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HOMOGENEOUS POLYNOMIAL HYPERSURFACE ISOLATED SINGULARITIES

TAKAO AKAHORI

ABSTRACT. The mirror conjecture means originally the deep relation between complex and symplectic geometry in Calabi-Yau manifolds. Recently, this conjecture is posed beyond Calabi-Yau, and even for open manifolds (e.g. A_n singularities and its resolution) is discussed. While if we treat open manifolds, we can't avoid the boundary (in our case, CR manifolds). Therefore we pose the more precise conjecture (mirror symmetry with boundaries). Namely, in mirror symmetry, for boundaries, what kind of structure should correspond? For this problem, the A_n case is studied.

Let $(V^{(n)}, o)$ be an isolated singularity with complex dimension n in a complex Euclidean space C^N. Let M be the intersection of this $V^{(n)}$ and $S_\epsilon^{2N-1}(o)$. Then, over M a CR structure is induced. Kuranishi initiated the study of the deformation theory of CR structures, and since then, much progress has been made. Especially, related with the Calabi-Yau manifolds, Z^1 space, a subspace of the Kohn-Rossi cohomology, has been found and studied (see [5], [6]). In this paper, we study the hypersurface singularity. For a hypersurface singularity in C^{n+1}, defined by a polynomial f, the parameter space of the versal family is

$$C[z_1, \ldots, z_{n+1}]/(f, \frac{\partial f}{\partial z_i}).$$

In this paper, if f is a homogeneous polynomial, the correspondence between the Kodaira-Spencer class and $C[z_1, \ldots, z_{n+1}]/(f, \frac{\partial f}{\partial z_i})$, is written down explicitly. Using this expression, the space Z^1 is determined. Recently, in mathematical physics, string theory is discussed not only on Calabi-Yau manifolds, but also on isolated singularities. I hope that our work in CR structures might be applicable in this direction.

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1. The deformation theory of CR structures

Let $(V^{(n)}, o)$ be an isolated singularity in a complex Euclidean space C^N with complex dimension n. Let $S_\epsilon^{2N-1}(o)$ be a hypersphere, centered at o with radius ϵ. Let M be the intersection of this V and $S_\epsilon^{2N-1}(o)$. Then our M is a non-singular real odd dimensional C^∞ manifold. Moreover, a CR structure, $(M, {}^0T'')$ is induced by;

$$ {}^0T'' = C \otimes TM \cap T''(V - o), $$

and satisfies

$$ {}^0T'' \cap \overline{{}^0T''} = 0, \ \dim_{\mathbf{C}} {}^0T'' = n - 1, $$
$$ [X, Y] \in \Gamma(M, {}^0T'') \text{ for all } X, Y \in \Gamma(M, {}^0T''). $$

For this CR structures, the deformation theory has been developed. We, briefly sketch the deformation theory of CR structures (for the notations, for example $\overline{\partial}_{T'}$, $\Gamma(M, E_i)$, etc., see [3]). We take a supplemental vector field ξ and set $T' = {}^0T' + C \otimes \xi$, where ${}^0T' = \overline{{}^0T''}$. Then, like the case compact complex manifolds, there is a "standard deformation complex",

$$ 0 \to \Gamma(M, T') \overset{\overline{\partial}_{T'}}{\to} \Gamma(M, T' \otimes ({}^0T'')^*) \overset{\overline{\partial}_{T'}^{(1)}}{\to} \Gamma(M, T' \otimes \wedge^2({}^0T'')^*) \to $$

$$ \to \Gamma(M, T' \otimes \wedge^p({}^0T'')^*) \overset{\overline{\partial}_{T'}^{(p)}}{\to} \Gamma(M, T' \otimes \wedge^{p+1}({}^0T'')^*) \to \cdots . \tag{1.1} $$

For the formal category, it is enough. But if we want to get a convergent solution, we have to introduce a "new complex" (see [1], [3])

$$ 0 \to \mathcal{H} \overset{\overline{\partial}}{\to} \Gamma(M, E_1) \overset{\overline{\partial}^{(1)}}{\to} \Gamma(M, E_2) \to \cdots $$

$$ \to \Gamma(M, E_p) \overset{\overline{\partial}^{(p)}}{\to} \Gamma(M, E_{p+1}) \to \cdots . \tag{1.2} $$

With this complex, the versal family of CR structure $(M, {}^0T'')$ is given by solving the following pair of partial differential equations (the existence of the solution is proved in [1], [3]),

$$ \begin{cases} \overline{\partial}^{(1)}\phi + R_2(\phi) = 0, & \phi \in \Gamma(M, E_1) \\ \overline{\partial}^* \phi = 0. \end{cases} $$

Here $\overline{\partial}^*$ means the adjoint operator with respect to (1.2). While, our $\mathcal{H}$ is not a standard one. In fact, $\mathcal{H}$ is defined as follows:

$$ \mathcal{H} = \{u : u \in \Gamma(M, T'), \overline{\partial} u = 0\}. $$

And in $\overline{\partial} u = 0$, the first order derivative appears. So this space is just a prehilbert (not a space consisting of C^∞ sections of a certain vector bundle). In the case strongly pseudo convex, our $\overline{\partial}^*$ is a second order differential operator. And the following holds (see Lemma 5.3 in [3]).

PROPOSITION 1.1. *$\overline{\partial}^*$ is expressed a composition of two differential operators*

$$\overline{\partial}^* = \pi_{\widetilde{H}_0} \circ \overline{\partial}^*_{T'} \text{ on } \Gamma(M, E_1).$$

Here $\overline{\partial}^*_{T'}$ means the formal adjoint operator and $\pi_{\widetilde{H}_0}$ includes the first order derivation. So our harmonic space is

$$\{u : u \in \Gamma(M, E_1), \overline{\partial}^{(1)} u = 0, \pi_{\widetilde{H}_0} \circ \overline{\partial}^*_{T'} u = 0\}$$

and the harmonic space includes as a subspace

$$\{u : u \in \Gamma(M, E_1), \overline{\partial}^{(1)} u = 0, \overline{\partial}^*_{T'} u = 0\}$$

because of Proposition 1.2. Surprisingly, the latter space comes in a different subject (see [6]). In fact, we assume that our CR structure admits a non vanishing CR form of $\Gamma(M, \wedge^n (T')^*)$. This is an analogy of Calabi-Yau manifold. And we assume that our ξ satisfies

$$[\xi, \Gamma(M, {}^0T'')] \subset \Gamma(M, {}^0T'').$$

Then by wedging $\Gamma(M, \wedge^n (T')^*)$ from the left hand side, T'-valued p form can be regarded as an element of $\wedge^{n-1}(T')^* \wedge^p ({}^0T'')^*)$. By this identification, infinitesimal deformation space $= \wedge^{n-1}(T')^*$ valued Kohn-Rossi cohomology (corresponds to the case $p = 1$). In this situation, a subspace Z^1 of infinitesimal deformation space is found. (see [6]). Here

$$Z^1 = \{u : u \in F^{n-1,1},\ d''u = 0, d'u = 0\}.$$

It is proved that this Z^1 = the image of wedge product of $\{u : u \in \Gamma(M, E_1), \overline{\partial}^{(1)} u = 0, \overline{\partial}^* u = 0\}$ (see Theorem 2.1 in [5]).

So the difference of both spaces seems an interesting problem. If infinitesimal deformation space Z^1 holds, then such an isolated singularity has a similar property as a Calabi-Yau manifold. So, what kind of isolated singularity has this property? And in a general isolated singularity, how Z^1 space is like These are next problems.

2. Compact complex manifold case

Let X be a compact complex manifold with complex dimension n. Let $(\mathfrak{X}, \pi, S)$ be a family of deformations of complex structures of X. This means that; $\mathfrak{X}$ is a complex manifold, S is also an analytic space with the origin o, and π is a smooth map (the rank of $\pi = \dim_C S$) from $\mathcal{X}$ to S. So we have the Kodaira Spencer map

$$\rho_o : T_o S \to H^1(X, T'X). \tag{2.1}$$

Assume that there is a holomorphic $(n, 0)$ form, ω, which is non-vanishing on X, and it can be extended to $\mathcal{X}$ holomorphically (we use the notation $\tilde{\omega}$ for this extension). In this situation, we write down the Kodaira-Spencer class. We take a C^∞ diffeomorphism map from $X \times S$ to $\mathcal{X}$,

$$\begin{array}{ccc} X \times S & \xrightarrow{i_s} & \mathcal{X} \\ \text{the projection to the second factor} \downarrow & & \downarrow \pi \\ S & \xrightarrow{\text{identity map}} & S. \end{array}$$

Then, we have

THEOREM 2.1.

$$\{\frac{\partial}{\partial s} i_s^*(\tilde{\omega} \mid_{X_s}) \mid_{s=o}\}^{(n-1,1)} = \omega \,\bar{\wedge}\, \rho_o((\frac{\partial}{\partial s})_o).$$

Here, $\{\frac{\partial}{\partial s} i_s^*(\tilde{\omega} \mid_{X_s}) \mid_{s=o}\}^{(n-1,1)}$ means the $(n-1, 1)$ part of $\frac{\partial}{\partial s} i_s^*(\tilde{\omega} \mid_{X_s}) \mid_{s=o}$ and $\bar{\wedge}$ means the inner product, and the equality holds in $H^1(X, \wedge^{n-1}(T'X)^*)$.

Obviously, this holds without the assumption of compactness, but in this case, we must slightly change the notion C^∞ diffeomorphism map.

3. Deformation theory on $\mathbf{C}[\mathbf{z_1}, \ldots, \mathbf{z_{n+1}}]/(\mathbf{f}, \frac{\partial \mathbf{f}}{\partial \mathbf{z_i}})$

Now in this section, we treat a hypersurface isolated singularity

$$V = \{(z_1, \ldots, z_{n+1}) : f(z_1, \ldots, z_{n+1}) = 0\}$$

and its boundary M

$$M = V \cap \{(z_1, \ldots, z_{n+1}) : | z_1 |^2 + \cdots + | z_{n+1} |^2 = 1\}.$$

Here we assume that; if $p \neq o$, then $df(p) \neq 0$. As is well known, the versal family of this isolated singularity is

$$\tilde{V} = \{(z_1, \ldots, z_{n+1}, t_1, \ldots, t_l) : f(z_1, \ldots, z_{n+1}) = t_1 g_1 + \cdots + t_l g_l\} \tag{3.1}$$

where g_i is a representative of deformation algebra $C[z_1, \ldots, z_{n+1}]/(f, \frac{\partial f}{\partial z_i})$, l means the dimension of this algebra. We would like to give the correspondence with this algebra and our deformation theory of CR structures.

Usually, in the algebraic geometry, people study the residue of $n+1$ form, $\frac{1}{f(z)-t}dz_1 \wedge \cdots \wedge dz_{n+1}$ but we adopt a different method.

For the C^{n+1}, we put a metric $<,>$ (if f is a quasi homogeneous polynomial, instead of the usual metric, we must consider the weighted metric). With this metric, we introduce type (1,0) vector X_f on $C^{n+1}-o$ by;

(3.2) $< Y, X_f >= Yf \quad \text{on} \quad C^{n+1}-o, \quad Y \in T'(C^{n+1}-o).$

LEMMA 3.1. *X_f does not vanish at every point of $C^{n+1}-o$.*

Proof. If X_f vanishes at a point p of $C^{n+1}-o$, by the definition of X_f, at p, $(Yf)(p) = 0$ for any $Y \in T'(C^{n+1}-o)$. This is absurd because of $df(p) \neq 0$ at every point p of $C^{n+1}-o$. □

Next we set $X'_f = \frac{1}{X_f f} X_f$ and consider a differential form on $C^{n+1}-o$.

(3.3) $$\omega' = X'_f \rfloor dz_1 \wedge \cdots \wedge dz_{n+1}.$$

By the definition, this is a form type $(n, 0)$. Moreover, we have

LEMMA 3.2.

$$dz_1 \wedge \cdots \wedge dz_{n+1} = df \wedge \omega'$$

on $C^{n+1}-o$.

From this lemma, the following proposition follows.

PROPOSITION 3.3.

$$d(\omega' \,|_{V_t}) = 0.$$

Here

$$V_t = \{(z_1, \ldots, z_{n+1}) : f(z) = t\}.$$

This means that $\omega' \,|_{V_t}$ is V_t non-vanishing holomorphic $(n, 0)$-form (canonical form).

Next we set C^∞ diffeomorphism map defined on a neighborhood of M $i_t : V - o \to V_t$ by;

$$(z_1, \ldots, z_{n+1}) \to (z_1 + (X'_f z_1)t, \ldots, z_{n+1} + (X'_f z_{n+1})t).$$

This map satisfies **Key relation 1**

$$i_t^* f \equiv f + t \quad \mod \ t^2.$$

Therefore this map makes sense as mod t^2. By pulling back $\omega' \,|_{V_t}$ by this map, we define a total degree n form ω'_1 on V by;

$$i_t^*(\omega' \,|_{V_t}) = \omega' \,|_{V-o} + \omega'_1 t \qquad \mod \ (\bar{t}, t^2).$$

Then by $d(\omega' \,|_{V_t}) = 0$ on V_t, we have the following proposition.

PROPOSITION 3.4.

$$d\omega_1' = 0 \quad on \quad V.$$

We denote the $(n-1,1)$ (resp. $(n,0)$) part of ω_1' by ${\omega_1'}^{(n-1,1)}$ (resp. ${\omega_1'}^{(n,0)}$). Then by $\omega_1' = {\omega_1'}^{(n,0)} + {\omega_1'}^{(n-1,1)}$, we have the following proposition.

PROPOSITION 3.5.

$$\bar{\partial}{\omega_1'}^{(n-1,1)} = 0$$

$$\partial{\omega_1'}^{(n-1,1)} + \bar{\partial}{\omega_1'}^{(n,0)} = 0.$$

This ${\omega_1'}^{(n-1,1)}$ must be Kodaira- Spencer class of a family of deformations $\{V_t\}$. More precisely,

$${\omega_1'}^{(n-1,1)} = \{\text{the Kodaira Spencer class, induced by the family } \{V_t\} \text{ on } V - o\} \wedge \{ \text{ the canonical form } \omega' \},$$

by the argument in Theorem 2.1.

Now we have determined the correspondence;

$$1 \in C[z_1, \dots, z_{n+1}]/(f, \frac{\partial f}{\partial z_i}) \to {\omega_1'}^{(n-1,1)} \in H^1(V-o, \wedge^{n-1}(T'(V-o))^*).$$

Then for

$$g(z) \in C[z_1, \dots, z_{n+1}]/(f, \frac{\partial f}{\partial z_i}),$$

we set

$$V_{g(z)t} = \{(z_1, \dots, z_{n+1}) : f(z) = g(z)t\}$$

and set C^∞ diffeomorphism map $i_t(g) : V - o \to V_{g(z)t}$ by

$$(z_1, \dots, z_{n+1}) \to (z_1 + (X_f' z_1)g(z)t, \dots, z_{n+1} + (X_f' z_{n+1})g(z)t).$$

So,

$$i_t(g)^* f(z_i) = f(z_i + (X_f' z_i)g(z)t).$$

By the Taylor expansion theorem,

$$i_t(g)^* f(z_i) = f(z_i) + \sum_{i=1}^{n+1} (X_f' z_i)g(z)t \cdot \frac{\partial f}{\partial z_i} + O(t^2).$$

While, because of $\sum_{i=1}^{n+1} X_f' z_i \cdot \frac{\partial f}{\partial z_i} = 1$,

$$f(i_t(g)(z)) \equiv f(z) + g(z)t \mod t^2.$$

Therefore on $V - o$, this map also makes sense as mod t^2, and by the same procedure, we take the $(n-1,1)$ part of the first order term of $i_t(g)^*(\omega' |_{V_t})$ with respect to t.

PROPOSITION 3.6. *The* $(n-1,1)$ *part of the linear term of* $i_t(g_1+g_2)^*(\omega'|_{V_t}) = i_t(g_1)^*(\omega'|_{V_t}) + i_t(g_2)^*(\omega'|_{V_t})$, *and the* $(n-1,1)$ *part of the linear term of* $i_t(h\cdot g)^*(\omega'|_{V_t}) =$ *the* $(n-1,1)$ *part of the linear term of* $h\cdot i_t(g)^*(\omega'|_{V_t})$.

By our definitions, this proposition is obvious. So, we omit this. Now, we must show that this correspondence makes sense mod $(f, \frac{\partial}{\partial z_i})$.

THEOREM 3.7. *The above correspondence makes sense as a linear map from* $C[z_1,\dots,z_{n+1}]/(f,\frac{\partial f}{\partial z_i})$ *to* $H^1(V-o,\wedge^{n-1}(T'(v-o))^*)$.

Proof. For the proof of this theorem, it suffices to see that: f and $\frac{\partial}{\partial z_i}$ *correspond to 0 in* $H^1(V-o,\wedge^{n-1}(T'(V-o))^*)$. For the case f, we compute $i_t(f)^*\omega'$. For this purpose, by Proposition 3.6, it is enough to check two cases, f, $\frac{\partial f}{\partial z_i}$.

(1) The case f.

By the definition of $i_t(f)$,

$$(z_1,\dots,z_{n+1}) \to (z_1 + f(X'_f z_1)t,\dots,z_{n+1}+f(X'_f z_{n+1})t).$$

So, on $V-o$, this map is nothing but the identity map. Hence,

$$i_t(f)(\omega'|_{V_t}) = \omega',$$

and so the $(n-1,1)$ part vanishes.

(2) The case $\frac{\partial f}{\partial z_i}$.

For simplicity, we assume $\frac{\partial f}{\partial z_1}$ without loss of generality. By Lemma 3.2,

$$i_t(\frac{\partial f}{\partial z_1})^*(dz_1\wedge\cdots\wedge dz_{n+1}) = i_t(\frac{\partial f}{\partial z_1})^* df \wedge i_t(\frac{\partial f}{\partial z_1})^*\omega'.$$

Hence

$$X'_f\rfloor i_t(\frac{\partial f}{\partial z_1})^*(dz_1\wedge\cdots\wedge dz_{n+1}) = X'_f\rfloor i_t(\frac{\partial f}{\partial z_1})^* df \wedge X'_f\rfloor i_t(\frac{\partial f}{\partial z_1})^*\omega'.$$

We are focussing on the $(n-1,1)$ part. We have equalities

$$\begin{aligned} i_t(\frac{\partial f}{\partial z_1})^* df &= d\{f(z_i + (\frac{\partial f}{\partial z_i})(X'_f z_i)t)\} \\ &= df + ((\frac{\partial f}{\partial z_1})X'_f\rfloor df)t \bmod t^2. \end{aligned}$$

Hence the corresponding $(n-1,1)$ term becomes

$$\begin{aligned} X'_f\rfloor i_t(\frac{\partial}{\partial z_1})^*\omega' &= \text{the } (n-1,1) \text{ part of the linear term of} \\ & \quad X'_f\rfloor\{i_t(\frac{\partial f}{\partial z_1})^*(dz_1\wedge\cdots\wedge dz_{n+1})\}. \end{aligned}$$

And by the direct substitution, this becomes

$$\begin{aligned} X'_f \rfloor \{ & d(z_1 + \frac{\partial f}{\partial z_1}(X'_f z_1)t) \wedge dz_2 \cdots \wedge dz_{n+1} \\ & + dz_1 \wedge d(z_2 + \frac{\partial f}{\partial z_1}(X'_f z_2)t) \wedge \cdots \wedge dz_{n+1} \\ & + \cdots + dz_1 \wedge \cdots \wedge dz_n \wedge d(z_{n+1} + (\frac{\partial f}{\partial z_{n+1}})(X'_f z_{n+1}))\}. \end{aligned}$$

So the $(n-1,1)$ part of the linear term becomes the $(n-1,1)$ part of

$$\begin{aligned} X'_f \rfloor \{ & d((\frac{\partial f}{\partial z_1})(X'_f z_1)) \wedge dz_2 \cdots \wedge dz_{n+1} \\ & + dz_1 \wedge d((\frac{\partial f}{\partial z_1})(X'_f z_2)) \wedge dz_2 \cdots \wedge dz_{n+1} + \cdots \}. \end{aligned}$$

Namely, the $(n-1,1)$ part of

$$\begin{aligned} & (\frac{\partial f}{\partial z_1}) d((X'_f z_1)) \wedge dz_2 \wedge \cdots \wedge dz_{n+1} \\ & + (\frac{\partial f}{\partial z_1}) dz_1 \wedge d((X'_f z_2)) \wedge \cdots \wedge dz_{n+1} \\ & + (\frac{\partial f}{\partial z_1}) dz_1 \wedge dz_2 \wedge d((X'_f z_3)) \wedge \cdots \wedge dz_{n+1} + \cdots . \end{aligned}$$

While,

$$df = \frac{\partial f}{\partial z_1} dz_1 + \frac{\partial f}{\partial z_2} dz_2 + \cdots .$$

We use this equality (we substitute $\frac{\partial f}{\partial z_1} dz_1$ by $df - \{\frac{\partial f}{\partial z_2} dz_2 + \cdots\}$. And if a term includes df, then on $V - o$, it must vanish. By using these facts, we can rewrite the corresponding term as follows.

Namely, the $(n-1,1)$ part of

$$X'_f \rfloor \{\{(\frac{\partial f}{\partial z_1}) d(X'_f z_1) + (\frac{\partial f}{\partial z_2}) d(X'_f z_2) + \cdots)\} dz_2 \wedge dz_3 \cdots \wedge dz_{n+1}\}$$

is the one. While, by the definition of X'_f,

$$(\frac{\partial f}{\partial z_1}) X'_f z_1 + (\frac{\partial f}{\partial z_2}) X'_f z_2 + \cdots = 1.$$

Therefore

$$\sum_{i=1}^{n+1} \{d(\frac{\partial f}{\partial z_i})(X'_f z_i) + (\frac{\partial f}{\partial z_i}) d(X'_f z_i)\} = 0.$$

Of course $d(\frac{\partial f}{\partial z_i})$ is of type $(1,0)$. Therefore the $(n-1,1)$ part never appear. This completes our proof. □

Injectivity is proved by the fact; (3.1) is an effective family. And the Kohn-Rossi cohomology group $H^1(M, \wedge^{n-1}(T')^*)$ is isomorphic to $H^1(V-o, \wedge^{n-1}(T')^*)$, via the restriction map to the boundary M. Hence, we have the explicit representation of the Kohn-Rossi cohomology (only type $(n-1,1)$).

4. Main result

In Section 3, we have the Kodaira-Spencer class of a family of deformations $\{V_{g(z)t}\}$. For a given polynomial f, this procedure is computable. With these, we can write down our Z^1 space in some cases. We assume that f is a special kind of homogeneous polynomial, namely, $f = z_1^m + \cdots + z_{n+1}^m$. Then, our main theorem is that;

THEOREM 4.1. *If f is the above, Z^1 space is nothing but the harmonic space of the Kohn Rossi cohomology. And so, In this case,*

$$\textit{infinitesimal deformation space} = Z^1.$$

This means that; in a general theory of CR structures, in the middle dimension case, we have to treat the "special harmonic space" (see [3], [4]). *But for the above two cases, we see that; the "ordinary harmonic space" is enough.*

By using the expression in Section 3, we might treat the cases general homogeneous and also general quasi homogeneous, and determineZ^1 space. In these proofs, the following key relation might be essential.

$$i_t^*(\sum_{i=1}^{n+1} \sqrt{-1}\bar{z}_i dz_i) \equiv \sum_{i=1}^{n+1} \sqrt{-1}\bar{z}_i dz_i \qquad \mathrm{mod}\ (\bar{t}, t^2) \quad \text{on } V-o.$$

However in order to give a complete proof, we need more pages. In the next section, we only give the proof of our main theorem for the case $f = z_1^3 + z_2^3 + z_3^3$ just by "the direct computation", and leave another paper for cases general homogeneous and quasi homogeneous.

5. $\mathbf{f = z_1^3 + z_2^3 + z_3^3}$

Consider the hypersurface isolated singularity

$$z_1^3 + z_2^3 + z_3^3 = 0,$$

and a CR manifold

$$\{(z_1, z_2, z_3) : z_1^3 + z_2^3 + z_3^3 = 0, \mid z_1 \mid^2 + \mid z_2 \mid^2 + \mid z_3 \mid^2 = 1\}.$$

For this example, we compute Z^1 in the way in Section 3.

First, we see the deformation algebra

$$\frac{C[z_1, z_2, z_3]}{(f, \frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \frac{\partial f}{\partial z_3})},$$

where $f = z_1^3 + z_2^3 + z_3^3$. The representatives are

$$(5.1) \qquad 1, z_1, z_2, z_3, z_1z_2, z_1z_3, z_2z_3, z_1z_2z_3.$$

In this case, as CR manifold M is of real dimension 3, the Kohn-Rossi cohomology is infinite dimensional. But as a cohomology group, it makes sense. Anyway, for an element in (5.1), we write down the corresponding $(1,1)$ form (Kodaira-Spencer class). By using the usual Euclidean metric,

$$ds^2 = dz_1 \otimes dz_1 + dz_2 \otimes dz_2 + dz_3 \otimes dz_3,$$

by the definition of X_f,

$$X_f = 3\bar{z}_1^2 \frac{\partial}{\partial z_1} + 3\bar{z}_2^2 \frac{\partial}{\partial z_2} + 3\bar{z}_3^2 \frac{\partial}{\partial z_3}.$$

Therefore

$$X_f f = 9(\mid z_1 \mid^4 + \mid z_2 \mid^4 + \mid z_3 \mid^4)$$

and so, by the definition of $\omega' (\omega' := X_f \rfloor dz_1 \wedge dz_2 \wedge dz_3)$,

$$\omega' = \frac{1}{9(\mid z_1 \mid^4 + \mid z_2 \mid^4 + \mid z_3 \mid^4)} (3\bar{z}_1^2 dz_2 \wedge dz_3 - 3\bar{z}_2^2 dz_1 \wedge dz_3 + 3\bar{z}_3^2 dz_1 \wedge dz_2).$$

For brevity, we use abbreviation

$$\psi = 9(\mid z_1 \mid^4 + \mid z_2 \mid^4 + \mid z_3 \mid^4).$$

And so, the corresponding C^∞ diffeomorphism map is

$$i_t : (z_1, z_2, z_3) \to \Big(z_1 + \frac{3\bar{z}_1^2}{\psi} t, z_2 + \frac{3\bar{z}_2^2}{\psi} t, z_3 + \frac{3\bar{z}_3^2}{\psi} t\Big).$$

This map induces $i_t : M \to M_t$ in the sense of mod $(\bar{t}, t^2)$, where

$$M_t = \{(z_1, z_2, z_3) : z_1^3 + z_2^3 + z_3^3 = t, \mid z_1 \mid^2 + \mid z_2 \mid^2 + \mid z_3 \mid^2 = 1\}.$$

In this case as a supplement vector field ζ by adopting

$$\zeta := \sqrt{-1}\Big(\sum_{i=1}^{3} z_i \frac{\partial}{\partial z_i} - \sum_{i=1}^{3} \bar{z}_i \frac{\partial}{\partial \bar{z}_i}\Big).$$

Now the corresponding form associated with the Kodaira-Spencer class is

$$\begin{aligned}\omega_1' &= \frac{1}{9(|z_1|^4+|z_2|^4+|z_3|^4)^2}\{18\bar{z}_1^2\bar{z}_2 d\bar{z}_2 \wedge dz_3 \\ &+ 18\bar{z}_1^2\bar{z}_3 dz_2 \wedge d\bar{z}_3 - 18\bar{z}_2^2\bar{z}_1 d\bar{z}_1 \wedge dz_3 - 18\bar{z}_2^2\bar{z}_3 dz_1 \wedge d\bar{z}_3 \\ &+ 18\bar{z}_3^2\bar{z}_1 d\bar{z}_1 \wedge dz_2 + 18\bar{z}_3^2\bar{z}_2 dz_1 \wedge d\bar{z}_2\}.\end{aligned}$$

In fact, this is shown as follows.

$$dz_1 \wedge dz_2 \wedge dz_3 = df \wedge \omega'. \tag{5.2}$$

Therefore

$$i_t^*(dz_1 \wedge dz_2 \wedge dz_3) = i_t^*(df \wedge \omega') \tag{5.3}$$

$$i_t^*(dz_1 \wedge dz_2 \wedge dz_3) = i_t^*(df) \wedge i_t^*(\omega') \tag{5.4}$$

$$i_t^*(dz_1 \wedge dz_2 \wedge dz_3) \equiv df \wedge i_t^*(\omega') \mod t^2 \tag{5.5}$$

at every point of $V - o$.

(because of Key relation 1).

Therefore

$$\begin{aligned} i_t^*\omega' &\equiv X_f' \rfloor i_t^*(dz_1 \wedge dz_2 \wedge dz_3) \bmod t^2 \\ &\equiv X_f' \rfloor (i_t^* dz_1 \wedge i_t^* dz_2 \wedge i_t^* dz_3) \bmod t^2. \end{aligned} \tag{5.6}$$

We compute the linear term of the right hand side of this equation with respect to t. By a direct computation, the linear term is

$$X_f' \rfloor \Big\{ d(\frac{3\bar{z}_1^2}{\psi}) \wedge dz_2 \wedge dz_3 + dz_1 \wedge d(\frac{3\bar{z}_2^2}{\psi}) \wedge dz_3 + dz_1 \wedge dz_2 \wedge d(\frac{3\bar{z}_3^2}{\psi}) \Big\},$$

so

$$\begin{aligned} X_f' \rfloor \Big\{ &(-\frac{d\psi}{\psi^2}) 3\bar{z}_1^2 \wedge dz_2 \wedge dz_3 + \frac{1}{\psi} d(3\bar{z}_1^2) \wedge dz_2 \wedge dz_3 \\ &+ dz_1 \wedge (-\frac{d\psi}{\psi^2}) 3\bar{z}_2^2 \wedge dz_3 + \frac{1}{\psi} dz_1 \wedge d(3\bar{z}_1^2) \wedge dz_3 \\ &+ dz_1 \wedge dz_2 \wedge (-\frac{d\psi}{\psi^2}) 3\bar{z}_3^2 + \frac{1}{\psi} dz_1 \wedge \wedge dz_2 \wedge d(3\bar{z}_1^2) \Big\}. \end{aligned} \tag{5.7}$$

PROPOSITION 5.1. (5.7) *is purely of type* $(1,1)$ *at every point of* p *of* M. *That is to say,* $(2,0)$ *term of* (5.7) *vanishes at every point of* p *of* M.

Proof. The problem term is

$$(5.8)\quad -X'_f\rfloor\frac{d\psi}{\psi^2}3\overline{z}_1^2dz_2\wedge dz_3+X'_f\rfloor\frac{d\psi}{\psi^2}3\overline{z}_2^2dz_1\wedge dz_3-X'_f\rfloor\frac{d\psi}{\psi^2}3\overline{z}_3^2dz_1\wedge dz_2$$

While the first term of this becomes

$$\begin{aligned}&X'_f\rfloor\frac{d\psi}{\psi^2}\wedge 3\overline{z}_1^2dz_2\wedge dz_3\\ =\;&\frac{1}{\psi^2}(X'_f\psi)3\overline{z}_1^2dz_2\wedge dz_3-\frac{3\overline{z}_1^2}{\psi^2}d\psi\wedge(X'_fz_2)\wedge dz_3\\ &+\frac{3\overline{z}_1^2}{\psi^2}d\psi\wedge(X'_fz_3)dz_2\\ =\;&\frac{1}{\psi^2}\frac{X_f\psi}{X_ff}3\overline{z}_1^2dz_2\wedge dz_3-\frac{3\overline{z}_1^2}{\psi^2}\frac{3\overline{z}_2^2}{\psi}d\psi\wedge dz_3+\frac{3\overline{z}_1^2}{\psi^2}\frac{3\overline{z}_3^2}{\psi}d\psi\wedge dz_2.\end{aligned}$$

By the same way,

$$\begin{aligned}&X'_f\rfloor\frac{d\psi}{\psi^2}\wedge 3\overline{z}_2^2dz_1\wedge dz_3\\ =\;&\frac{1}{\psi^2}\frac{X_f\psi}{X_ff}3\overline{z}_2^2dz_1\wedge dz_3-\frac{3\overline{z}_2^2}{\psi^2}\frac{3\overline{z}_2^2}{\psi}d\psi\wedge dz_3+\frac{3\overline{z}_2^2}{\psi^2}\frac{3\overline{z}_3^2}{\psi}d\psi\wedge dz_1,\end{aligned}$$

and

$$\begin{aligned}&X'_f\rfloor\frac{d\psi}{\psi^2}\wedge 3\overline{z}_2^2dz_1\wedge dz_2\\ =\;&\frac{1}{\psi^2}\frac{X_f\psi}{X_ff}3\overline{z}_3^2dz_1\wedge dz_2-\frac{3\overline{z}_1^2}{\psi^2}\frac{3\overline{z}_3^2}{\psi}d\psi\wedge dz_2+\frac{3\overline{z}_2^2}{\psi^2}\frac{3\overline{z}_3^2}{\psi}d\psi\wedge dz_1.\end{aligned}$$

By summing up these, we have that: (5.8) becomes

$$\begin{aligned}&-\frac{1}{\psi^2}\frac{X_f\psi}{X_ff}3\overline{z}_1^2dz_2\wedge dz_3\\ &+\frac{1}{\psi^2}\frac{X_f\psi}{X_ff}3\overline{z}_2^2dz_1\wedge dz_3\\ (5.9)\qquad &-\frac{1}{\psi^2}\frac{X_f\psi}{X_ff}3\overline{z}_3^2dz_1\wedge dz_2.\end{aligned}$$

While, we have the following lemma.

LEMMA 5.2.

$$X'_f(|\,z_1\,|^2+|\,z_2\,|^2+|\,z_3\,|^2-1)=0\ \textit{on}\ V-o.$$

By this lemma,

$$\begin{aligned}
X'_f\psi &= X'_f\{9(\mid z_1\mid^4+\mid z_2\mid^4+\mid z_3\mid^4)\}\\
&= X'_f\{9((1-\mid z_2\mid^2-\mid z_3\mid^2)^2+\mid z_2\mid^4+\mid z_3\mid^4)\\
&\quad +9(\mid z_1\mid^4-(1-\mid z_2\mid^2-\mid z_3\mid^2)^2)\}\\
&= X'_f\{9((1-\mid z_2\mid^2-\mid z_3\mid^2)^2+\mid z_2\mid^4+\mid z_3\mid^4)\\
&\quad +9(\mid z_1\mid^2+\mid z_2\mid^2+\mid z_3\mid^3-1)(\mid z_1\mid^2-\mid z_2\mid^2-\mid z_3\mid^2+1)\}.
\end{aligned}$$

We set ψ_1 by;

$$\psi_1=9(\mid(1-\mid z_2\mid^2-\mid z_3\mid^2)\mid^2+\mid z_2\mid^4+\mid z_3\mid^4).$$

Hence

$$\psi=\psi_1+\psi-\psi_1,$$

so

$$X'_f\psi=X'_f\psi_1+X'_f(\psi-\psi_1).$$

The latter term vanishes on M, because

$$X'_f(\mid z_1\mid^2+\mid z_2\mid^2+\mid z_3\mid^2-1)=0,\ \text{on } V-o$$
$$\mid z_1\mid^2+\mid z_2\mid^2+\mid z_3\mid^2-1=0,\ \text{on } S_1^5(o).$$

And so

$$\begin{aligned}
X'_f\psi &= X'_f\psi_1+X'_f(\psi-\psi_1)\\
&= X'_f\psi_1 \text{ at every point of } M. \qquad (5.10)
\end{aligned}$$

While, on C^3-o,

$$X'_f\psi_1=\frac{3\bar{z}_2^2}{\psi_1}\frac{\partial\psi_1}{\partial z_2}+\frac{3\bar{z}_3^2}{\psi_1}\frac{\partial\psi_1}{\partial z_3}. \qquad (5.11)$$

So

$$\begin{aligned}
&\frac{1}{X_ff}(X_f\psi)3\bar{z}_1^2dz_2\wedge dz_3\\
&= \frac{3\bar{z}_2^2}{\psi}\frac{\partial\psi_1}{\partial z_2}3\bar{z}_1^2dz_2\wedge dz_3+\frac{3\bar{z}_3^2}{\psi}\frac{\partial\psi_1}{\partial z_3}3\bar{z}_1^2dz_2\wedge dz_3\\
&= \frac{3\bar{z}_2^23\bar{z}_1^2}{\psi}d\psi_1\wedge dz_3+\frac{3\bar{z}_3^23\bar{z}_1^2}{\psi}dz_2\wedge d\psi_1\\
&= \frac{3\bar{z}_2^23\bar{z}_1^2}{\psi}d\psi_1\wedge dz_3-\frac{3\bar{z}_3^23\bar{z}_1^2}{\psi}d\psi_1\wedge dz_2. \qquad (5.12)
\end{aligned}$$

On $M=V\cap S_1^5(o)$, obviously, $\psi_1=\psi$. By the same line, we calculate $(X_f\psi)3\bar{z}_2^2dz_1\wedge dz_3$, $(X_f\psi)3\bar{z}_3^2dz_1\wedge dz_2$, and summing up these, we see that these cancel. So ω'_1 is purely type $(1,1)$ on $M=V\cap S_1^5(o)$. □

Now we see that $\omega_1' \mid_M$ satisfies

$$\omega_1' \mid_M \in \Gamma(M, \theta \wedge ({}^0T'')^*) \quad \text{and} \quad d'\omega_1' \mid_M = 0.$$

In the sense of $\mathrm{mod}(\bar{t}, t^2)$, our map is the contact transform from M to M_t. So

$$i_t^*(\omega' \mid_{M_t}) \mid M \in \Gamma(M, \theta \wedge ({}^0T')^* + \theta \wedge ({}^0T'')^*).$$

For our purpose, we must see $\zeta \rfloor \omega_1' \in \Gamma(M, ({}^0T'')^*)$. However, by the definition of ω_1', it is easily seen. So, $\omega_1' \in \Gamma(M, \theta \wedge ({}^0T'')^*)$. Because of $d(i_t^* \omega' \mid_{M_t}) \mid M = 0$, we have

$$d'\omega_1' = 0.$$

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Department of Mathematics
Himeji Institute of Technology
Shosha 2167, Himeji, Japan
E-mail: akahorit@sci.himeji-tech.ac.jp