{-1}\right\}$ are normal families, where $f_j = \Lambda_j \circ A_j \circ T_j \circ h_j$. By Cartan Uniqueness Theorem, we can deduce the following result.

THEOREM 2.6 (Gaussier). *Let Ω be a domain in $\mathbb{C}^n$ and $p \in S(\Omega)$ an orbit accumulation point with $\mathcal{M}(p) = (1, \alpha_2, \ldots, \alpha_n)$. Assume that there is a neighborhood U of p such that $\partial\Omega \cap U$ is of class C^∞ and of finite D'Angelo type. If there is a coordinate system centered at p such that $\Omega \cap U$ is convex in this system, then Ω is biholomorphic to its model domain M_H at p defined by*

$$M_H = \{(z_1, \ldots, z_n) \mid Re\, z_1 + H(z_2, \ldots, z_n) < 0\},$$

where H is a non-degenerate polynomial of degree one with respect to $\alpha_2, \ldots, \alpha_n$.

This reveals the uniqueness of model domain in $\mathbb{C}^n$ with respect to the automorphism orbit accumulation point p. F. Berteloot proved that the uniqueness of model domain in $\mathbb{C}^2$ as follows.

THEOREM 2.7 (Berteloot). *Let Ω be a domain in $\mathbb{C}^2$ and let p be a point on $\partial\Omega$. Suppose that $\partial\Omega$ is of class C^∞, pseudoconvex and of D'Angelo finite type τ in a neighborhood of p. Let $\varphi_j \in \mathrm{Aut}(\Omega)$ satisfy*

$$\lim_{j \to \infty} \varphi_j(q) = p$$

for a point $q \in \Omega$. Then Ω is biholomorphically equivalent to the model domain M_H, where H is a homogeneous subharmonic polynomial of degree $\tau(p)$ such that the local defining function ρ of Ω near p is represented by

$$\rho(z_1, z_2) = Re\, z_1 + H(z_2, \bar{z}_2) + \cdots .$$

3. Proofs

Let Δ denote the open unit disc in $\mathbb{C}$. The following lemma is contained in [6]. For a smooth exposition, we prove it.

LEMMA 3.1. *Let Ω be a domain in $\mathbb{C}^n$ and let p be a boundary point of Ω. Assume that there are an open neighborhood U of p in $\mathbb{C}^n$ and a sequence of injective proper holomorphic maps $g_j : \Delta \to \Omega$ satisfying the following conditions*

1) *$U \cap \Omega$ is pseudoconvex,*
2) *$\lim_{j \to \infty} g_j(0) = p$.*

Let $E = \overline{\bigcup_{j=1}^{\infty} g_j(\Delta)} \cap \partial\Omega$. Then p is not an isolated point of E.

Proof. Expecting a contradiction, we suppose that p is an isolated point of E. So, there exists $\delta > 0$ such that

$$\|p-q\| \geq \delta, \ \forall q \in E \setminus \{p\}.$$

Choosing a subsequence if necessary, we may assume that

$$g_j(0) \in B(p; \frac{\delta}{4}), \ \forall j = 1, 2, \ldots.$$

Now, for each t with $\delta/3 < t < 2\delta/3$, we let

$$\begin{aligned} S_t &= \{z \in \mathbb{C}^n \mid \|p-z\| = t\} \\ B_t &= \{z \in \mathbb{C}^n \mid \|p-z\| < t\}. \end{aligned}$$

Applying Morse-Sard theorem to the smooth map

$$F : g_m(\Delta) \to \mathbb{R}$$

defined by $F(\zeta) = \|\zeta - p\|^2$, we infer that *for each positive integer m, and for almost all values of t, the set $S_t \cap g_m(\Delta)$ is in fact a real 1-dimensional manifold without boundary.* We also can conclude that $\overline{B_t} \cap g_m(\Delta)$ is a smooth submanifold with boundary in $S_t \cap g_m(\Delta)$.

Moreover, we shall verify that $S_t \cap g_m(\Delta)$ is a compact set. Since it is a bounded subset of $\mathbb{C}^n$, we are only to prove that it is closed.

Let $x \in \overline{S_t \cap g_m(\Delta)}$. Then there is a sequence $x_k \in S_t \cap g_m(\Delta)$ such that $x_k \to x$ as $k \to \infty$. Since $x_k \in g_m(\Delta)$, there exists a sequence $\zeta_k \in \Delta$ such that $g_m(\zeta_k) = x_k$ for each k. Because $\overline{\Delta}$ is compact, there are a point $\zeta \in \overline{\Delta}$ and a subsequence ζ_{kl} such that $\zeta_{kl} \to \zeta$ as $l \to \infty$. If $\zeta \in \partial\Delta$, by virtue of the properness of g_m, we have

$$\begin{aligned} x &= \lim_{l \to \infty} x_{kl} \\ &= \lim_{l \to \infty} g_m(\zeta_{kl}) \\ &\in \partial\Omega. \end{aligned}$$

This leads us to $\|x - p\| = t$ and $x \in E \setminus \{p\}$. Since this is impossible, we must have $\zeta \in \Delta$. Hence $x = g_m(\zeta) \in g_m(\Delta)$. This implies that $S_t \cap g_m(\Delta)$ is closed. □

Let X_m denote the connected component of $B_t \cap g_m(\Delta)$ with $g_m(0) \in X_m$ and let $G_m = g_m^{-1}(X_m)$. Then G_m is a domain in Δ. We now show

CLAIM. *There exists a simple closed curve γ_m in Δ satisfying*

(1) $g_m(\gamma_m) \subset S_t$,

(2) 0 *is an interior point of* γ_m.

Proof of the claim. Notice that 0 is an interior point of G_m and $g_m^{-1}(S_t \cap g_m(\Delta))$ is a finite union of simple closed curves. Therefore, the claim follows by the argument principle as soon as we prove that $\partial G_m \subset g_m^{-1}(S_t \cap g_m(\Delta))$.

Step 1. If $x \in \partial G_m \cap \Delta$, then $x \in g_m^{-1}(S_t \cap g_m(\Delta))$.

Since $x \in \Delta \cap \partial G_m$, we obtain $g_m(x) \in \overline{X_m} \setminus X_m$. Since $g_m(x) \in \Omega$, we have $g_m(x) \in \Omega \cap \overline{X_m} \setminus X_m \subset S_t \cap g_m(\Delta)$. Therefore, $x \in g_m^{-1}(S_t \cap g_m(\Delta))$.

Step 2. $\partial G_m \subset g_m^{-1}(S_t \cap g_m(\Delta))$.

In order to prove this, we suppose that there is a point $x \in \partial\Delta \cap \partial G_m$. We can choose $r_0 < 1$ so that $g_m^{-1}(S_t \cap g_m(\Delta)) \subset \{z \in \mathbb{C} \mid |z| < r_0\}$. Now, we are only to show that $C_r = \{z \in \mathbb{C} \mid |z| = r\}$ is contained in G_m if $r_0 < r < 1$. The existence of $x \in \partial\Delta$ guarantees that $C_r \cap G_m$ is nonempty. If $C_r \not\subset G_m$, then there is a point $q \in C_r \cap \partial G_m$. By step 1, $q \in g_m^{-1}(S_t \cap g_m(\Delta))$ and $|q| = r$. This is a contradiction. □

Let γ_m be the simply connected curve selected in the preceding claim. Let Γ_m be the set of all interior points of γ_m. It contains the origin by construction. We then choose a Riemann map $f_m : \Delta \to \Gamma_m$ with $f_m(0) = 0$. Then the composition $h_m = g_m \circ f_m : \Delta \to \Omega$ defines an analytic disc satisfying

$$\begin{aligned} h_m(0) &= g_m(0) \\ h_m(\partial\Delta) &= g_m(\gamma_m) \subset S_t. \end{aligned}$$

Since Ω is pseudoconvex, we have that $-\log d(x, \partial\Omega) : \Omega \to \mathbb{R}$ is a plurisubharmonic function, where $d(x, \partial\Omega)$ denotes the Euclidean distance from x to the boundary $\partial\Omega$. Consequently,

$$\begin{aligned} d(h_m(\partial\Delta), \partial\Omega) &= d(h_m(\Delta), \partial\Omega) \\ &\leq d(h_m(0), \partial\Omega) \\ &\leq d(g_m(0), p) \\ &\to 0 \text{ as } m \to \infty. \end{aligned}$$

In particular, there exists $q_m \in h_m(\partial\Delta) \subset S_t$ such that

$$d(q_m, \partial\Omega) = d(h_m(\partial\Delta), \partial\Omega) \to 0$$

as $m \to \infty$. Thus, there exists $q \in S_t \cap \partial\Omega$ such that q is a limit point of the sequence q_m. Namely, we have found a point $q \in E \cap S_t$. By the choice of t, we have arrived at the desired contradiction.

3.1. The proof of Theorems 1.1 and 1.2

Suppose that there are a sequence $\{\varphi_j\} \subset \mathrm{Aut}(\Omega)$ and a point $x \in \Omega$ such that

$$\lim_{j\to\infty} \varphi_j(x) = p.$$

By Theorem 2.6, there is a biholomorphism Ψ between Ω and the domain M_H, where H is a non-degenerate weighted homogeneous polynomial of degree one with respect to $\alpha_2, \dots, \alpha_n$. Since H has some homogeneous property, the model domain M_H contains the real half plane $P = \{(z_1, \dots, z_n) \mid \mathrm{Re}\, z_1 < 0, z_2 = \cdots = z_n = 0\}$ which is contained in the orbit of $(-1, 0, \dots, 0)$ by the action of the automorphism group $\mathrm{Aut}(M_H)$.

Define an injective proper holomorphic map $\mu : \Delta \to M_H$ by

$$\mu(z) = \left(\frac{z-1}{z+1}, 0, \dots, 0\right)$$

for every $z \in \Delta$. We consider a sequence of injective proper holomorphic maps $g_j := \varphi_j \circ \Psi^{-1} \circ \mu$ from the unit disc into Ω satisfying

$$g_j(0) = \varphi_j(q),$$

where $q = \Psi^{-1}(-1, 0, \dots, 0)$. Moreover $g_j(\Delta)$ is contained in the orbit of q by an action of $\mathrm{Aut}(\Omega)$.

By lemma 3.1, p is not an isolated point of $\overline{\bigcup_{j=1}^{\infty} g_j(\Delta)} \cap \partial\Omega \subset S(\Omega)$. Therefore, the proof of Theorem 1.2 was done.

For the proof of Theorem 1.1, we will suppose that there are a sequence $\{\varphi_j\} \in \mathrm{Aut}(\Omega)$ and a point $x \in \Omega$ such that

$$\lim_{j\to\infty} \varphi_j(x) = p.$$

By Theorem 1.1, there is a sequence $\{p_k\}$ of points in $S(\Omega)$ converging to p. By condition (2) in Theorem 1.1, we have $\mathcal{M}(p_N) \neq \mathcal{M}(p)$ for a sufficient large N. By Theorem 2.6, there are biholomorphisms Ψ, $\tilde{\Psi}$ and model domains M_H, M_{H_N} with respect to p, p_N in $S(\Omega)$ respectively. Note that a map $\tilde{\Psi} \circ \Psi^{-1}$ is a biholomorphism between M_H and M_{H_N}. By [8], we obtain $\mathcal{M}(p_N) = \mathcal{M}(p)$. This is a desired contradiction.

4. The Hausdorff set convergence

In this section, we introduce the Hausdorff set distance and convergence theorem which is called the Blaschke selection theorem. The content of this section is contained in [4, 13, 17, 21].

Let (X, d) be a metric space. For a point $x \in X$ and a positive real number r, we denote by

$$B(x; r) = \{y \in X \mid d(y, x) < r\}$$

and

$$\overline{B}(x; r) = \{y \in X \mid d(y, x) \leq r\}.$$

for a subset $A \subset X$ and $r > 0$, we denote the *r-neighborhood* of A by

$$N_r(A) = \bigcup_{a \in A} B(a; r).$$

Let A and B be subsets of X. We define the quantity $d^h(A, B)$ by

$$d^h(A, B) = \inf\{r > 0 \mid A \subset N_r(B) \text{and} B \subset N_r(A)\}.$$

Let $\mathcal{F}(X)$ be the family of all closed subsets of X. Then $(\mathcal{F}(X), d^h)$ is a metric space. Moreover, if X is complete, then $(\mathcal{F}(X), d^h)$ is a complete metric space.

THEOREM 4.1 (Blaschke's Selection Theorem). *Let (X, d) be a metric space, and let $K \subset X$ be a compact subset. Then the metric space $(\mathcal{F}(K), d^h)$ of closed subsets of K equipped with the Hausdorff metric d^h is compact.*

In a metric space, the compactness implies sequential compactness. We have the following statement.

THEOREM 4.2 (Generalized Blaschke's Selection Theorem). *Let (X, d) be a metric space. Let K be a compact subset of X. Then every sequence of closed subset of K with respect to the Hausdorff distance d^h.*

4.1. The proof of Proposition 1.3

For the proof, we present a fact in [6].

THEOREM 4.3. *Let Ω be a domain in $\mathbb{C}^2$ with a boundary point $p \in \partial\Omega$ admitting an open neighborhood U in which $\partial\Omega$ is C^∞ smooth pseudoconvex of finite type in the sense of D'Angelo. If p is an automorphism orbit accumulation point, then p is also an accumulation point of the set $S(\Omega)$.*

By the above theorem, we may suppose that $S(\Omega)$ contains infinitely many points.

Expecting a contradiction, we will suppose that $S(\Omega)$ is disconnected. We can choose two points p_1, p_2 and two distinct connected components

S_1, S_2 of $S(\Omega)$ satisfying $p_1 \in S_1$ and $p_2 \in S_2$. There are two automorphism orbits $\varphi_j^1(q)$ and $\varphi_j^2(q)$ converging to p_1, p_2 respectively.

By Theorem 2.7, there is a biholomorphism Ψ between Ω and the model domain M_H, where H is a homogeneous subharmonic polynomial without harmonic terms. By [20], $\mathrm{Aut}(\Omega)$ has finitely many connected components. Without the loss of generality, we may assume that $\Psi \circ \varphi_j^1(q)$, $\Psi \circ \varphi_j^2(q)$ is contained in the orbit of $\Psi(q)$ under action the connected component $\mathrm{Aut}_o(\Omega)$ of $\mathrm{Aut}(\Omega)$ containing the identity map.

We may construct a sequence of curves γ_j from $[0,1]$ into M_H such that

1) $\gamma_j(0) = \Psi \circ \varphi_j^1(q)$ and $\gamma_j(1) = \Psi \circ \varphi_j^2(q)$,
2) $\lim_{j\to\infty} \gamma_j(I) \subset \partial M_H \cup \{\infty\}$,
3) $\gamma_j(I)$ is a subset of the orbit of $\Psi(q)$ under the action $\mathrm{Aut}_o(\Omega)$,

where the convergence in 2) means the local Hausdorff set convergence.

Since Ψ is proper, the sequence $\Psi^{-1} \circ \gamma_j(I)$ is in the $\mathcal{F}(\overline{\Omega})$. By Theorem 4.2, there is a subsequence $\Psi^{-1}\gamma_{j_k}(I)$ whose Hausdorff set limit exists, denoted by $\tilde{I}$. Therefore, $\tilde{I} \subset S(\Omega)$. Since $\Psi^{-1} \circ \gamma_{j_k}(I)$ is connected, so is $\tilde{I}$. Since p_1, p_2 are contained in the $\tilde{I}$, we have a contradiction.

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