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THE EINSTEIN-KÄHLER METRICS ON HUA DOMAIN

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ABSTRACT. In this paper we describe the Einstein-Kähler metric for the Cartan-Hartogs of the first type which is the special case of the Hua domains. Firstly, we reduce the Monge-Ampère equation for the metric to an ordinary differential equation in the auxiliary function $X = X(z,w) = |w|^2[\det(I - ZZ^T)]^{-\frac{1}{K}}$ (see below). This differential equation can be solved to give an implicit function in X. Secondly, we get the estimate of the holomorphic section curvature under the complete Einstein-Kähler metric on this domain.

Let M be a complex manifold. Then a Hermitian metric $\sum_{i,j} g_{i,\overline{j}} dz^i \bigotimes d\overline{z}^j$ defined on M is said to be Kähler if the Kähler form $\Omega = \sqrt{-1}\sum_{i,j} g_{i,\overline{j}} dz^i \bigwedge d\overline{z}^j$ is closed. The Ricci form of this metric is defined to be $-\partial\overline{\partial}\log\det(g_{i,\overline{j}})$. If the Ricci form of the Kähler metric is proportional to the Kähler form, the metric is called Einstein-Kähler. If the manifold is not compact, we require the metric to be complete. Clearly for a noncompact complex manifold to admit such a metric, it is necessary that there exists a volume form, the negative of whose Ricci tensor defines a complete Kähler metric. The volume form of this Kähler metric must be equivalent to the original volume form. If we normalize the metric by requiring the scalar curvature to be minus one, then the Einstein-Kähler metric is unique. Cheng and Yau [1] proved that any bounded domain D which is the intersection of domain with C^2 boundary admits a complete Einstein-Kähler. Without any regularity assumption on the domain D, Mok and Yau [5] proved that the complete Einstein-Kähler metric always exists. This Einstein-Kähler metric is given by

$$E_D(z) := \sum \frac{\partial^2 g}{\partial z_i \partial \overline{z}_j} dz_i \overline{dz_j},$$

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where g is the unique solution to the boundary problem of the Monge-Ampère equation:

$$\begin{cases} \det\left(\dfrac{\partial^2 g}{\partial z_i \partial \overline{z}_j}\right) = e^{(n+1)g} & z \in D, \\ g = \infty & z \in \partial D, \end{cases}$$

and g is called generating function of $E_D(z)$.

The explicit formulas for the Einstein-Kähler metric, however, are only known in the simplest cases. The purpose of this paper is to describe the Einstein-Kähler metric for the Cartan-Hartogs domain of the first type:

$$\begin{aligned} &Y_I(1, m, n; K) \\ &:= \{w \in \mathbf{C}, Z \in R_I(m, n) : |w|^{2K} < \det(I - ZZ^T), K > 0\} := Y_I, \end{aligned}$$

which is the special case of the Hua domains.

The Hua domains are introduced by the second author in 2000. They can be written as follows:

$$\begin{aligned} & HE_I(N_1, \ldots, N_r, m, n; p_1, \ldots, p_r) \\ := \ & \{W_j \in \mathbf{C}^{N_j}, Z \in R_I(m, n) : \sum_{j=1}^{r} \|W_j\|^{2p_j} \\ & < \det(I - ZZ^T), p_j > 0, j = 1, 2, \ldots, r\}, \end{aligned}$$

$$\begin{aligned} & HE_{II}(N_1, \ldots, N_r, p; p_1, \ldots, p_r) \\ := \ & \{W_j \in \mathbf{C}^{N_j}, Z \in R_{II}(p) : \sum_{j=1}^{r} \|W_j\|^{2p_j} \\ & < \det(I - ZZ^T), p_j > 0, j = 1, 2, \ldots, r\}, \end{aligned}$$

$$\begin{aligned} & HE_{III}(N_1, \ldots, N_r, q; p_1, \ldots, p_r) \\ := \ & \{W_j \in \mathbf{C}^{N_j}, Z \in R_{III}(q) : \sum_{j=1}^{r} \|W_j\|^{2p_j} \\ & < \det(I - ZZ^T), p_j > 0, j = 1, 2, \ldots, r\}, \end{aligned}$$

$$\begin{aligned} & HE_{IV}(N_1, \ldots, N_r, n; p_1, \ldots, p_r) \\ := \ & \{W_j \in \mathbf{C}^{N_j}, Z \in R_{IV}(n) : \sum_{j=1}^{r} \|W_j\|^{2p_j} \\ & < 1 - 2ZZ^T + |ZZ'|^2, p_j > 0, j = 1, 2, \ldots, r\}, \end{aligned}$$

where $\|W_j\|^2 = \sum_{k=1}^{N_j} |w_{jk}|^2$, $R_I(m,n), R_{II}(p), R_{III}(q)$ and $R_{IV}(n)$ are the first, second, third, fourth Cartan domain respectively in the sense of Loo-Keng HUA [3], $\bar{Z}^T$ indicates the conjugate and transpose of Z, det indicates the determinant. $r, N_1, \dots, N_r, m, n, p, q$ are positive integers, $p_1, \dots, p_r$ are positive real numbers. Some results on Hua domains can be found in [6, 7, 8].

This paper is organized as follows. Section 1 presents some background material and known results from domain Y_I needed for us. In section 2, by using the noncompact automorphism group of Y_I and the biholomorphic invariance of the Einstein-Kähler metric, we reduce the Monge-Ampère equation for the metric to an ordinary differential equation in the auxiliary function $X = X(z,w) = |w|^2[\det(I - Z\bar{Z}^T)]^{-\frac{1}{K}}$, this differential equation can be solved to give an implicit function in X. In section 3, the estimate of the holomorphic section curvature under the Einstein-Kähler metric on Y_I is given.

1. Preliminaries

LEMMA 1. $\mathrm{Aut}(Y_I)$ *indicates the holomorphic automorphism group of* Y_I *consisting of the following mappings* $F(z, w; z_0, \theta_0)$:

$$\begin{cases} w^* = e^{i\theta_0} w \det(I - Z_0\bar{Z}_0^T)^{\frac{1}{2K}} \det(I - Z\bar{Z}_0^T)^{-\frac{1}{K}}, \\ Z^* = A(Z - Z_0)(I - \bar{Z}_0^T Z)^{-1} D^{-1}, \end{cases}$$

where $\bar{A}^T A = (I - Z_0\bar{Z}_0^T)^{-1}, \bar{D}^T D = (I - \bar{Z}_0^T Z_0)^{-1}, \quad Z_0, Z \in R_I(m,n)$, $\theta_0 \in \mathbf{R}$.

Proof. See [6].

Obviously, the $F(z, w; z_0, \theta_0)$ maps point (Z_0, w) on to point $(0, w^*)$ and $Z^* = A(Z - Z_0)(I - \bar{Z}_0^T Z)^{-1} D^{-1}$ is holomorphic automorphism of $R_I(m,n)$.

LEMMA 2. *Let* $X = X(Z,w) = |w|^2[\det(I - Z\bar{Z}^T)]^{-\frac{1}{K}}$. *Then* X *is invariant under the mapping of* $\mathrm{Aut}(Y_I)$. *That is* $X(Z^*, w^*) = X(Z, w)$.

Proof. See [6]. □

LEMMA 3A. *If $F = F(z, w; z_0, \theta_0) \in \mathrm{Aut}(Y_I)$. Let J_F be the Jacobi matrix of $F(z, w; z_0, \theta_0)$, i.e.*

$$J_F = \begin{pmatrix} \dfrac{\partial z^*}{\partial z} & \dfrac{\partial w^*}{\partial z} \\ 0 & \dfrac{\partial w^*}{\partial w} \end{pmatrix},$$

where $z = (z_{11}, \ldots, z_{1n}, z_{21}, \ldots, z_{2n}, \ldots, z_{m1}, \ldots, z_{mn})$ is a vector, z_{jk} is the element of Z and $Z = (z_{jk})_{m\times n} \in R_I(m, n)$. Then one has

$$\frac{\partial z^*}{\partial z}|_{z_0=z} = (A' \cdot\!\times D^T)|_{z_0=z},$$

$$\frac{\partial w^*}{\partial z}|_{z_0=z} = \frac{1}{K} e^{i\theta} \det(I - ZZ_0^T)^{-\frac{1}{2K}} E(Z)' w,$$

$$\frac{\partial w^*}{\partial w}|_{z_0=z} = e^{i\theta} \det(I - ZZ_0^T)^{-\frac{1}{2K}},$$

where $E(Z) = (\mathrm{tr}[(I - ZZ^T)^{-1} I_{11} Z^T], \mathrm{tr}[(I - Z^T)^{-1} I_{12} Z^T], \ldots, \mathrm{tr}[(I - ZZ^T)^{-1} I_{mn} Z^T])$ is column vector with mn entries. I_{pq} is defined as a $m \times n$ matrix, the (p, q)-th entry of I_{pq}, i.e., the entry located at the junction of the p-th row and q-th column of I_{pq}, is 1, and others entries of I_{pq} are zero. The meaning of $\times$ can be found in [4].

Proof. It can be got by direct computations. □

LEMMA 3B. *If $F = F(z, w; z_0, \theta_0) \in \mathrm{Aut}(Y_I)$ and $T = T[(z, w), \overline{(z, w)}]$ is the metric matrix of the Einstein-Kähler metric of Y_I, one has*

$$T[(z, w), \overline{(z, w)}] = J_F|_{z_0=z} T[(0, w^*), \overline{(0, w^*)}] \overline{J_F}'|_{z_0=z},$$

and $|J_F|^2_{z_0=z} = \det(I - ZZ^T)^{-(m+n+\frac{1}{K})}$, where $|J_F| = \det J_F$.

Proof. It can be proved by using the invariance of the Einstein-Kähler metric under the holomorphic automorphism of Y_I. □

LEMMA 4. *Let B be a $m \times n$ matrix. Then*

$$\mathrm{tr}(B\overline{B}'B\overline{B}') \le \mathrm{tr}(B\overline{B}')\mathrm{tr}(B\overline{B}') \le (m^2 - m + 1)\mathrm{tr}(B\overline{B}'B\overline{B}').$$

Proof. It is obvious. □

2. Reduce the Monge-Ampère equation to an ordinary differential equation

Let $Z = (z_{jk})_{m\times n} \in R_I(m,n)$. We denote

$$\begin{aligned}(z,w) &= (z_{11},\dots,z_{1n},z_{21},\dots,z_{2n},\dots,z_{m1},\dots,z_{mn},w)\\ &= (z_1,z_2,\dots,z_N),\end{aligned}$$

where $N = mn+1$, and

$$g_{\alpha\overline{\beta}}(z,w) = \frac{\partial^2 g}{\partial z_\alpha \partial \overline{z}_\beta} \qquad \alpha,\beta = 1,2,\dots,N.$$

Note that

$$\frac{\partial g}{\partial z_N} = \frac{\partial g}{\partial w}.$$

Suppose $g(z,w)$ generates the Einstein-Kähler metric of Y_I. Then $g(z,w)$ is a solution to the boundary problem of the Monge-Ampère equation:

$$(1) \qquad \begin{cases} \det(g_{\alpha\overline{\beta}}(z,w)) = e^{(N+1)g(z,w)} & (z,w)\in Y_I,\\ g=\infty & (z,w)\in \partial Y_I.\end{cases}$$

Let $F:(z,w)\longrightarrow (z^*,w^*)$, $F = F(z,w;z_0,\theta_0)\in \mathrm{Aut}(Y_I)$. Because of the invariance of the metric, it is easy to show that $\det(g_{\alpha\overline{\beta}}(z,w)) = |J_F|^2\det(g_{\alpha\overline{\beta}}(z^*,w^*))$. So

$$e^{(N+1)g(z,w)} = |J_F|^2 e^{(N+1)g(z^*,w^*)}.$$

Thus

$$e^{-g(z^*,w^*)} = |J_F|^{\frac{2}{N+1}} e^{-g(z,w)}.$$

For arbitrary $(z,\ w)\in Y_I$, especially take $z_0 = z$, $\theta_0 = -\mathrm{Arg}w$, that is $F_0 = F(z,w;z,-\mathrm{Arg}w)$. We have

$$e^{-g(0,w^*)} = |J_{F_0}|^{\frac{2}{N+1}} e^{-g(z,w)} = \det(I-ZZ^T)^{-\frac{(m+n+1/K)}{N+1}} e^{-g(z,w)},$$

where $w^* = X^{\frac12}$. If $\lambda = K(m+n)+1$, then $|J_F|^2_{Z_0=Z} = X^\lambda |w|^{-2\lambda}$. Let $h(X) = e^{-g(0,X^{\frac12})} = e^{-g(0,w^*)}$. We obtain

$$h(X) = |J_{F_0}|^{\frac{2}{N+1}} e^{-g(z,w)} = X^{\frac{\lambda}{N+1}} |w|^{-2\frac{\lambda}{N+1}} e^{-g(z,w)}.$$

Hence

$$\frac{\partial h}{\partial w} = h'(X)\frac{\partial X}{\partial w} = |J_F|^{\frac{2}{N+1}} e^{-g(z,w)}\frac{\partial(-g)}{\partial w}.$$

It is obvious that

$$(2)\qquad \begin{aligned}\frac{\partial X}{\partial w} &= \frac{X}{w},\\ \frac{\partial X}{\partial \overline{w}} &= \frac{X}{\overline{w}}.\end{aligned}$$

For the sake of convenience, let

$$\frac{X}{w} = \begin{cases} \overline{w}\det(I - ZZ^T)^{-1/K} & w \neq 0,\\ 0 & w = 0.\end{cases}$$

Then

$$\frac{\partial g}{\partial w} = -\frac{X}{w}\cdot\frac{h'(X)}{h(X)}.$$

And due to $h(X) = X^{\frac{\lambda}{N+1}}|w|^{-2\frac{\lambda}{N+1}}e^{-g(z,w)}$, we have

$$\frac{\partial h}{\partial z_\alpha} = h'(X)\frac{\partial X}{\partial z_\alpha} = -h(X)\frac{\partial g}{\partial z_\alpha} + \frac{\lambda}{N+1}X^{-1}\frac{\partial X}{\partial z_\alpha}h(X)$$

Where $\alpha = 1, 2, \ldots, N-1$. Then

$$\frac{\partial g}{\partial z_\alpha} = \left(\frac{\lambda}{N+1}X^{-1} - \frac{h'(X)}{h(X)}\right)\frac{\partial X}{\partial z_\alpha}.$$

Let

$$(3)\qquad Y(X) = \frac{\lambda}{N+1} - X\frac{h'(X)}{h(X)},$$

then

$$\frac{\partial g}{\partial z_\alpha} = YX^{-1}\frac{\partial X}{\partial z_\alpha}, \qquad \alpha = 1, 2, \ldots, N-1.$$

$$\frac{\partial g}{\partial w} = \left(Y - \frac{\lambda}{N+1}\right)\frac{1}{w}.$$

So, we have

$$(4)\qquad \frac{\partial^2 g}{\partial w \partial \overline{w}} = Y'\frac{X}{|w|^2},$$

$$\frac{\partial^2 g}{\partial w \partial \overline{z}_\beta} = \frac{1}{w} Y' \frac{\partial X}{\partial \overline{z}_\beta},$$

$$\frac{\partial^2 g}{\partial z_\alpha \partial \overline{w}} = Y' X^{-1} \frac{\partial X}{\partial \overline{w}} \cdot \frac{\partial X}{\partial z_\alpha} + Y X^{-1} \frac{\partial^2 X}{\partial z_\alpha \partial \overline{w}} - Y X^{-2} \frac{\partial X}{\partial \overline{w}} \cdot \frac{\partial X}{\partial z_\alpha},$$

$$\frac{\partial^2 g}{\partial z_\alpha \partial \overline{z}_\beta} = Y' X^{-1} \frac{\partial X}{\partial \overline{z}_\beta} \cdot \frac{\partial X}{\partial z_\alpha} + Y X^{-1} \frac{\partial^2 X}{\partial z_\alpha \partial \overline{z}_\beta} - Y X^{-2} \frac{\partial X}{\partial \overline{z}_\beta} \cdot \frac{\partial X}{\partial z_\alpha},$$

where $\alpha, \beta = 1, 2, \ldots, N-1$. Because

(5)
$$\frac{\partial X}{\partial z_{pq}} = \frac{1}{K} X \operatorname{tr}[(I - ZZ^T)^{-1} I_{pq} Z^T],$$
$$\frac{\partial X}{\partial \overline{z}_{st}} = \frac{1}{K} X \operatorname{tr}[(I - ZZ^T)^{-1} Z I'_{st}].$$

From formula (2) and formula (5), we have

(6)
$$\left.\frac{\partial X}{\partial w}\right|_{z=0} = \overline{w}, \qquad \left.\frac{\partial X}{\partial \overline{w}}\right|_{z=0} = w,$$
$$\left.\frac{\partial X}{\partial z_\alpha}\right|_{z=0} = \left.\frac{\partial X}{\partial z_{pq}}\right|_{z=0} = 0, \quad \left.\frac{\partial X}{\partial \overline{z}_\beta}\right|_{z=0} = \left.\frac{\partial X}{\partial \overline{z}_{st}}\right|_{z=0} = 0,$$

where $\alpha = n(p-1)+q, \beta = n(s-1)+t$. And from formula (5), we have

(7)
$$\frac{\partial^2 X}{\partial z_{pq} \overline{z}_{st}} = \frac{1}{K} \frac{\partial X}{\partial \overline{z}_{st}} \operatorname{tr}[(I - ZZ^T)^{-1} I_{pq} Z^T] + \frac{X}{K} \operatorname{tr}[(I - ZZ^T)^{-1} I_{pq} I'_{st}]$$
$$+ \frac{X}{K} \operatorname{tr}[(I - ZZ^T)^{-1} Z I'_{st} (I - ZZ^T)^{-1} I_{pq} Z^T].$$

From formula (2), (5), (6), (7), by computation, we obtain

(8)
$$\left.\frac{\partial^2 X}{\partial z_{pq} \overline{z}_{st}}\right|_{z=0} = \frac{X}{K} \delta_{ps} \delta_{qt},$$
$$\left.\frac{\partial^2 X}{\partial z_{pq} \overline{w}}\right|_{z=0} = 0,$$
$$\left.\frac{\partial^2 X}{\partial w \overline{z}_{st}}\right|_{z=0} = 0,$$

where

$$\delta_{ps} = \begin{cases} 1 & p = s, \\ 0 & p \neq s. \end{cases} \qquad \delta_{qt} = \begin{cases} 1 & q = t, \\ 0 & q \neq t. \end{cases}$$

From formula (6), (8) and formula (4), we obtain

$$\left.\frac{\partial^2 g}{\partial w \partial \overline{w}}\right|_{z=0} = Y',$$

$$\left.\frac{\partial^2 g}{\partial w \partial \overline{z}_\beta}\right|_{z=0} = 0,$$

$$\left.\frac{\partial^2 g}{\partial z_\alpha \partial \overline{w}}\right|_{z=0} = 0,$$

$$\left.\frac{\partial^2 g}{\partial z_\alpha \partial \overline{z}_\beta}\right|_{z=0} = YK^{-1}\delta_{\alpha\beta},$$

where $\alpha, \beta = 1, 2, \ldots, N-1$. Therefore

$$\det(g_{\alpha\overline{\beta}}(0, w^*)) = \det \begin{pmatrix} \frac{Y}{K} & & & 0 \\ & \ddots & & \\ & & \frac{Y}{K} & \\ 0 & & & Y' \end{pmatrix} = \left(\frac{Y}{K}\right)^{N-1} Y'.$$

Due to the

$$\det(g_{\alpha\overline{\beta}}(z, w)) = \det(g_{\alpha\overline{\beta}}(0, w^*))|J_{F_0}|^2 = \left(\frac{Y}{K}\right)^{N-1} Y'|J_{F_0}|^2,$$

and

$$e^{(N+1)g(z,w)} = |J_{F_0}|^2 e^{(N+1)g(0,w^*)} = |J_{F_0}|^2 h^{-(N+1)}.$$

we reduce the Monge-Ampère equation to an ordinary differential equa tion:

$$\left(\frac{Y}{K}\right)^{N-1} Y' = h^{-(N+1)}.$$

It is equivalent to

$$\log(Y^{N-1}Y') + (N+1)\log h - \log K^{N-1} = 0.$$

Derivatives to X

$$\frac{(Y^{N-1}Y')'}{(Y^{N-1}Y')} + (N+1)\frac{h'}{h} = 0.$$

By computation and formula (3), we obtain

$$\frac{(Y^{N-1}Y')'}{(Y^{N-1}Y')} + \frac{\lambda}{X} - (N+1)\frac{Y}{X} = 0.$$

So we have

$$[X(Y^{N-1}Y')]' = (Y^{N+1})' - \frac{\lambda-1}{N}(Y^N)'.$$

It is equivalent to

$$XY^{N-1}Y' = Y^{N+1} - \frac{\lambda-1}{N}Y^N + C, \tag{9}$$

where C is a constant. Because

$$\frac{\partial g}{\partial w} = (Y - \frac{\lambda}{N+1})\frac{1}{w}.$$

So

$$Y = w\frac{\partial g}{\partial w} + \frac{\lambda}{N+1},$$

it holds for $\forall(z,w) \in Y_I$. If take $(z,0) \in Y_I$, then $X = 0$, thus $w\frac{\partial g}{\partial w}|_{w=0} = 0$, therefore $Y(0) = \frac{\lambda}{N+1}$. We obtain $C = \frac{(\lambda-N-1)\lambda^N}{N(N+1)^{N+1}}$ from formula (9).

Suppose g is a solution of Monge-Ampère equation. Then from the above we have

$$g = \frac{1}{N+1}\log[(\frac{Y}{K})^{N-1}Y'\det(I - ZZ^T)^{-(m+n+\frac{1}{K})}],$$

where Y are the solutions to the following problem:

$$\begin{cases} XY^{N-1}Y' = Y^{N+1} - \frac{\lambda-1}{N}Y^N + \frac{(\lambda-N-1)\lambda^N}{N(N+1)^{N+1}}, \\ Y(0) = \frac{\lambda}{N+1}. \end{cases} \tag{10}$$

The solutions of above problem are not unique. But the solution of the problem (1) is unique. If g is a generating function of the complete Einstein-Kähler metric on Y_I, then g is unique and has the form as follows:

$$g = \frac{1}{N+1}\log[(\frac{Y}{K})^{N-1}Y'\det(I - ZZ^T)^{-(m+n+\frac{1}{K})}],$$

where the Y is also the solution of problem (10). Therefore, the estimate for the holomorphic sectional curvature of the complete Einstein-Kähler metric of Y_I is included in the estimates provided in next section.

Because $\det(g_{\alpha\overline{\beta}}) = K^{1-N}Y^{N-1}Y'|J_{F_0}|^2 > 0$ for $\forall(z,w) \in Y_I$. Let

$$G(Y) = Y^{N+1} - \frac{\lambda-1}{N}Y^N + \frac{(\lambda-N-1)\lambda^N}{N(N+1)^{N+1}}.$$

From the first formula of (10), we have $G(Y) > 0$ for $\forall X \in (0,1)$. Because $G(\frac{\lambda}{N+1}) = 0$ and

$$G'(\frac{\lambda}{N+1}) = \left(\frac{\lambda}{N+1}\right)^{N-1} > 0.$$

Therefore there exists $\delta > 0$ such that $G(\frac{\lambda}{N+1} - \delta) < 0$. Thus, from the fact that $G(Y(X)) > 0$ for $\forall X \in (0,1)$, we have $Y(X) > \frac{\lambda}{N+1}$. And from the first formula of (10), we have $Y'(X) > 0$. Finally, for $\forall(z,w) \in Y_I$, we have $X \in [0,1)$, $Y(X) \geq \frac{\lambda}{N+1}$ and $Y'(X) > 0$.

For the problem (10), we obtain

$$C^*X = (Y - Y(0))\left(Y^N - \frac{\lambda - N - 1}{N(N+1)}\sum_{k=1}^{N} Y(0)^{k-1}Y^{N-k}\right)e^{\varphi(Y)},$$

where

$$\varphi(Y) = -\int_{Y(0)}^{Y} \frac{(N+1)Y^{N-1}}{Y^N - \frac{\lambda-N-1}{N(N+1)}\sum_{k=1}^{N} Y(0)^{k-1}Y^{N-k}}dY.$$

C^* is a positive constant, and Y is the function in X. Therefore the problem (10) can be solved to give an implicit function in X.

3. The estimates of holomorphic sectional curvatures

Recall that

$$g(z,w) = \frac{1}{N+1}\log[(\frac{Y}{K})^{N-1}Y' \det(I - ZZ^T)^{-(m+n+\frac{1}{K})}],$$

where Y are the solutions of (10). Then $g(z,w)$ generate the Einstein-Kähler metrics of Y_I. The holomorphic sectional curvatures of the Einstein-Kähler metrics have the following form by definition:

$$\omega[(z,w), d(z,w)] = \frac{d(z,w)[-\overline{d}dT + dTT^{-1}\overline{dT}']\overline{d(z,w)}'}{[d(z,w)T\overline{d(z,w)}']^2},$$

where

$$d = \sum \frac{\partial}{\partial z_\alpha}dz_\alpha, \qquad \overline{d} = \sum \frac{\partial}{\partial \overline{z}_\alpha}\overline{dz}_\alpha.$$

And the metric matrix $T = (g_{\alpha\overline{\beta}})$.

Note that the holomorphic sectional curvature of the Einstein-Kähler metric is invariant under the holomorphic automrphism of Y_I, and due to the Lemma 1, for $\forall(z,w) \in Y_I$ there exist $F \in \mathrm{Aut}(Y_I)$ such that

$F(z,w) = (0,w^*)$. So it suffices to calculate the $\omega[(z,w),d(z,w)]$ on point $(0,w^*)$. By Lemma 3b,

$$T[(z,w),\overline{(z,w)}] = J_F|_{z_0=z}T[(0,w^*),\overline{(0,w^*)}]\overline{J_F}'|_{z_0=z}$$

$$= J\begin{pmatrix} \frac{Y}{K}I & 0 \\ 0 & Y' \end{pmatrix}\overline{J}',$$

we have

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix},$$

where

$$T_{11} = \frac{Y}{K}(A'\overline{A} \cdot\times \overline{D}'D) + \frac{XY'}{K^2}E(Z)'\overline{E(Z)},$$

$$T_{12} = \frac{Y'}{K}w\det(I - ZZ^T)^{-\frac{1}{K}}E(Z)',$$

$$T_{21} = \overline{T}'_{12},$$

$$T_{22} = Y'\det(I - ZZ^T)^{-\frac{1}{K}},$$

where $A, D, E(Z)$ see Lemma 1 and Lemma 3a. Because

$$dT = \begin{pmatrix} dT_{11} & dT_{12} \\ dT_{21} & dT_{22} \end{pmatrix}, \overline{d}dT = \begin{pmatrix} \overline{d}dT_{11} & \overline{d}dT_{12} \\ \overline{d}dT_{21} & \overline{d}dT_{22} \end{pmatrix}.$$

By computations, we obtain

$$\begin{aligned} dT_{11}|_{z=0} &= \frac{Y'}{K}\overline{w}dwI, \\ dT_{12}|_{z=0} &= 0, \\ dT_{21}|_{z=0} &= \frac{Y'}{K}\overline{w}dz, \\ dT_{22}|_{z=0} &= Y''\overline{w}dw, \end{aligned}$$

$$\begin{aligned}
\bar{\partial}\partial T_{11}|_{z=0} &= \frac{XY''+Y'}{K}|dw|^2 I + \frac{XY'}{K^2}|dz|^2 I + \frac{XY'}{K^2}\overline{dz}' dz, \\
&+\frac{Y}{K}(\overline{dZ}dZ' \cdot\times I + I\cdot\times \overline{dZ}' dZ), \\
\bar{\partial}\partial T_{12}|_{z=0} &= \frac{XY''+Y'}{K} dw\overline{dz}', \\
\bar{\partial}\partial T_{21}|_{z=0} &= \frac{XY''+Y'}{K}\overline{dw}dz, \\
\bar{\partial}\partial T_{22}|_{z=0} &= (XY'''+Y'')|dw|^2 + \frac{XY''+Y'}{K}|dz|^2.
\end{aligned}$$

Let

$$-\bar{\partial}\partial T + \partial T T^{-1}\overline{\partial T}'|_{z=0} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix},$$

then we have

$$\begin{aligned}
&(dz,dw)[-\bar{\partial}\partial T + \partial T T^{-1}\overline{\partial T}']\overline{(dz,dw)}'|_{z=0} \\
&= (dz,dw)\begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}\overline{(dz,dw)}' \\
&= dzR_{11}\overline{dz'} + dwR_{21}\overline{dz'} + dzR_{12}\overline{dw'} + dwR_{22}\overline{dw'}.
\end{aligned}$$

Because

$$T^{-1}|_{z=0} = \begin{pmatrix} \frac{K}{Y}I & 0 \\ 0 & \frac{1}{Y'} \end{pmatrix},$$

by computations, we obtain

$$\begin{aligned}
R_{11} &= \left(\frac{XY'^2}{KY} - \frac{XY''+Y'}{K}\right)|dw|^2 I - \frac{XY'}{K^2}|dz|^2 I - \frac{XY'}{K^2}\overline{dz}' dz \\
&-\frac{Y}{K}(\overline{dZ}dZ' \cdot\times I + I\cdot\times \overline{dZ}' dZ), \\
R_{12} &= \left(\frac{XY'^2}{KY} - \frac{XY''+Y'}{K}\right) dw\overline{dz}', \\
R_{21} &= \overline{R_{12}'}, \\
R_{22} &= \left(\frac{XY'^2}{KY} - \frac{XY''+Y'}{K}\right)|dz|^2 + \left(\frac{XY''^2}{Y'} - XY''' - Y''\right)|dw|^2
\end{aligned}$$

From (10), by computations, we obtain

$$XY' = Y^2 - \frac{\lambda-1}{N}Y + \frac{C}{Y^{N-1}},$$

$$\frac{XY'^2}{KY} - \frac{XY''+Y'}{K} = -\frac{YY'}{K}(1-\frac{NC}{Y^{N+1}}),$$

$$\frac{XY''^2}{Y'} - XY''' - Y'' = -(2+\frac{N(N-1)C}{Y^{N+1}})Y'^2.$$

It is easy to show that

$$dz(\overline{dZ}dZ' \cdot\times I + I \cdot\times \overline{dZ'}dZ)\overline{dz'} = 2\mathrm{tr}(dZ\overline{dZ'}dZ\overline{dZ'}).$$

In fact, let

$$du = dz(\overline{dZ}dZ' \cdot\times I)$$

and rewrite du as an $m \times n$ matrix dU, then $dU = dZ\overline{dZ'}dZ$. So

$$dz(\overline{dZ}dZ' \cdot\times I)\overline{dz'} = du\overline{dz'} = \mathrm{tr}(dU\overline{dZ'}) = \mathrm{tr}(dZ\overline{dZ'}dZ\overline{dZ'}).$$

Similarly

$$dz(I \cdot\times \overline{dZ'}dZ)\overline{dz'} = \mathrm{tr}(dZ\overline{dZ'}dZ\overline{dZ'}).$$

Therefore we obtain

$$\begin{aligned}
&(dz, dw)[-\bar{d}dT + dTT^{-1}\overline{dT'}]\overline{(dz,dw)}' \\
&= dzR_{11}\overline{dz'} + dwR_{21}\overline{dz'} + dzR_{12}\overline{dw'} + dwR_{22}\overline{dw'} \\
&= -\frac{2Y^2}{K^2}(1-\frac{\lambda-1}{NY}+\frac{C}{Y^{N+1}})|dz|^4 - \frac{2Y}{K}\mathrm{tr}(dZ\overline{dZ'}dZ\overline{dZ'}) \\
&- 4\frac{YY'}{K}(1-\frac{NC}{Y^{N+1}})|dz|^2|dw|^2 - (2+\frac{N(N-1)C}{Y^{N+1}})Y'^2|dw|^4,
\end{aligned}$$

we have

$$\begin{aligned}
(11) \qquad & \omega[(z,w), d(z,w)]|_{z=0} \\
= \quad & -2\Big(\frac{\frac{Y^2}{K^2}(1-\frac{\lambda-1}{NY}+\frac{C}{Y^{N+1}})|dz|^4}{(\frac{Y}{K}|dz|^2+Y'|dw|^2)^2} + \frac{\frac{Y}{K}\mathrm{tr}(dZ\overline{dZ'}dZ\overline{dZ'})}{(\frac{Y}{K}|dz|^2+Y'|dw|^2)^2} \\
& + \frac{2\frac{YY'}{K}(1-\frac{NC}{Y^{N+1}})|dz|^2|dw|^2}{(\frac{Y}{K}|dz|^2+Y'|dw|^2)^2} + \frac{(1+\frac{N(N-1)C}{2Y^{N+1}})Y'^2|dw|^4}{(\frac{Y}{K}|dz|^2+Y'|dw|^2)^2}\Big),
\end{aligned}$$

where $C = \frac{(\lambda-N-1)\lambda^N}{N(N+1)^{N+1}}$.

We estimate the $\omega[(z,w), d(z,w)]|_{z=0}$ in three cases respectively.

(i) The case of $K \geq \frac{mn+1}{m+n}$. Then $\lambda \geq N+1$ and $C \geq 0$.

Let $\varepsilon = \frac{2N}{\lambda(N-1)}$.

If $N > 2$, that is $mn > 1(\ (m,n) \neq (1,1))$. We have

$$0 < \frac{2N}{\lambda(N-1)} \leq \frac{2N}{N^2-1} < 1.$$

That is $0 < \varepsilon < 1$. From formula (11), we have

$$\begin{aligned}
-\frac{1}{2}\omega = \varepsilon + &\frac{1}{(\frac{Y}{K}|dz|^2 + Y'|dw|^2)^2} \\
&\times \left[\frac{Y^2}{K^2}\left(1 - \varepsilon - \frac{\lambda - 1}{NY} - \frac{(N+1)C}{(N-1)Y^{N+1}}\right)|dz|^4\right. \\
&+ \frac{Y}{K}\operatorname{tr}(dZ\overline{dZ'}dZ\overline{dZ'}) \\
&+ \frac{N(N-1)C}{2Y^{N+1}}\left(\frac{2Y}{K(N-1)}|dz|^2 - Y'|dw|^2\right)^2 \\
&+ \left.(1-\varepsilon)\frac{2YY'}{K}|dz|^2|dw|^2 + (1-\varepsilon)Y'^2|dw|^4\right].
\end{aligned}$$

Because $Y(X) \geq \frac{\lambda}{N+1}$, so

$$\begin{aligned}
-\frac{1}{2}\omega \geq \varepsilon + &\frac{1}{(\frac{Y}{K}|dz|^2 + Y'|dw|^2)^2} \\
&\times \left[\frac{Y^2}{K^2}\left(1 - \varepsilon - \frac{\lambda-1}{NY} - \frac{(N+1)C}{(N-1)Y}(\frac{N+1}{\lambda})^N\right)|dz|^4\right. \\
&+ \left.\frac{Y}{K}\operatorname{tr}(dZ\overline{dZ'}dZ\overline{dZ'})\right] \\
\geq \varepsilon + &\frac{\frac{Y}{K^2}\left((1-\varepsilon)\frac{\lambda}{N+1} - \frac{\lambda-2}{N-1}\right)|dz|^4 + \frac{Y}{K}\operatorname{tr}(dZ\overline{dZ'}dZ\overline{dZ'})}{(\frac{Y}{K}|dz|^2 + Y'|dw|^2)^2} \\
= \varepsilon + &\frac{Y}{K^2}\frac{K\operatorname{tr}(dZ\overline{dZ'}dZ\overline{dZ'}) - \frac{2(\lambda-1)}{N^2-1}|dz|^4}{(\frac{Y}{K}|dz|^2 + Y'|dw|^2)^2}.
\end{aligned}$$

Because $|dz|^2 = \operatorname{tr}(dZ\overline{dZ'})$, according to Lemma 4, we have

$$\begin{aligned}
-\frac{1}{2}\omega &\geq \varepsilon + \frac{Y}{K^2}\frac{(K - \frac{2(\lambda-1)}{N^2-1}(m^2-m+1))\operatorname{tr}(dZ\overline{dZ'}dZ\overline{dZ'})}{(\frac{Y}{K}|dz|^2 + Y'|dw|^2)^2} \\
&= \varepsilon + \frac{Y}{K}\frac{(1 - \frac{2(m+n)(m^2-m+1)}{N^2-1})\operatorname{tr}(dZ\overline{dZ'}dZ\overline{dZ'})}{(\frac{Y}{K}|dz|^2 + Y'|dw|^2)^2}.
\end{aligned}$$

Next, we proof the inequality:

$$\frac{2(m+n)(m^2-m+1)}{N^2-1} \leq 1. \tag{12}$$

It is equivalent to $2(m+n)(m^2-m+1) \leq mn(mn+2)$. When $m = 1, n > m$, inequality (12) becomes $2(n+1) \leq n^2+2n$, so inequality (12) is true. When $m = 2$, inequality (12) becomes $(n-2)(2n+3) \geq 0$, so inequality (12) is true. When $m = 3$, inequality (12) becomes $(n-3)(9n+14)+5n \geq 0$, inequality (12) is true. If $m \geq 4$, then $2(m+n)(m^2-m+1) \leq 2m^2(m+n) \leq 2mn2n \leq mn(mn+2)$, so inequality (12) is also true. Therefore $\omega \leq -2\varepsilon = -\frac{4N}{\lambda(N-1)} < 0$. When $m = n = 1$, we have $N = 2, \lambda = 2K+1, Y \geq \frac{2K+1}{3}, \frac{C}{Y^{N+1}} < \frac{1}{2}$. So

$$\begin{aligned}
-\frac{1}{2}\omega &= 1 + \frac{C}{Y^3}\frac{\frac{Y^2}{K^2}|dz|^4 - \frac{4YY'}{K}|dz|^2|dw|^2 + Y'^2|dw|^4}{(\frac{Y}{K}|dz|^2 + Y'|dw|^2)^2} \\
&\geq 1 + \frac{C}{Y^3}\frac{-\frac{2YY'}{K}|dz|^2|dw|^2}{(\frac{Y}{K}|dz|^2 + Y'|dw|^2)^2} \\
&\geq \frac{\frac{Y^2}{K^2}|dz|^4 + \frac{YY'}{K}|dz|^2|dw|^2 + Y'^2|dw|^4}{(\frac{Y}{K}|dz|^2 + Y'|dw|^2)^2} \geq \frac{3}{4}.
\end{aligned}$$

Therefore $\omega \leq -\frac{3}{2} < 0$.
(ii) The case of $\frac{mn-1}{m+n} < K < \frac{mn+1}{m+n}$. Then $N-1 < \lambda < N+1, C < 0$,

$$\begin{aligned}
1 + \frac{N(N-1)C}{2Y^{N+1}} &> 1 + \frac{N(N-1)C}{2}\left(\frac{N+1}{\lambda}\right)^{N+1} \\
&= \frac{(N+1)(\lambda-N+1)}{2\lambda} > 0.
\end{aligned}$$

So

$$\begin{aligned}
\omega < -2\Bigg[&\frac{\frac{Y^2}{K^2}(1 - \frac{\lambda-1}{NY} + \frac{C}{Y^{N+1}})|dz|^4 + \frac{Y}{K}\mathrm{tr}(dZ\overline{dZ'}dZ\overline{dZ'})}{(\frac{Y}{K}|dz|^2 + Y'|dw|^2)^2} \\
&+ \frac{(1 + \frac{N(N-1)C}{2Y^{N+1}})Y'^2|dw|^4}{(\frac{Y}{K}|dz|^2 + Y'|dw|^2)^2}\Bigg].
\end{aligned}$$

According to Lemma 4 and $Y \geq \frac{\lambda}{N+1}$ we have

$$\omega < -2\left[\frac{\frac{Y^2}{K^2}(1-\frac{\lambda-1}{NY}+\frac{C}{Y^{N+1}}+\frac{K}{Y(m^2-m+1)})|dz|^4}{(\frac{Y}{K}|dz|^2+Y'|dw|^2)^2} + \frac{(1+\frac{N(N-1)C}{2Y^{N+1}})Y'^2|dw|^4}{(\frac{Y}{K}|dz|^2+Y'|dw|^2)^2}\right].$$

It is easy to show that $1-\frac{\lambda-1}{NY}+\frac{C}{Y^{N+1}}+\frac{K}{Y(m^2-m+1)}$ is an increasing function for $Y > \frac{\lambda}{N+1}$. So

$$1-\frac{\lambda-1}{NY}+\frac{C}{Y^{N+1}}+\frac{K}{Y(m^2-m+1)} > \frac{K(N+1)}{\lambda(m^2-m+1)}.$$

So

$$\omega < -2\frac{\frac{Y^2}{K^2}(\frac{K(N+1)}{\lambda(m^2-m+1)})|dz|^4+(1+\frac{N(N-1)C}{2Y^{N+1}})Y'^2|dw|^4}{(\frac{Y}{K}|dz|^2+Y'|dw|^2)^2}.$$

Therefore $\omega < -\min(\frac{K(N+1)}{\lambda(m^2-m+1)}, \frac{(N+1)(\lambda-N+1)}{2\lambda})$.

(iii) The case of $K \leq \frac{mn-1}{m+n}$. Then $\lambda \leq N-1, C<0$, from formula (11), we have

$$-\frac{1}{2}\omega-1 = \left(\frac{\frac{Y^2}{K^2}(-\frac{\lambda-1}{NY}+\frac{C}{Y^{N+1}})|dz|^4}{(\frac{Y}{K}|dz|^2+Y'|dw|^2)^2} + \frac{\frac{Y}{K}\mathrm{tr}(dZ\overline{dZ'}dZ\overline{dZ'})}{(\frac{Y}{K}|dz|^2+Y'|dw|^2)^2} - \frac{2\frac{YY'}{K}(\frac{NC}{Y^{N+1}})|dz|^2|dw|^2}{(\frac{Y}{K}|dz|^2+Y'|dw|^2)^2} + \frac{(\frac{N(N-1)C}{2Y^{N+1}})Y'^2|dw|^4}{(\frac{Y}{K}|dz|^2+Y'|dw|^2)^2}\right),$$

according to Lemma 4,

$$\frac{Y}{K}\mathrm{tr}(dZ\overline{dZ'}dZ\overline{dZ'}) \geq \frac{Y}{K}\frac{1}{m^2-m+1}|dz|^4,$$

$$-\frac{1}{2}\omega-1 \geq \frac{(-\frac{\lambda-1}{NY}+\frac{C}{Y^{N+1}}+\frac{K}{Y(m^2-m+1)})\frac{Y^2}{K^2}|dz|^4}{(\frac{Y}{K}|dz|^2+Y'|dw|^2)^2} + \frac{-2\frac{YY'}{K}(\frac{NC}{Y^{N+1}})|dz|^2|dw|^2+(\frac{N(N-1)C}{2Y^{N+1}})Y'^2|dw|^4}{(\frac{Y}{K}|dz|^2+Y'|dw|^2)^2}.$$

Let

$$u = \frac{\frac{Y}{K}|dz|^2}{\frac{Y}{K}|dz|^2+Y'|dw|^2}.$$

Obviously $0 \le u \le 1$, then

$$\begin{aligned} & -\frac{1}{2}\omega - 1 \\ \ge & -\left(\frac{\lambda-1}{NY} - \frac{C}{Y^{N+1}} - \frac{K}{Y(m^2-m+1)}\right)u^2 - \frac{2NC}{Y^{N+1}}u(1-u) \\ + & \frac{N(N-1)C}{2Y^{N+1}}(1-u)^2. \end{aligned}$$

That is

$$\begin{aligned} & \frac{1}{2}\omega + 1 \\ \le & \left(\frac{\lambda-1}{NY} - \frac{C}{Y^{N+1}} - \frac{K}{Y(m^2-m+1)}\right)u^2 + \frac{2NC}{Y^{N+1}}u(1-u) \\ - & \frac{N(N-1)C}{2Y^{N+1}}(1-u)^2. \end{aligned}$$

Because $-C > 0$, and $\lambda = K(m+n)+1, N = mn+1$, so

$$\begin{aligned} & \frac{\lambda-1}{NY} - \frac{K}{Y(m^2-m+1)} = \frac{K}{Y}\left(\frac{m+n}{N} - \frac{1}{m^2-m+1}\right) \\ = & \frac{K}{Y}\frac{(m+n)(m^2-m+1)-(mn+1)}{(m^2-m+1)N} \\ \ge & \frac{K}{Y}\frac{(m+n)m-(mn+1)}{(m^2-m+1)N} \ge 0. \end{aligned}$$

Let

$$a^* = \frac{\lambda-1}{NY} - \frac{C}{Y^{N+1}} - \frac{K}{Y(m^2-m+1)},$$

$$b^* = -\frac{2NC}{Y^{N+1}},$$

$$c^* = -\frac{N(N-1)C}{2Y^{N+1}},$$

then they are positive. Then,

$$\frac{1}{2}\omega+1 \le a^*u^2 - b^*u(1-u) + c^*(1-u)^2 = (a^*+b^*+c^*)u^2 - (b^*+2c^*)u + c^*.$$

Now we consider the maximum value of $\frac{1}{2}\omega + 1$ on $u = 0$ or $u = 1$. If $u = 0$, then $Y(0) \le Y(X)$,

$$\frac{1}{2}\omega + 1 = c^* \le -\frac{N(N-1)C}{2Y(0)^{N+1}} = \frac{(N-1)(N+1-\lambda)}{2\lambda}.$$

If $u = 1$, then

$$\frac{1}{2}\omega + 1 = a^* \le \frac{\lambda - 1}{NY(0)} - \frac{C}{Y(0)^{N+1}} - \frac{K}{Y(0)(m^2 - m + 1)}$$
$$= 1 - \frac{K(N+1)}{\lambda(m^2 - m + 1)}.$$

From $\lambda \le (N-1)$, we obtain $\frac{(N-1)}{\lambda} \ge 1$. From $N+1-\lambda \ge 2$, we obtain $\frac{N+1-\lambda}{2} \ge 1$. So

$$c^* = -\frac{N(N-1)C}{2Y^{N+1}} \ge 1.$$

Because $a^*(0) < 1$, so the maximum value of $\frac{1}{2}\omega + 1$ is $c^*(0)$ on $u = 0$. Therefore, in the case of (iii), we have

$$\omega \le -2 + \frac{(N-1)(N+1-\lambda)}{\lambda},$$

the equal is true at the point $(z, 0)$ in the direction $(0, dw)$. The $-2 + \frac{(N-1)(N+1-\lambda)}{\lambda}$ is nonnegative and in some cases it can be positive.

Finally we get the conclusions as follows. In the case of $K \le \frac{mn-1}{m+n}$, the holomorphic section curvature under the complete Einstein-Kähler metric of Y_I can get the positive value. If $K > \frac{mn-1}{m+n}$, then the holomorphic section curvature is bounded from above by a negative constant, due to [2], we obtain the following comparison theorem.

COMPARISON THEOREM. *Let*

$$Y_I := \{W \in \mathbf{C}, Z \in R_I(m,n) : |W|^{2K} < \det(I - ZZ^T), K > 0\}.$$

If $E_{Y_I}(z, w; y)$ and $R_{Y_I}(z, w; y)$ denote the complete Einstein-Kähler metric and Kobayashi metric of Y_I respectively. When $K > \frac{mn-1}{m+n}$, then there exists a positive constant c such that

$$E_{Y_I}(z, w; y) \le cR_{Y_I}(z, w; y)$$

for all $(z, w) \in Y_I, y \in C^N$.

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