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THE FORMULA FOR THE SINGULARITY OF SZEGÖ KERNEL : I

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ABSTRACT. We develop a method of calculating explicitly the singularity of Szegö and Bergman kernel by using the process developed by Boutet de Monvel and Sjöstrand.

The Bergman kernel and the Szegö kernel are important invariants attached to complex domains. The global aspect of the kernels is difficult to grasp, but their singularity is a local invariant and we can handle it to some extent.

We consider a strongly pseudoconvex hypersurface M to be the boundary of $\Omega \subset C^n$, where Ω is open. Fefferman [8] found the formula for the Bergman kernel of Ω. Boutet de Monvel and Sjöstrand [6] noted that the similar formula holds also for the Szegö kernel K^S near M.

To write down the formula denote by x (resp. z) the standard real chart (resp. complex chart) of C^n. Let $r = 0$ be one of the defining equations of M such that $r > 0$ on Ω. Denote by $r(x, x')$ a function on $\Omega \times \Omega$ satisfying the conditions: (i) $r(x, x) = r(x)$, (ii) $\overline{\partial}_z r(x, x')$ is zero up to of order ∞ at $x = x'$, (iii) $r(x, x') = \overline{r(x', x)}$. Then the formula for the Szegö kernel of Ω is

$$K^S(x, x') = F(x, x')(r(x, x'))^{-n} + G(x, x') \log r(x, x'). \tag{1}$$

The above three also developed methods to calculate the singularity of the kernels. The program of Fefferman ([9], [10]) was developed by Bailey, Eastwood, Gover, Graham, and others ([1], [2], [7], [12], [13], [14]). It was completed by Hirachi and Komatsu ([17], [18]) by using Kashiwara approach [21] based on Sato's hyperfunctions. Kashiwara approach was also used by Boutet of Monvel ([4], [5]) and Hirachi-Komatsu-Nakazawa

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([19], [20]) for calculation of the singularity. Boutet de Monvel and Sjöstrand [6] used Hörmander's Fourier integral operators.

The kernel is a global object. However, its singularity at a point is determined locally. Therefore, we fix a reference point, say p_*, in M, and try to write down explicitly the formula for the singularity near p_* using the local data. In this enterprise we follow the method developed by Boutet de Monvel and Sjöstrand. In this calculation we ignore the CR or the conformal geometric aspect of the formula for the singularity. At present the formula is crude. It will be refined and the geometric formula will be developed in the subsequent papers.

It is hoped that the singularity furnishes new invariants when applied to the unit ball bundles of hermitian vector bundles, or isolated singularities.

In §1 we give the outline of the construction. In §2 we write down the equations which characterize the functions we need in the construction. In §3 we develop the method to solve the equations given in §2. In §4 we summarize our construction.

§1. The outline

A) We consider C^{n+1} instead of C^n. We choose the origin as our reference point p_* and consider a small piece of strongly pseudoconvex hypersurface, say M, containing p_*. Since we are interested in the local aspect, we may shrink M.

We have the standard chart (z, w) of $C^{n+1} = C^n \times C$ where $w = x^0 + iv^0$ with real x^0, v^0. A $\overline{\partial}_b$ operator is a vector field on M which is of type $(0,1)$ when it is written using the complex chart (z, w) of the ambiant space. Let $r = 0$ be a local equation of M in C^{n+1}. By making a holomorphic linear change of our chart if necessary, we may assume $r_w \neq 0$ at p_*. Then the $\overline{\partial}_b$ operators of M are generated by

$$(1) \qquad Q^\alpha = \frac{\partial}{\partial \overline{z^\alpha}} - ih^\alpha \frac{\partial}{\partial \overline{w}}, \quad ih^\alpha = \frac{r_{\bar{\alpha}}}{r_{\bar{w}}}$$

$(\alpha = 1, \ldots, n)$, where $r_{\bar{\alpha}} = \partial r / \partial \overline{z^\alpha}, r_{\bar{w}} = \partial r / \partial \bar{w}$.

In the following we always consider the case

(2)

$$r = \frac{1}{i}(w - \bar{w}) - (|z|^2 + N(z, x^0)), \quad \text{where} \quad N \equiv 0 \pmod{(z, \bar{z}, x^0)^4}.$$

Hence the restriction of (z, x^0) to M is a chart of M, and we see by calculation that its $\overline{\partial}_b$ operators are generated by

$$(3)\quad Q^\alpha = \frac{\partial}{\partial \overline{z^\alpha}} - \frac{i}{2} h^\alpha \frac{\partial}{\partial x^0}, \quad h^\alpha = z^\alpha + \text{higher order} \quad (\alpha = 1, \dots, n).$$

When $N = 0$, it is the model case. Namely, it is the homogeneous space $\mathcal{M}$, i.e. the Heisenberg space. Therefore $\overline{\partial}_b$ operators for $\mathcal{M}$ are generated by

$$(4)\qquad P^\alpha = \frac{\partial}{\partial \overline{z^\alpha}} - \frac{i}{2} z^\alpha \frac{\partial}{\partial x^0}.$$

We may regard the CR structures M and $\mathcal{M}$ are defined on the same neighborhood, say M, of the origin, which we still denote by p_*, in $C^n \times R$ with the standard chart (z, x^0). We have the standard real chart $x = (x^1, \dots, x^{2n}, x^0)$, $z^\alpha = x^\alpha + i x^{n+\alpha}$. Let $(x, \xi), \xi = (\xi_1, \dots, \xi_{2n}, \xi_0)$ be the standard real chart of the cotangent bundle $T^*(C^n \times R)$ induced by the chart x.

For later discussion it is more convenient to introduce a copy y of x, and consider P^α (resp. Q^α) operating on x-space (resp. y-space). We also introduce a copy η of ξ so that η is the fiber chart of the cotangent bundle of y-space. Both x and y represent points in the space M.

Set $\zeta_\alpha = (\xi_\alpha - i\xi_{n+\alpha})/2$. The copy of ζ^α induced by η will be written as $\zeta^\alpha(\eta)$. Then the symbols of Q^α and P^α are given by

$$(5)\qquad q^\alpha(y, \eta) = i\overline{\zeta_\alpha}(\eta) + \frac{1}{2} h^\alpha(y)\eta_0, \quad p^\alpha(x, \xi) = i\overline{\zeta_\alpha} + \frac{1}{2} z^\alpha \xi_0.$$

We may omit y in $h(y)$, when there is no possibility of confusion. Set

$$(6)\qquad g^\alpha(y, \eta) = \frac{q^\alpha}{i\eta_0} = \frac{\overline{\zeta_\alpha}(\eta)}{\eta_0} - \frac{i}{2} h^\alpha, \quad f^\alpha(x, \xi) = \frac{p^\alpha}{i\xi_0} = \frac{\overline{\zeta_\alpha}}{\xi_0} - \frac{i}{2} z^\alpha.$$

We have conic neighborhoods

$$(7)\qquad \begin{aligned} \Gamma^\delta_{\mathcal{M}}(M) =& \{(x, \xi) : x \in M,\ |f^\alpha| < \delta\},\ \Gamma^\delta_M(M) \\ &= \{(y, \eta) : y \in M,\ |g^\alpha| < \delta\} \end{aligned}$$

of the characteristics at p_* of $\overline{\partial}_b$ operators.

B) Let Ω be a bounded domain in C^{n+1} with smooth strongly pseudoconvex boundary $\partial\Omega$. Denote by H the L_2-closure of the space of the restrictions of the holomorphic functions on $\overline{\Omega}$. We define the Szegö projection operator as the orthogonal projection operator of L_2 to H.

We construct a Fourier integral operator $\tilde{S} : C_0^\infty(M') \to C_0^\infty(M)$ (for a small neighborhood M' of p_*) such that, in the case M is regarded as an open subset of Ω as above, it differs from the restriction to $C_0^\infty(M')$ of the Szegö projection operator of $\partial\Omega$ by an operator of order $-\infty$. If this is the case, we say that $\tilde{S}$ has the same singularity as the Szegö projection near p_*. The construction of such $\tilde{S}$ is done by the following steps:

I) we say that a symplectic map $\chi(x,\xi) = (y(x,\xi), \eta(x,\xi))$ is homogeneous, when the domain of definition is a cone, and $y(x,\xi)$ (resp. $\eta(x,\xi)$) is homogeneous of degree 0 (resp. of degree -1) in ξ.

We construct a homogeneous symplectic map

$$\chi : \Gamma^{\delta_1}_{\mathcal{M}}(M') \to \Gamma^{\delta_2}_{M}(M), \tag{8}$$

for suitable $\delta_1,\ \delta_2 > 0$, which transforms the module generated by q^α to the module generated by p^α. Namely, χ is such that

$$q^\alpha(\chi(x,\xi)) = r^\alpha_\beta(x,\xi) p^\beta(x,\xi) \tag{9}$$

for a suitable symbol $r^\alpha_\beta(x,\xi)$ of order zero. We have the graph $\mathcal{G}$, a subspace of (x,ξ,y,η)-space, of χ. We construct χ in such a way that (y,ξ) is a chart of $\mathcal{G}$,

II) for each symbol $a(y,\xi)$ defined on the range of the above chart (y,ξ) we construct the Fourier integral operator F^χ_a with the symbol a induced by χ,

III) we construct a symbol $a(y,\xi)$ such that F^χ_a is unitary up to an operator of order $-\infty$ and such that, with a suitable matrix (B^α_β) of pseudo-differential operators,

$$F^\chi_a \circ B^\alpha_\beta \circ P^\beta \equiv Q^\alpha \circ F^\chi_a \tag{10}$$

modulo operators of order $-\infty$ (when restricted on a small neighborhood of p_*),

IV) $\tilde{S} = F^\chi_a f_1 \mathcal{S}_0 f_2 (F^\chi_a)^*$, where $\mathcal{S}_0$ is the Szegö projection operator of $\mathcal{M}$ and f_1, f_2 are cup-off functions, i.e. smooth functions with compact support which is equal to 1 on a neighborhood of p_*,

V) finally we write down the singularity of $\tilde{S}$ by applying "the method of stationary phase for complex phase functions" [22] to $\tilde{S}$.

C) Our idea is to use an one-parameter family M_t of CR structures with the conditions: $M_1 = M, M_0 = \mathcal{M}$, and, for $t \neq 0$, M_t is isomorphic to M restricted to a neighborhood of p_*. Namely, under the chart $(z_t, x_t^0) = (tz, t^2x^0)$

$$(11) \qquad Q^\alpha = \frac{1}{t}\left(\frac{\partial}{\partial \overline{z_t^\alpha}} - \frac{i}{2}h_t^\alpha \frac{\partial}{\partial x_t^0}\right), \quad h_t^\alpha = \frac{1}{t}h^\alpha(tz, t^2x^0).$$

Therefore the CR structure, say M_t, generated by

$$(12) \qquad Q_t^\alpha = \frac{\partial}{\partial \overline{z^\alpha}} - \frac{i}{2}h_t^\alpha \frac{\partial}{\partial x^0}$$

is isomorphic to M and, because of the formula in (3) in §1, h_t^α extends smoothly to $t = 0$ and $h_0^\alpha = z^\alpha$.

We also construct $r = r(x, x')$ in (1) in Introduction for M_t depending smoothly in t. The Szegö kernel $S_t = F_t r_t^{-n} + G_t \log r_t$ depends smoothly on t. Actually we calculate their restriction to $M \times M$. The advantage of using M_t is that we may use differential equations in t, x to construct functions we need. For simplicity of notations we often omit t.

§2. The equations which appears in the construction

A) We first construct a one parameter family of symplectic maps χ_t mentioned in (I) in §1. We do this by writing down the equation which must be satisfied by χ_t.

For any one parameter family of transformations f_t, with $f_0 =$ the identity map, we associate a one parameter family of infinitesimal transformations v_t by

$$(1) \qquad f_{t+\epsilon} \circ (f_t)^{-1} \equiv \text{ the identity map} + \epsilon\, v_t \qquad (\text{mod } \epsilon^2).$$

We say that v_t is associated to f_t, vice versa. When we have a one-parameter family v_t we get back f_t by solving the ordinary differential equation:

$$(2) \qquad \frac{df_t}{dt} = (v_t \text{ at } f_t), \quad f_0 = \text{the identity map}.$$

Therefore to construct χ_t it is enough to construct the associated one-parameter family v_t of infinitesimal symplectic transformations, say

$$(3) \qquad v_t(y, \eta) = v^j(y, \eta, t)\frac{\partial}{\partial x_j} + v_j(y, \eta, t)\frac{\partial}{\partial \eta_j}.$$

It is well known that v_t as above is an infinitesimal symplectic transformation if and only if there is a real valued function $\lambda_t(y,\eta)$ such that

$$v^j(y,\eta,t)=\frac{\partial\lambda_t}{\partial\eta_j},\quad v_j(y,\eta,t)=-\frac{\partial\lambda_t}{\partial y^j}. \tag{4}$$

We call the function λ_t a potential for v_t, also for the associated χ_t. Therefore it is enough to find a potential λ_t for χ_t so that the associated χ_t satisfies the condition (9) in §1. Since we want χ to be homogeneous, we need λ which is homogeneous of degree 1.

B) We have the symbol $q_t^\alpha(y,\eta)$ of Q_t^α. Assume that a one-parameter family of symplectic maps χ_t is given. Let λ_t be its potential. We then find easily

$$\frac{dq_t^\alpha(\chi_t(x,\xi))}{dt}=(\{q_t^\alpha,\lambda_t\}+\frac{1}{2}\dot{h}_t^\alpha)(\chi_t(x,\xi)), \tag{5}$$

where $\dot{h}_t^\alpha=dh_t^\alpha/dt$ and $\{\phi,\psi\}$ is the Poisson bracket of ϕ and ψ. As for the right-hand side of (9) in §1, assume that χ_t and a non-singular matrix valued function $r_t=(r_\beta^\alpha(x,\xi))$ are given. Define a matrix valued function $s_t=(s_\beta^\alpha(y,\eta))$ by

$$\frac{dr_\beta^\alpha(x,\xi)}{dt}=s_\gamma^\alpha(\chi(x,\xi))r_\beta^\gamma(x,\xi). \tag{6}$$

Then, when (9) in §1 is satisfied, we find that

$$\frac{dr_\beta^\alpha(x,\xi)p^\beta(x,\xi)}{dt}=s_\beta^\alpha(\chi(x,\xi))q^\beta(\chi(x,\xi)). \tag{7}$$

Therefore we find the following:

THEOREM 1. *Assume that a one parameter family of symplectic maps χ_t as in § 1-(8) (with χ_0 = the identity map) of potential λ_t satisfies the equality* (9) *in* § 1. *Then for a suitable matrix valued function $s_t=(s_\beta^\alpha(y,\eta))$ we have*

$$\{q^\alpha,\lambda\}+\dot{q}^\alpha=s_\beta^\alpha\,q^\beta. \tag{8}$$

Tracing back the above argument we find the following:

THEOREM 2. *Let a real valued $\lambda_t=\lambda(y,\eta)$ and a complex valued $s_t=(s_\beta^\alpha(y,\eta))$ satisfy the above equation* (8) *on $\Gamma_M^\delta(M)$. Consider*

v^j, v_j given in (4). Let $\chi_t(x,\xi) = (y(x,\xi), \eta(x,\xi))$ be the solution of the equation:

$$(9) \quad \frac{dy^j(x,\xi)}{dt} = v^j(y(x,\xi),\eta(x,\xi)), \quad \frac{d\eta_j(x,\xi)}{dt} = v_j(y(x,\xi),\eta(x,\xi))$$

with $y(x,\xi) = x$, $\eta(x,\xi) = \xi$ when $t = 0$. Then χ_t is a symplectic map. Define $r^\alpha_\beta(x,\xi)$ as the solution of the equation:

(10)

$$\frac{dr^\alpha_\beta(x,\xi)}{dt} = s^\alpha_\gamma(y(x,\xi),\eta(x,\xi))r^\gamma_\beta(x,\xi), \quad r^\alpha_\beta(x,\xi) = \delta^\alpha_\beta \text{ when } t = 0.$$

Then for sufficiently small δ_1 and M' we have the equality

$$(11) \qquad q^\alpha(y(x,\xi),\eta(x,\xi)) = r^\alpha_\beta(x,\xi)p^\beta(x,\xi),$$

on $\Gamma^{\delta_1}_{\mathcal{M}}(M')$. We thus reduced the problem of constructing our χ to the problem of finding the solution of the equation (8).

C) A real valued function $S(y,\xi)$ is called a generating function of a symplectic map $\chi(x,\xi) = (y(x,\xi),\eta(x,\xi))$, when (x,ξ,y,η) is in the graph of χ if and only if

$$(12) \qquad x = S'_\xi(y,\xi), \quad \eta = S'_y(y,\xi).$$

We see easily that a generating function exists locally if and only if (y,ξ) is a chart of the graph. Since our χ is a small deformation of the identity map, it has a generating function. Since our χ is homogeneous, choose one which is homogeneous of degree 1, say S. We then define

$$(13.1) \qquad \phi(y,x,\xi) = S(y,\xi) - x\cdot\xi.$$

For any symbol $a(y,\xi)$ (defined on the domain of the definition of $S(y,\xi)$) we define the associated Fourier integral operator F^χ_a by

$$(13.2) \quad F^\chi_a u(y) = c_m \int e^{i\phi(y,x,\xi)}\, a(y,\xi)\, u(x)\, dxd\xi, \quad c_m = (2\pi)^{-m},$$

where, in our case $m = 2n+1$, and the integral is the oscillatory integral. For the general properties of Fourier integral operators, we refer to Hömander [16], or Grigis-Sjöstrand [15]. We note that the calculus of Fourier integral operators is a calculus modulo operators of order $-\infty$, and that the formal sum $\sum_{l=l_1}^{\infty} a_{(-l)}$ of symbols, where $a_{(k)}$ is of oder k, always converges.

D) We next write down the equation which must be satisfied by the symbol $a(y,\xi)$ so that the (10) in §1 is satisfied. The idea is to write down the symbol of the form $l(y,\xi)$ for the operator in the each side of the equation (10) in §1. Then the equation holds when the symbols coincide.

The pseudo-differential operator B with a symbol $b(y,\xi)$ is given by

$$Bu(y) = c_m \int e^{(y-x)\cdot\xi}\, b(y,\xi)\, u(x)\, dx d\xi. \tag{14}$$

Then by the well known formula for the composition of pseudo-differential operators we see that $B^{\alpha}_{\beta} \circ P^{\beta}$ is a pseudo-differential operator with the symbol

$$\hat{b}^{\alpha}(y,\xi) = b^{\alpha}_{\beta}(y,\xi) p^{\beta}(y,\xi) - \frac{i}{2}\frac{\partial b^{\alpha}_{\beta}(y,\xi)}{\partial \zeta_{\beta}}\xi_0. \tag{15}$$

We find by calculation that

$$(F_a \circ B^{\alpha}_{\beta} \circ P^{\beta})u(y) = c_m \int e^{i\phi(y,x,\xi)} a^{\alpha}_{P}(y,\xi) u(x) dx d\xi, \quad \text{where} \tag{16}$$

$$\begin{aligned} a^{\alpha}_{P}(y,\xi) &= c_m \int e^{i\psi(y,v,\theta,\xi)}\, a(y,\theta)\hat{b}^{\alpha}(v,\xi) dv d\theta, \\ &\text{where} \quad \psi(y,v,\theta,\xi) = S(y,\theta) - S(y,\xi) - v\cdot(\theta-\xi), \end{aligned} \tag{17}$$

where $\hat{b}^{\alpha}$ is given in (15). In order to have a formula for a^{α}_{P} which is more convenient for us, we need to replace $\hat{b}^{\alpha}(v,\xi)$ in the above by a symbol of the form $\tilde{b}^{\alpha}(y,\theta,\xi)$. We achieve this by using the Tayler series with center $v = S'_{\theta}(y,\theta)$ of $\hat{b}^{\alpha}(v,\xi)$. Namely, when we set for a function $f(v)$, a multi-index $I = (i_1, \dots, i_l)$, $i_1, \dots, i_l = 1, \dots, 2n, 0$, and for indeterminates w^i

$$\partial_I f = \frac{\partial^l f}{\partial v^{i_1} \cdots \partial v^{i_l}}, \quad w^I = w^{i_1} \cdots w^{i_l}, \quad \text{we have} \tag{18.1}$$

$$\hat{b}^{\alpha}(v,\xi) = \sum_I \hat{b}^{\alpha}_I(v,y,\theta,\xi), \quad \text{where} \tag{18.2}$$

$$\hat{b}_I^\alpha(v,y,\theta,\xi) = (\partial_I \hat{b}^\alpha(v,\xi))_{v=S'_\theta(y,\theta)}\ (v - S'_\theta(y,\theta))^I. \tag{18.3}$$

Note that

$$(v^j - S'_{\theta_j}(y,\theta))e^{i\psi(y,v,\theta,\xi)} = -\frac{\partial e^{i\psi(y,v,\theta,\xi)}}{\partial\theta_j}. \tag{19}$$

Hence, by integration by parts we find for example when $I = (\ j\ )$

$$\begin{aligned}&\sum_j \int e^{i\psi(y,v,\theta,\xi)}\hat{b}_{(j)}^\alpha(v,y,\theta,\xi)a(y,\theta)dvd\theta \\ &= \int e^{i\psi(y,v,\theta,\xi)}\frac{\partial}{\partial\theta_j}\Big\{\big(\frac{\partial\hat{b}^\alpha(v,\xi)}{\partial v^j}\big)_{v=S'_\theta(y,\theta)}a(y,\theta)\Big\}dvd\theta.\end{aligned} \tag{20}$$

Therefore repeating this process we find after a considerable amount of calculation a symbol $\tilde{b}^\alpha(y,\theta,\xi)$ such that

$$a_P^\alpha(y,\xi) = c_m \int e^{i\psi(y,v,\theta,\xi)}\tilde{b}^\alpha(y,\theta,\xi)dvd\theta. \tag{21}$$

Since $c_m \int e^{-i<v,\tau>}g(\tau)dvd\tau = g(0)$, we see by the change of variable $\tau = \theta - \xi$ that

$$a_P^\alpha(y,\xi) = \tilde{b}^\alpha(y,\xi,\xi). \tag{22}$$

For the left-hand side of (10) in §1 we see

$$\begin{aligned}Q^\alpha \circ F_a u(y) &= c_m \int d^{i\phi(y,x,\xi)}a_Q(y,\xi)u(x)d\xi dx, \\ a_Q(y,\xi) &= Q_y^\alpha a(y,\xi) + q^\alpha(y, S'_y(y,\xi))a(y,\xi).\end{aligned} \tag{23}$$

Therefore (10) in §1 is satisfied when with a_P, a_Q given in (22), (23)

$$a_P(y,\xi) = a_Q(y,\xi). \tag{24}$$

E) When we analyse the above construction more carefully, we find after a great deal of calculation and cancellation that the above equation (24) (mod. a symbol of order $-\infty$) is satisfied if and only if the following *inductive conditions are satisfied:* Set

$$a(y,\xi) = \sum_{l=0}^{\infty} a_{(-l)}(y,\xi), \tag{25}$$

where $a_{(-l)}$ is of homogeneous degree $-l$. Then with unknown auxiliary symbols $C^{\alpha}_{\beta(-l)}(y,\xi)$ of homogeneous degree $-l$

$$(26)_0 \qquad \begin{aligned} &\frac{1}{2i}\frac{\partial}{\partial\zeta_\beta}\{a_{(0)}(y,\xi)\; r^{\alpha}_{\beta}(S'_{\xi}(y,\xi),\xi)\} \\ &= \frac{1}{\xi_0}\,(Q^{\alpha}_{y}a_{(0)}(y,\xi) + C^{\alpha}_{\beta(-1)}(y,\xi)p^{\beta}(S'_{\xi}(y,\xi),\xi)). \end{aligned}$$

and for $l > 0$

$$(26)_l \qquad \begin{aligned} &\frac{1}{2i}\frac{\partial}{\partial\zeta_\beta}\{a_{(-l)}(y,\xi)\; r^{\alpha}_{\beta}(S'_{\xi}(y,\xi),\xi) - C^{\alpha}_{\beta(-l)}(y,\xi)\} \\ &= \frac{1}{\xi_0}\,(Q^{\alpha}_{y}a_{(-l)}(y,\xi) + C^{\alpha}_{\beta(-l-1)}(y,\xi)p^{\beta}(S'_{\xi}(y,\xi),\xi)). \end{aligned}$$

We can also write down the equation for the symbol b^{α}_{β} of the pseudo-differential operator B^{α}_{β}. It is written using the above $C(y,\xi)$ and $a(y,\xi)$. However, since B^{α}_{β} does not contribute to the formula for singularity of the Szegö kernel, we omit it.

F) We have the graph $\mathcal{G}$ of our map $\chi:\ (x,\xi)\to(y,\eta)$. In our case (y,ξ) is a chart of $\mathcal{G}$. Hence $a(y,\xi)$ may be regarded as the expression of a function a on $\mathcal{G}$ in terms of the chart (y,ξ). Then we may regard the above equations $(26)_l$ as equations defined on $\mathcal{G}$. It turns out that the equation written in terms of the chart $(x,\xi_0,f,\bar{f})$ of $\mathcal{G}$ is easier to solve.

We will consider the equation (26) on a conic open set $\Gamma^{\delta}_{\mathcal{M}}(M)$ (cf. (7) in §1). The charts (y,ξ) and $(x,\xi_0,f,\bar{f})$ are related by :

$$(27) \qquad x = S'_{\xi}(y,\xi), \quad f^{\alpha} = \frac{1}{i\xi_0}p^{\alpha}(S'_{\xi}(y,\xi),\xi) = \frac{\overline{\zeta_\alpha}}{\xi_0} - \frac{i}{2}S'_{\zeta_\alpha}(y,\xi).$$

Since $r^{\alpha}_{\beta}(x,\xi)$ is of order 0, it is a function of $(x,f,\bar{f})$. We define

$$(28.1) \qquad (R^{\alpha}_{\beta}(x,f,\bar{f})) = \text{the inverse matrix of } (r^{\alpha}_{\beta}(x,\xi)),$$

$$(28.2) \qquad R^{\alpha}(x,f,\bar{f}) = \frac{i}{2}R^{\alpha}_{\gamma}(x,f,\bar{f})\;\xi_0\;\frac{\partial r^{\gamma}_{\beta}(S'_{\xi}(y,\xi),\xi)}{\partial\zeta_\beta},$$

$$(29)_l \qquad C^{l,\alpha}(x,f,\bar{f}) = \frac{i}{2}R^{\alpha}_{\gamma}(x,f,\bar{f})\;(\xi_0)^{l+1}\;\frac{\partial C^{\gamma}_{\beta(-l)}(S'_{\xi}(y,\xi),\xi)}{\partial\zeta_\beta},$$

where $C^{\alpha}_{\beta(0)} = 0$ and $C^{\alpha}_{\beta(-l)}$ for $l \geq 1$ are in (26). Set

$$a(y,\xi) = \sum_{l=0}^{\infty} a(y,\xi)_{(-l)}, \quad a(y,\xi)_{(-l)} = (\xi_0)^{-l} a_{<l>}(x,f,\bar{f}). \tag{30}$$

We denote by

$$\frac{\tilde{\partial}}{\partial x^j}, \quad \frac{\tilde{\partial}}{\partial \xi_0}, \quad \frac{\partial}{\partial f^{\alpha}}, \quad \frac{\partial}{\partial \overline{f^{\alpha}}}$$

the partial derivatives with respect to the chart $(x, \xi_0, f, \bar{f})$. Denote by $\tilde{P}^{\alpha}_x$ the operator obtained from P^{α}_x by replacing $\partial/\partial x$ by $\tilde{\partial}/\partial x$.

We find after calculation using the change of charts formula that our equation $(26)_l$ for $a_{<l>}$ takes the form:

$$(\tilde{P}^{\alpha}_x + i\frac{\partial}{\partial \overline{f^{\alpha}}} + R^{\alpha})a_{<l>} = C^{l,\alpha} + D^{\alpha}_{\beta<l>} f^{\beta} \tag{$31)_l$}$$

for an auxiliary unknown symbol $D^{\alpha}_{\beta<l>}$. In the above $C^{0,\alpha} = 0$ and, when we know the equation $(31)_l$, we know the equation $(31)_{l+1}$ by using $C^{l+1,\alpha}$ given in $(29)_{l+1}$ where

$$C^{\alpha}_{\beta(-l-1)} = i(\frac{1}{\xi_0})^{l+1} r^{\alpha}_{\gamma} D^{\gamma}_{\beta<l>} + i(\frac{1}{\xi_0})^{l} V^{\alpha}_{\beta} a_{<l>}, \tag{$32)_{l+1}$}$$

$$V^{\alpha}_{\beta} = \frac{\tilde{\partial}\, r^{\alpha}_{\beta}}{\partial \xi_k}\frac{\tilde{\partial}}{\partial x^k} - \frac{i}{2}\frac{\tilde{\partial}\, r^{\alpha}_{\beta}}{\partial \zeta_{\gamma}}\frac{\partial}{\partial f^{\gamma}} + \frac{i}{2}\frac{\tilde{\partial}\, r^{\alpha}_{\beta}}{\partial \overline{\zeta_{\gamma}}}\frac{\partial}{\partial \overline{f^{\gamma}}}. \tag{33}$$

We thus have an inductive equations for $a_{<l>}$ and $D^{\alpha}_{\beta<l>}$ by $(31)_l$ – $(32)_{l+1}$ and $(29)_{l+1}$.

G) Besides satisfying the above conditions, a has to be such that F_a is unitary up to an operator of order $-\infty$, when restricted to a small neighborhood of p_*. We next write down the equation for $a(y,\xi)$ to satisfy this condition. Assume that $a(y,\xi)$ is compactly supported in $y \in M$.

We fix a volume element in y-space, say $V(y)dy$. The volume element depends on t, and it is the standard volume element when $t = 0$. Let $< f, g >_V$ denote the L_2 inner product with respect to the volume element. We have

$$\begin{aligned} F^{\chi}_a &: C^{\infty}_0(R^{2n+1}) \to C^{\infty}_0(M), \\ F^{\chi}_a u(y) &= c_m \int e^{i\phi(y,x,\xi)} a(y,\xi) u(x)\, dx d\xi. \end{aligned} \tag{34}$$

Then $(F_a^\chi)_V^*$ is given by

$$(35)\qquad \begin{aligned} &(F_a^\chi)_V^* \;:\; C_0^\infty(M) \to C^\infty(R^{2n+1}),\\ &(F_a^\chi)_V^* v(x) = c_m \int e^{-i\phi(y,x,\xi)}\, \overline{a(y,\xi)}\, V(y)v(y)dyd\xi. \end{aligned}$$

Let f be a cut-off function. Set $f_\epsilon(x) = f(x/\epsilon)$ so that f_ϵ converges to the constant function 1 as $\epsilon \to 0$. Then we see easily by calculation that $F_a^\chi \circ f_\epsilon (F_a^\chi)_V^* : C_0^\infty(M) \to C_0^\infty(M)$ converges to an operator, say $F_a^\chi \circ (F_a^\chi)_V^*$, given by

$$(36)\quad F_a \circ (F_a^\chi)_V^* v(y) = c \int e^{i(S(y,\xi)-S(\tilde{y},\xi))} a(y,\xi)\, \overline{a(\tilde{y},\xi)}\, v(\tilde{y})V(\tilde{y})d\tilde{y}d\xi.$$

We define $\kappa = \kappa(y,\tilde{y},\xi)$ by

$$(37)\qquad S(y,\xi) - S(\tilde{y},\xi) = < y - \tilde{y}, \kappa(y,\tilde{y},\xi) > .$$

Solving ξ in κ, we have $\xi = \xi(y,\tilde{y},\kappa)$. It then follows that

$$(38)\qquad \begin{aligned} &F_a \circ (F_a^\chi)_V^* v(y) = c_m \int e^{i<y-\tilde{y},\kappa>} b(y,\tilde{y},\kappa)v(\tilde{y})d\tilde{y}d\kappa, \quad \text{where}\\ &b(y,\tilde{y},\kappa) = a(y,\xi(y,\tilde{y},\kappa))\overline{a(\tilde{y},\xi(y,\tilde{y},\kappa))}V(\tilde{y}) \det \frac{\partial \xi(y,\tilde{y},\kappa)}{\partial \kappa}. \end{aligned}$$

To replace the above $b(y,\tilde{y},\kappa)$ by a symbol of the form $k(y,\kappa)$ we need

LEMMA. *Let $a(y,x,\theta)$ be a symbol. Then*

(39)
$$\int e^{i(y-x)\cdot\theta} a(y,x,\theta)u(x)dxd\theta \equiv \int e^{i(y-x)\cdot\theta} b(y,\theta)u(x)dxd\theta, \quad \textit{where}$$

(40.1)
$$a(y,x,\theta) = \sum (-1)^l \frac{1}{l!} a_{j_1,\dots,j_l}(y,\theta)(y^{j_1} - x^{j_1})\dots(y^{j_l} - x^{j_l}), \quad \textit{with}$$
$$a_{j_1,\dots,j_l}(y,\theta) = (\frac{\partial^l a(y,x,\theta)}{\partial x^{j_1}\dots\partial x^{j_l}})_{x=y}$$

is the Taylor series centered at $x = y$ of $a(y,x,\theta)$ as a function of x, and

$$(40.2)\qquad b(y,\theta) = \sum (\frac{1}{i})^l \frac{1}{l!} \frac{\partial}{\partial \theta_{j_1}} \cdots \frac{\partial}{\partial \theta_{j_l}} a_{j_1,\dots,j_l}(y,\theta).$$

We see the above easily by integration by parts.
We then find that

$$(41)\qquad F_a \circ F_a^{V*} v(y) = c\int e^{i<y-\tilde{y},\kappa>} \sum_{l=0}^{\infty} k_l(y,\kappa) v(\tilde{y}) d\tilde{y} d\kappa, \quad \text{where}$$

$(42)_l$

$$k_l(y,\kappa) = \Big[(\frac{\partial^2}{\partial \kappa_j \partial \tilde{y}^j})^l \{ a(y,\xi(y,\tilde{y},\kappa)) \overline{a(\tilde{y},\xi(y,\tilde{y},\kappa))} V(\tilde{y}) \det \frac{\partial \xi(y,\tilde{y},\kappa)}{\partial \kappa} \} \Big]_{\tilde{y}=y}.$$

Clearly, $F_a \circ F_a^{V*}$ has the same singularity as $F_a \circ f F_a^{V*}$ when restricted to a small neighborhood of p_*. Therefore our condition is

$$(43)\qquad \sum (\frac{1}{i})^l \frac{1}{l!}\, k_l(y,\kappa) = 1 \text{ when } y \text{ is in a neighborhood of } p_*.$$

We thus find

THEOREM. *Assume that a symbol $a(y,\xi)$ satisfies the following conditions:* (i) (25)-(26), *(or equivalently* (29)-(33)*), and* (ii) (42)-(43). *Then, when restricted to a small neighborhood of p_*, F_a induces an unitary operator of L_2 space with respect to the volume element V and for a suitable matrix (B^α_β) of pseudo-differential operators with compact support we have $F_a \circ B^\alpha_\beta \circ P^\beta \equiv Q^\alpha \circ F_a$ modulo an operator of order $-\infty$.*

H) We next construct a Fourier integral operator ρ_M which has the same singularity near p_* as the Szegö projection. Set

$$(44)\qquad \psi_H(x,\tilde{x}) = x^0 - \tilde{x}^0 + \frac{i}{2}(|z|^2 + |\tilde{z}|^2 - 2 < z, \tilde{z} >)$$

where $< z, v >= z^\alpha \overline{v^\alpha}$. Then the Szegö projection operator ρ_H for the model case is given by

$$(45)\qquad \rho_H u(x) = C_H \int \frac{1}{\psi_H(x,\tilde{x})^{n+1}} u(\tilde{x}) d\tilde{x}, \quad C_H = \frac{n!}{(\pi)^{n+1}}.$$

Noting that

$$(46)\qquad \Re i\psi_H(x,\tilde{x}) = -\frac{1}{2}|z-\tilde{z}|^2 \le 0,$$

we find by integration by parts that we can rewrite

$$\rho_H u(x) = c_H \int e^{i\tau\psi_H(x,\tilde{x})}\tau^n u(\tilde{x})d\tilde{x}d\tau, \quad c_H = \frac{1}{(i\pi)^{n+1}}, \tag{47}$$

where the integral for τ is for $\tau > 0$.

Let $a(y,\xi)$ be a symbol given in the theorem. Clearly with a cut-off function f

$$\rho_M = F_a \circ f\rho_H \circ fF_a^{V*} \tag{48}$$

is the Fourier integral operator having the same singularity (near p_*) as the Szegö projection of the CR structure M with respect to the volume element $V(y)dy$.

In order to write ρ_M more explicitly we set

$$\begin{aligned} &\Psi\ (y,\tilde{y},x,\tilde{x},\theta,\tilde{\theta},\tau) \\ &= S(y,\theta) - x\cdot\theta\ + \tau\psi_H(x,\tilde{x}) - S(\tilde{y},\tilde{\theta}) + \tilde{x}\cdot\tilde{\theta}. \end{aligned} \tag{49}$$

We then find by calculation that with

$$c_M = (2\pi)^{-2(2n+1)}c_H, \tag{50.1}$$

(50.2)

$$\rho_M v(y) = c_M \int e^{i\Psi(y,\tilde{y},x,\tilde{x},\theta,\tilde{\theta},\tau)}a(y,\theta)\overline{a(\tilde{y},\tilde{\theta})}\tau^n V(\tilde{y})v(\tilde{y})d\tilde{y}dxd\tilde{x}d\theta d\tilde{\theta}d\tau.$$

In the above the integral is over $(\tilde{y},\theta,\tilde{\theta},x,\tilde{x},\tau)$ where θ and $\tilde{\theta}$ are in a small cone neighborhood of e_0, where $(e_0)^0 = 1$, $(e_0)^j = 0$ for $j > 0$. Therefore we may set $\theta = \theta_0\nu$, $\tilde{\theta} = \tilde{\theta}_0\tilde{\nu}$ where $\nu,\tilde{\nu}$ are in a small neighborhood of e_0. On the other hand the critical cone of our phase function Ψ is contained in the submanifold: $\theta_0 - \tau = 0, \tilde{\theta}_0 - \tau = 0$. Since the singularity comes from the integral on a cone neighborhood of the critical cone of the phase function, we may set

$$\theta = \tau\nu, \quad \tilde{\theta} = \tau\tilde{\nu}, \tag{51}$$

and use a new chart σ of dimension $m_1 = 4(2n+1)$ given by

$$\sigma = (x,\nu,\tilde{x},\tilde{\nu}), \quad \sigma^{(1)} = x,\ \sigma^{(2)} = \nu,\ \sigma^{(3)} = \tilde{x},\ \sigma^{(4)} = \tilde{\nu}, \tag{52}$$

where $\nu, \tilde{\nu}$ are in a small neighborhood of e_0. Then we can rewrite (50) as

$$\rho_M v(y) = c_M \int e^{i\tau\Phi(w,\sigma)} A(w,\sigma,\tau) v(\tilde{y}) d\sigma d\tau d\tilde{y}, \quad \text{where} \tag{53}$$

$$w = (y,\tilde{y}), \quad \Phi(w,\sigma) = S(y,\nu) - x\cdot\nu + \psi_H(x,\tilde{x}) - S(\tilde{y}.\tilde{\nu}) + \tilde{x}\cdot\tilde{\nu}, \tag{54.1}$$

$$A(w,\sigma,\tau) = a(y,\tau\nu)\overline{a(\tilde{y},\tau\tilde{\nu})} V(\tilde{y})\tau^{m_0}, \quad m_0 = 5n+2. \tag{54.2}$$

I) We are thus lead to consider an integral of the form

$$I(w,\tau) = \int e^{i\tau\Phi(w,\sigma)} A(w,\sigma,\tau) d\sigma, \tag{55}$$

with a reference point (w_*, σ_*) where $w_* = (0,0)$, and $\sigma_*^{(1)} = \sigma_*^{(3)} = 0$, $\sigma_*^{(2)} = \sigma_*^{(4)} = e_0$. Note that Φ satisfies the following conditions:

(56)
$$\Phi(w_*,\sigma_*) = 0, \ \Phi'_\sigma(w_*,\sigma_*) = 0, \ \det \Phi''_{\sigma\sigma}(w_*,\sigma_*) \neq 0, \ \Im\Phi(w,\sigma) \geq 0.$$

We apply the method of stationary phase. We see by calculation that the equation for the critical set of our complex phase function is given by

(57)
$$S'_\xi(y,\theta) = x, \ \theta = e_0 + ix_+ - i\tilde{x}_+ + \tilde{x}_\sharp, \ S'_\xi(\tilde{y},\tilde{\theta}) = \tilde{x}, \ \tilde{\theta} = e_0 + ix_+ - i\tilde{x}_+ + x_\sharp,$$

where

$$x_+ = (x^1, \dots, x^{2n}, 0), \quad x_\sharp = (\dots, -x^{n+\alpha}, \dots, x^\alpha, \dots, 0). \tag{58}$$

In order to have a nice solution of the above equation we complexify the variable σ, and consider the almost analytic extensions of $S'_\xi(y,\xi)$, $S'_\xi(\tilde{y},\xi)$. We keep $w = (y,\tilde{y})$ as a real vector variable. Then the equation (57) is the same as an equation in $(x,\tilde{x})$ given by

$$S'_\xi(y, e_0 + ix_+ - i\tilde{x}_+ + \tilde{x}_\sharp) = x, \quad S'_\xi(\tilde{y}, e_0 + ix_+ - i\tilde{x}_+ + x_\sharp) = \tilde{x}, \tag{59}$$

Since $S'_\xi(y,\xi)$ is s small deformation of y, the above equation has clearly a unique complex solution. We denote the unique solution by

$$x = X(y,\tilde{y}), \quad \tilde{x} = \tilde{X}(y,\tilde{y}). \tag{60}$$

Until further notice σ is a complex vector variable. (57) suggests that we set

$$U(w) = e_0 + iX_+(w) - i\tilde{X}_+(w) + \tilde{X}_\sharp(w), \quad \tilde{U}(w) = e_0 + iX_+(w) - i\tilde{X}_+(w) + X_\sharp(w). \tag{61}$$

Hence the equation (59) is rewritten as

$$S'_\xi(y, U(w)) = X(w), \quad S'_\xi(\tilde{y}, \tilde{U}(w)) = \tilde{X}(w). \tag{62}$$

We define a map Z of w-space to complexified σ-space by

$$x = X(w), \quad \nu = U(w), \quad \tilde{x} = \tilde{X}(w), \quad \tilde{\nu} = \tilde{U}(w). \tag{63}$$

$(w, \tau Z(w))$ is a parametrization of the complex critical cone of our phase function $\Phi(w, \sigma)$. Note that $Z(w_*) = \sigma_*$. We have a crucial estimate: for a constant $c > 0$,

$$\Im\Phi(w, Z(w)) \geq c|\Im Z(w)|^2 \tag{64}$$

(cf. the formula (2.5) in [22]).

J) Since $\Phi'_\sigma(w, Z(w)) = 0$,

$$\Phi(w, Z(w) + \sigma) = \Phi(w, Z(w)) + \frac{1}{2}Q_{ab}(w, \sigma)\sigma^a\sigma^b + \rho, \quad \text{where} \tag{65.1}$$

$$Q_{ab}(w, \sigma) = 2\int_0^1 (1 - s)\Phi''_{\sigma^a\sigma^b}(w, Z(w) + s\sigma)ds, \tag{65.2}$$

$$|\rho| < C_N(|\Im\sigma|^N + |\Im Z(w)|^N) \tag{65.3}$$

for any natural number N. ρ with an estimate as above does not contribute to the singularity of our formula. Hence we are going to ignore it, using $\equiv$ instead of $=$.

We have for a choice of $\Psi^a_b(w, \sigma)$

$$-iQ_{ab}(w, \sigma) = \sum_c \Psi^c_a(w, \sigma)\Psi^c_b(w, \sigma). \tag{66}$$

Define a map $\sigma \to \theta = (\dots, \theta^a, \dots) \in C^{m_1}$ by

$$\theta^a(w,\sigma) = \Psi^a_b(w,\sigma)\sigma^b. \tag{67}$$

Let $\theta \to \sigma = \sigma(w,\theta)$ be the inverse map. Hence

$$\sigma = \sigma(w,\theta) \quad \text{if and only if} \quad \theta = \theta(w,\sigma). \tag{68}$$

Regarding σ as a real vector variable, we have by (65.1), (66), and (67)

$$i\Phi(w,\sigma) \equiv i\psi_M(w) - \frac{1}{2}\theta(w,\sigma - Z(w)) \cdot \theta(w,\sigma - Z(w)), \tag{69.1}$$

$$\psi_M(w) = \Phi(w, Z(w)). \tag{69.2}$$

Therefore modulo an operator of order $-\infty$

$$I(w,\tau) = e^{i\tau\psi_M(w)} \int_{R^m} e^{-\frac{\tau}{2}\theta(w,\sigma-Z(w))\cdot\theta(w,\sigma-Z(w))} A(w,\sigma,\tau)d\sigma, \tag{70}$$

where the integral is over the real variable σ.

In order to avoid confusion we introduce a complex variable σ_C, a copy of the complex variable σ. From now on in this section σ is always a real vector variable. Similarly, we have θ_C, and a real vector variable θ. We also have the maps $\sigma_C \to \theta_C = \theta(w,\sigma_C)$ and $\theta_C \to \sigma(w,\theta_C)$.

Denote by Θ the 4(2n+1)- form on σ_C-space given by

$$\Theta = e^{-\frac{\tau}{2}\theta(w,\sigma_C)\cdot\theta(w,\sigma_C)} A(w,\sigma_C + Z(w),\tau)d\sigma_C, \tag{71}$$

We have for each w a submanifold $\Gamma(w)$ of real dimension 4(2n+1) defined by

$$\Gamma(w) = \{\sigma_C = \sigma - Z(w) \ : \ \sigma \text{ a real vector variable}\}. \tag{72}$$

Then by (70)

$$I(w,\tau) = e^{i\tau\psi_M(w)} \int_{\Gamma(w)} \Theta. \tag{73}$$

We now change our variable from σ_C to $\theta_C = \theta(w, \sigma_C)$. Then with a submanifold

$$\Gamma'(w) = \theta(w, \Gamma(w)) = \{\theta(w, \sigma - Z(w))\}, \tag{74}$$

we see by the change of variables formula

$$I(w,\tau) \equiv e^{i\tau\psi_M(w)} \int_{\Gamma'(w)} e^{-\frac{\tau}{2}\theta_C\cdot\theta_C} A(w, \sigma(w,\theta_C) + Z(w)) \det \frac{\partial\sigma(w,\theta_C)}{\partial\theta_C} d\theta_C. \tag{75}$$

We then note that in our case the submanifold $\Gamma'(w)$ has a parameterization:

$$\Gamma'(w) = \{\rho + iB(w,\rho) \ : \ \rho \text{ is a real vector variable}\}. \tag{76}$$

with a suitable real vector valued function B. Introduce a real parameter s in $[0,1]$. We move by homotopy the domain $\Gamma'(w)$ of the integral in (75) to $\Gamma'_s(w) \ : \ \theta = \rho + isB$. Then we find by the Stokes Theorem and the almost analyticity that the integral does not change (mod. a smooth operator) under the homotopy. Therefore we conclude that

$$I(w,\tau) \equiv e^{i\tau\psi_M(w)} \int e^{-\frac{\tau}{2}\rho\cdot\rho} A(w, Z(w) + \sigma(w,\rho), \tau) \left(\det \frac{\partial\sigma(w,\theta_C)}{\partial\theta_C}\right)_{\theta_C=\rho} d\rho. \tag{77}$$

Set

$$\Delta = \sum_{a=1}^{m} \frac{\partial^2}{\partial\rho^a\partial\rho^a}, \quad \frac{1}{i}\psi_M(w) = p, \tag{78}$$

$$f(w,\rho,\tau) = A(w, Z(w) + \sigma(w,\rho), \tau) \left(\det \frac{\partial\sigma(w,\theta_C)}{\partial\theta_C}\right)_{\theta_C=\rho}, \tag{79}$$

so that (cf. (64))

$$I(w,\tau) = \int e^{-\tau p} e^{-\frac{\tau}{2}} f(w,\rho,\tau) d\rho. \quad \Re p \geq 0. \tag{80}$$

Set (cf. (54.2))

$$f(w,\rho,\tau)=\sum_{s=0}^{\infty} f_{<s>}(w,\rho)\tau^{m_0-s}, \quad m_0=5n+2. \tag{81}$$

Let the strongly pseudo convex inside tubular neighborhood of M in C^{n+1} be defined by $r>0$. We find that we can find such $r(x)$ so that with $r(x,x')$ given in (1) in the Introduction we have

$$\frac{1}{i}\psi_M(x,x')=r(x,x'). \tag{82}$$

Therefore we conclude:

THEOREM. *Let $I(w,\tau)$ be given by* (70). *Then with $r(w)$ defined as above and $f(w,\rho,\tau)$ given in* (79) *we have asymptotically*

$$\int I(w,\tau)d\tau=\sum_{l=0}^{n} f_l^0 r^{-l-1}+\log p\sum_{l=0}^{\infty} f_l r^l, \tag{83}$$

$$f_l^0=(2\pi)^{\frac{2n+1}{2}}\sum_{s+k=n-l}\frac{l!}{k!}\left(\frac{1}{2}\right)^k\Delta^k f_{<s>}(0). \tag{84}$$

$$f_l=(2\pi)^{\frac{2n+1}{2}}\sum_{s+k=n+l+1}\frac{1}{l!}(-1)^{l+1}\frac{1}{k!}\left(\frac{1}{2}\right)^k\Delta^k f_{<s>}(0). \tag{85}$$

To write down the singularity of the Szegö kernel we apply the above formula in the case $A(w,\sigma,\tau)$ is given by (54.2), where the symbol $a(y,\xi)$ is in the theorem in p.653. We then find the following: Using $U(w),\tilde{U}(w)$ given in (60)-(61) and $\sigma(w,\theta_C)$ given in (66)-(68), we set

$$\begin{aligned} a(y,\tau U(w)&+\tau\sigma^{(2)}(w,\rho))\,\overline{a(\tilde{y},\tau\tilde{U}(w)+\tau\sigma^{(4)}(w,\rho))}\\ &(\det\frac{\partial\sigma(w,\theta_C)}{\partial\theta_C})_{\theta_C=\rho}=\sum_{s=0}^{\infty}B_{<s>}(w,\rho)\tau^{-s}. \end{aligned} \tag{86}$$

Then

$$\rho_M v(y)=\int K^S(y,\tilde{y})v(\tilde{y})V(\tilde{y})d\tilde{y} \tag{87}$$

where the Szegö kernel $K^S(y,\tilde{y})$ has the expression

(88) $$K^S(y,\tilde{y}) \equiv \sum_{k=0}^{n} F_k(w)(r(w))^{-k-1} + \sum_{k=0}^{\infty} G_k(w)\ (r(w))^k \log r(w),$$

(89) $$F_k(w) = (\frac{1}{\pi})^{n+1} k!\, i^{k-n} \sum_{n-s-l=k} \frac{l}{l!}(\frac{1}{2})^l [(\Delta_r)^l B_{<s>}(w,r)]_{r=0}.$$

(90)
$$G_k(w) = (\frac{1}{\pi})^{n+1} \frac{(-1)^{k+1}}{k!} i^{-k-n-1} \sum_{s+k=n+l+1} \frac{1}{l!}(\frac{1}{2})^l [(\Delta_\rho)^l B_{<s>}(w,\rho)]_{\rho=0}.$$

§3. The constructions for solutions

A) We construct a solution of the equation (8) in §2. It has many solutions. For simplicity we construct a solution of the form:

(1) $$\lambda = \eta_0 \sum_{p>0} (\lambda_{(0,p)} + \lambda_{(p,0)}), \quad \lambda_{(p,0)} = \overline{\lambda_{(0,p)}},$$

where $\lambda_{(p,q)}$ is of type (p,q) in $(g,\bar{g})$, $g = g(y,\eta)$, with smooth functions in y as coefficients. Define $c^{\alpha\bar{\beta}}(y)$ and $c^\alpha(y)$ by

(2) $$[Q^\alpha, \overline{Q^\beta}] = ic^{\alpha\bar{\beta}} \frac{\partial}{\partial y^0}, \quad [Q^\alpha, \partial/\partial y^0] = c^\alpha \frac{\partial}{\partial y^0}.$$

Let $(c_{\alpha\bar{\beta}})$ denote the inverse matrix of $(c^{\alpha\bar{\beta}})$. $h^\alpha, \dot{h}^\alpha$ are functions in y. We have

(3) $$\begin{aligned} &\{g^\alpha, g^\beta\} = ic^\alpha g^\beta, \quad \{q^\alpha, \overline{g^\beta}\} = c^{\alpha\bar{\beta}} + ic^\alpha \overline{g^\beta}, \quad \{q^\alpha, z^\beta(y)\} = 0, \\ &\{q^\alpha, \overline{z^\beta}\} = \frac{1}{i}\delta^\alpha_\beta, \quad \{q^\alpha, y^0\} = -\frac{1}{2}h^\alpha(y), \quad \{g^\alpha, \eta_0\} = -c^\alpha. \end{aligned}$$

Note that $(y,\eta_0,g,\bar{g})$ is a chart of the complexified cotangent space of M. We denote by $\tilde{\partial}/\partial y, \tilde{\partial}/\partial\eta_0, \partial/\partial g, \partial/\partial\bar{g}$ the partial derivatives with respect to the chart. We set $\tilde{q}^\alpha = q^\alpha(y, 1/i\tilde{\partial}/\partial y)$. We then find that $\lambda_{(0,p)}$ in (1) is given inductively as follows:

(4.1) $$\lambda_{(0,1)} = i\overline{g^\beta}\frac{i}{2}c_{\alpha\bar{\beta}}\dot{h}^\alpha, \quad \dot{h}^\alpha = \frac{\partial h^\alpha}{\partial t}, \quad \text{and for } p > 0$$

$$(4.2) \qquad \lambda_{(0,p)} = \frac{i}{p}\overline{g^{\alpha}}c_{\beta\bar{\alpha}}(\tilde{Q}^{\beta} - (p-2)c^{\beta})\lambda_{(0,p-1)}.$$

We also define

$$(5) \qquad s^{\alpha}_{\beta} = \eta_0 \sum_{p\geq 0}(s^{\alpha}_{\beta})_{(p,0)}, \quad (s^{\alpha}_{\beta})_{(p,0)} = \frac{1}{p+1}(pc^{\alpha} - \tilde{Q}^{\alpha})\frac{\partial\lambda_{(p+1,0)}}{\partial g^{\beta}}.$$

The above λ and s^{α}_{β} satisfy the equation (8) in §2.

B) The generating function $S(y,\xi)$ for our symplectic map χ can be constructed as follows: for each ξ consider the map : $x \to y = y(x,\xi)$. Let

$$(6) \qquad y \to x = x(y,\xi)$$

be the inverse map. Then

$$(7) \qquad S(y,\xi) = \xi_j x^j(y,\xi), \quad \text{and}$$

$$(8) \qquad \frac{\partial S(y,\xi)}{\partial \xi_j} = x^j(y,\xi).$$

C) We construct a symbol $a(y,\xi)$ satisfying the conditions (31)-(32) in §2 as well as the conditions (42)-(43) in §2. We use the notations in G) in §2.

By (42)-(43) in §2 we see that $a_{(0)}(y,\xi)$ must satisfy

$$(9) \qquad a_{(0)}(y,\xi(y,y,\kappa))\overline{a_{(0)}(y,\xi(y,y,\kappa))}V(y)\det\frac{\partial\xi(y,y,\kappa)}{\partial\kappa} = 1.$$

Note that

$$(10) \qquad (\det\frac{\partial\xi(y,x,\kappa)}{\partial\kappa})^{-1} = [\det\frac{\partial\kappa(y,x,\xi)}{\partial\xi}]_{\xi=\xi(y,y,\kappa)}.$$

Therefore, when we set

$$(11) \qquad G(y,\xi) = V(y)\ \det\frac{\partial\kappa(y,y,\xi)}{\partial\xi},$$

the condition (9) is equivalent to the condition:

$$(12) \qquad a_{(0)}(y,\xi)\overline{a_{(0)}(y,\xi)}V(y) = G(y,\xi).$$

In view of (42)-(43) in §2 we set

$$\sum_{l=0}^{\infty} \frac{1}{i}\frac{1}{l!} k_l(y,\kappa) = \tilde{k}(y,\kappa). \tag{13}$$

Denote by $\tilde{k}_{(l)}(y,\kappa)$ the homogeneous degree l part of $\tilde{k}$. We then find that we can write down a function $k^{\sharp}_{(-l)}(y,\xi,a_{(0)},\dots,a_{(-l+1)})$ for $l \geq 1$ such that

$$\begin{aligned}\tilde{k}_{(-l)}(y,\kappa(y,y,\xi)) = (a_{(-l)}(y,\xi)\overline{a_{(0)}(y,\xi)} + a_{(0)}(y,\xi)\overline{a_{(-l)}(y,\xi)})\\ V(y)G(y,\xi)^{-1} + k^{\sharp}_{(-l)}(y,\xi,a_{(0)},\dots,a_{(-l+1)}).\end{aligned} \tag{14}$$

Then the (unitary) condition for $a_{(-l)}(y,\xi)$ is given by

$$\begin{aligned}(a_{(-l)}(y,\xi)\overline{a_{(0)}(y,\xi)} + a_{(0)}(y,\xi)\overline{a_{(-l)}(y,\xi)})V(y)\\ + k^{\sharp}_{(-l)}(y,\xi,a_{(0)},\dots,a_{(-l+1)})G(y,\xi) = 0.\end{aligned} \tag{15}$$

D) $a(y,\xi)$ must also satisfy the condition (31) in §2. We use the notation in F) in §2. We set

$$a_{<l>}(x,f,\bar{f}) = a_{<l>\phi}(x) + \sum a_{<l>[p,q]}(x,f,\bar{f}), \tag{16}$$

where $a_{<l>[p,q]}$ is of type (p,q) in $f,\bar{f}$. We then see that the condition $(31)_l$ in §2 is a condition for $a_{<l>\phi}$, $a_{<l>[0,q]}$, and $D^{\alpha}_{\beta<l>}$. We also see that we can write down the solutions of (31) in §2 easily. Namely, we define as above $R^{\alpha}_{\phi}(x)$, $R^{\alpha}_{[p,q]}$, $C^{l,\alpha}_{\phi}$, $C^{l,\alpha}_{[p,q]}$, $D^{\alpha}_{\beta<l>\phi}$, $D^{\alpha}_{\beta<l>[p,q]}$, where $D^{\alpha}_{\beta<l>\phi}$, $D^{\alpha}_{\beta<l>[p,q]}$ are unknown. Then for arbitrary choice of $a_{<l>\phi}(x)$, we have a solution. Namely

$$a_{<l>[0,1]} = \overline{f^{\alpha}} H^{l,\alpha}_{\phi}, \quad H^{l,\alpha}_{\phi} = C^{l,\alpha}_{\phi} - \tilde{P}^{\alpha}_{x} a_{<l>\phi} - R^{\alpha}_{\phi}. \tag{17}$$

$a_{<l>[0,q]}$ for $q \geq 1$ is similarly determined inductively, i.e.

$$\begin{aligned}a_{<l>[0,q+1]} = \frac{1}{q+1}\overline{f^{\alpha}} H^{l,\alpha}_{q}, \quad H^{l,\alpha}_{q} = C^{l,\alpha}_{[0,q]} - \tilde{P}^{\alpha}_{x} a_{<l>[0,q]}\\ -R^{\alpha}_{\phi} a_{<l>[0,q]} - \sum_{q_1+q_2=q)} R^{\alpha}_{[0,q_1]} a_{<l>[0,q_2]}.\end{aligned} \tag{18}$$

To see this actually works, we need to show

$$\frac{\partial H^{l,\alpha}_{q-1}}{\partial \overline{f^{\beta}}} = \frac{\partial H^{l,\beta}_{q-1}}{\partial \overline{f^{\alpha}}}. \tag{19}$$

We find the above is valid by calculation using the inductive assumption. When $a_{<l>\phi}$, $a_{<l>[0,q]}$, $q = 1, 2, \ldots$ satisfy the above, we then see easily that $a_{<l>}$ with arbitrary $a_{<l>[p,q]}$, $p \neq 0$, satisfies the equation $(31)_l$ in §2 by choosing $D^{\alpha}_{\beta<l>}$ suitably.

The conditions (13)-(15), (17), and (18) in this section must be satisfied by our $a(y,\xi)$. We see easily how to write down $a(y,\xi)$, not uniquely, satisfying the conditions.

E) We next make the construction of $\theta^a(w,\sigma)$, with a complex vector variable σ, in (65)-(67) in §2 more explicit. We find by calculation that (using x', y' instead of $\tilde{x}, \tilde{y}$, and μ instead of $\tilde{\nu}$).

$-i\, Q_{ab}(w,\sigma)\ \underline{\sigma}_a \underline{\sigma}_b = Q^1(\underline{\sigma},\underline{\sigma}) + R_{ab}(w,\sigma)\ \underline{\sigma}_a\underline{\sigma}_b$, where

$$\begin{aligned} Q^1(\underline{\sigma},\underline{\sigma}) &= i\ \underline{x}\cdot\{2\underline{\nu} - i\ \underline{x}_+ + i\ \underline{x}'_+ - \underline{x}'_\sharp\} \\ &\quad + i\underline{x}'\cdot\{-2\underline{\mu} - i\underline{x}'_+ + i\underline{x}_+ + \underline{x}_\sharp\}, \\ R_{ab}(w,\sigma)\ \underline{\sigma}_a\underline{\sigma}_b &= 2i\int_0^1 (1-s)\{S''_{\mu_j\mu_k}(y', s\hat{\mu} + \mu(y,y'))\underline{\mu}_j\underline{\mu}_k \\ &\quad - S''_{\nu_j\nu_k}(y, s\hat{\nu} + \nu(y,y'))\underline{\nu}_j\underline{\nu}_k\} ds, \end{aligned} \tag{20}$$

$$\hat{\nu} = (\nu_0 - 1)e_0 + \nu_+, \quad \hat{\mu} = (\mu_0 - 1)e_0 + \mu_+. \tag{21}$$

We also find by calculation (cf. (58) in §2)

$$Q^1(\underline{\sigma},\underline{\sigma}) = \sum_{p=1}^{4} \frac{1}{4}(L^{(p)}\underline{\sigma})\cdot(L^{(p)}\underline{\sigma}), \quad \text{where with} \tag{22}$$

$$x_* = ix_+ + x_\sharp, \quad x'_* = ix'_+ - x'_\sharp, \tag{23.1}$$

$$\begin{aligned} L^{(1)}\sigma &= x\ +\ x_+\ +\ i\ x'_*\ +\ i\ 2\nu, \\ L^{(2)}\sigma &= i\ x\ -\ i\ x_+\ +\ x'_*\ +\ 2\nu, \\ L^{(3)}\sigma &= x'\ +\ x'_+\ +\ i\ x_*\ -\ i\ 2\mu, \\ L^{(4)}\sigma &= i\ x'\ -\ i\ x'_+\ +\ x_*\ -\ 2\mu. \end{aligned} \tag{23.2}$$

We consider $\theta(w,\sigma)$ of the form

$$\theta(w,\sigma) = L\sigma + K(w,\sigma)\sigma, \tag{24}$$

where we regard L and $K(w,\sigma)$ as $m_1 \times m_1$ matrixes, $m_1 = 4(2n+1)$. We also regard the quadratic form R as a matrix, say $[R]$ representing the map : $\sigma \to \tilde{\sigma}$, where $\tilde{\sigma}^a = R_{ab}(w,\sigma)\sigma^b$. Then our equation is

$$(L^{\text{tr}} + K^{\text{tr}})K = [R]. \tag{25}$$

The equation has a formal solution. Namely,

$$\begin{aligned} K &= \sum_m K_m, \quad K_1 = (L^{\text{tr}})^{-1}[R], \\ K_m &= -\,(L^{\text{tr}})^{-1} \sum_{j=1}^{m-1} K_j K_{m-j} \text{ for } m > 1. \end{aligned} \tag{26}$$

Since $S(y,\xi) \equiv y \cdot \xi \pmod{t}$, $R_{ab} \equiv 0 \pmod{t}$. Hence we see that the above gives a solution of (26) as a formal power series in t. We then find by calculation

$$\theta(w,\sigma) = L\sqrt{\frac{1}{4} + L^{-1}(L^{\text{tr}})^{-1}[R]}\;\sigma. \tag{27}$$

Note that the inverse map L^{-1} of the map $\sigma \to l = (l^{(1)}, l^{(2)}, l^{(3)}, l^{(4)})$, $l^{(p)} = L^{(p)}\sigma$, is given by

$$\begin{aligned} \sigma^{(1)} &= x = \frac{1}{2}(l^{(1)} - il^{(2)}), \\ \sigma^{(2)} &= \nu = \frac{1}{4}\{(-il^{(1)} + l^{(2)}) + i(l_+^{(1)} - il_+^{(2)}) - i(l_+^{(3)} - il_+^{(4)}) \\ &\qquad + (l_\sharp)^{(3)} - i(l_\sharp)^{(4)})\} \\ \sigma^{(3)} &= x' = \frac{1}{2}(l^{(3)} - il^{(4)}), \\ \sigma^{(4)} &= \mu = \frac{1}{4}\{(il^{(3)} - l^4) - i(l_+^{(3)} - il_+^{(4)}) + i(l_+^{(1)} - il_+^{(2)}) \\ &\qquad + (l_\sharp)^{(1)} - i(l_\sharp)^{(2)})\}. \end{aligned} \tag{28}$$

This complete the construction of $\theta(w,\sigma)$.

§4. The construction of the singularity

For convenience we write down our process of finding the formula for the singularity of the Szegö kernel.

$y, \tilde{y}, y'$ represent points on a strongly pseudoconvex CR manifold $M = M_t$. $x, \tilde{x}, x'$ are for the model case $\mathcal{M}$. $w = (y, \tilde{y})$, $\sigma = (x, \tilde{x}, \nu, \tilde{\nu})$.

1) Define $\lambda(y, \eta)$ by (1)-(4) in §3. Define $s^{\alpha}_{\beta}(y, \eta)$ by (2) in §3 and (5) in §3.

2) Define $y(x, \xi)$ by (4) in §2 and (9) in §2. Let $y \to x(y, \xi)$ be the inverse map of $x \to y(x, \xi)$. Set $S(y, \xi) = \xi_j x^j(y, \xi)$. Then $\partial S(y, \xi)/\partial \xi_j = x^j(y, \xi)$.

3) Define $\kappa(w, \xi)$ by (37) in §2. Let $\kappa \to \xi(w, \kappa)$ be the inverse map of $\xi \to \kappa(w, \xi)$. Define $k_l(y, \kappa)$ by $(42)_l$ in §2 and $\tilde{k}(y, \kappa)$ by (13) in §3.

4) The symbol $a(y, \xi)$ is chosen to satisfy (11)-(12) in §3, (14)-(15) in §3, (26) in §2 (equivalently (30)-(33) in §2, or $a_{<l>}$ in D) in §3).

5) Define $X(w), \tilde{X}(w), U(w), \tilde{U}(w)$ by (61)-(62) in §2.

6) Define $\Phi(w, \sigma)$ by (54.1) in §2, and $Z(w)$ by (63) in §2.

7) Define $\theta(w, \sigma)$ by (65.2)-(66)-(67) in §2 (or E) in §3). Let $\theta \to \sigma(w, \theta)$ be the inverse map of $\theta \to \theta(w, \sigma)$.

8) The formula for the singularity is given in (86)-(90) in §2.

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