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NOTE ON SRIVASTAVA'S TRIPLE HYPERGEOMETRIC SERIES H_A AND H_C

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ABSTRACT. The aim of this note is to consider some interesting reducible cases of H_A and H_C introduced by Srivastava who actually noticed the existence of three additional complete triple hypergeometric functions H_A, H_B, and H_C of the second order in the course of an extensive investigation of Lauricella's fourteen hypergeometric functions of three variables.

1. Introduction and results required

Lauricella [2], in 1893, generalized Appell's four functions to functions of n variables. For $n = 3$, he conjectured the existence of ten hypergeometric functions of three variables in addition to F_A, F_B and F_D defined by himself. These ten functions, namely, F_E, F_F, F_G, F_K, F_M, F_N, F_P, F_R, F_S, and F_T were defined, and their properties were studied by Saran [3]. In the course of Saran's function, Srivastava came across three additional new hypergeometric functions H_A, H_B, and H_C of three variables which are given in Srivastava and Manocha [6, pp. 68-69, Eqs. (36)-(38)]. The H_A is recalled here:

(1.1)

$$H_A\left(\alpha,\beta,\beta';\gamma,\gamma';x,y,z\right)=\sum_{m,\,n,\,p=0}^{\infty}\frac{(\alpha)_{m+p}(\beta)_{m+n}(\beta')_{n+p}}{(\gamma)_m(\gamma')_{n+p}\;m!\;n!\;p!}x^m y^n z^p,$$

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whose region of convergence is $|x| < r$, $|y| < s$, $|z| < t$, and $r + s + t = 1 + st$, and $(\alpha)_n$ denotes the Pochhammer symbol defined by

$$(\alpha)_n = \begin{cases} 1, & \text{if } n = 0, \\ \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1), & \text{if } n \in \mathbb{N} = \{1, 2, 3, \cdots\} \end{cases}$$
$$= \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)},$$

where $\Gamma(z)$ is the well-known Gamma function.

The following well-known formulas are recalled for our present investigation (see [1]).

Gauss theorem:

$$(1.2) \qquad {}_2F_1\left[\begin{matrix} a, b\,; \\ c\,; \end{matrix}\; 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

provided $\Re e(c - a - b) > 0$.

Kummer's theorem:

$$(1.3) \qquad {}_2F_1\left[\begin{matrix} a, b\,; \\ 1+a-b\,; \end{matrix}\; -1\right] = \frac{\Gamma(1+a-b)\Gamma(1+\frac{1}{2}a)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)}.$$

Gauss's second summation theorem:

$$(1.4) \qquad {}_2F_1\left[\begin{matrix} a, b\,; \\ \frac{1}{2}(a+b+1)\,; \end{matrix}\; \frac{1}{2}\right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})}.$$

Bailey's formula:

$$(1.5) \qquad {}_2F_1\left[\begin{matrix} a, 1-a\,; \\ c\,; \end{matrix}\; \frac{1}{2}\right] = \frac{\Gamma(\frac{1}{2}c)\Gamma(\frac{1}{2}c+\frac{1}{2})}{\Gamma(\frac{1}{2}c+\frac{1}{2}a)\Gamma(\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2})}.$$

Appell's function F_1:

(1.6)

$$F_1(a, b, b'; 1+a+b-b'; 1, -1) = \frac{\Gamma(1-b')\Gamma(1+\frac{1}{2}a)\Gamma(1+a+b-b')}{\Gamma(1+a)\Gamma(1+b-b')\Gamma(1+\frac{1}{2}a-b')},$$

where F_1 is defined as in [6, p. 53, Eq.(4)].

The object of this note is to derive some interesting reducible cases of H_A and H_C.

2. Main results and their derivations

We first show that

$$\begin{aligned} &H_A\left(\alpha,\beta,\beta';\gamma,1+\beta+\beta'-\alpha;x,1,-1\right) \\ &= \frac{\Gamma(1-\alpha)\Gamma(1+\frac{1}{2}\beta')\Gamma(1+\beta+\beta'-\alpha)}{\Gamma(1+\beta')\Gamma(1+\beta-\alpha)\Gamma(1+\frac{1}{2}\beta'-\alpha)}\,{}_2F_1\left[\begin{matrix}\beta,\alpha-\frac{1}{2}\beta' \\ \gamma\end{matrix};x\right]. \end{aligned} \tag{2.1}$$

Indeed, by noting $(\lambda)_{m+\ell}=(\lambda)_m(\lambda+m)_\ell$ and starting with (1.1), we have

$$\begin{aligned} &H_A\left(\alpha,\beta,\beta';\gamma,\gamma';x,y,z\right) \\ &= \sum_{m,\,n,\,p=0}^{\infty} \frac{(\alpha)_m(\alpha+m)_p(\beta)_m(\beta+m)_n(\beta')_{n+p}}{(\gamma)_m(\gamma')_{n+p}\;m!\;n!\;p!}x^m y^n z^p \\ &= \sum_{m=0}^{\infty} \frac{(\alpha)_m(\beta)_m}{(\gamma)_m\,m!}\left(\sum_{n,\,p=0}^{\infty}\frac{(\beta')_{n+p}(\alpha+m)_p(\beta+m)_n}{(\gamma')_{n+p}\;n!\;p!}y^n z^p\right)x^m \\ &= \sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\gamma)_m\;m!}F_1\left(\beta',\beta+m,\alpha+m;\gamma';y,z\right)x^m. \end{aligned} \tag{2.2}$$

Setting $y=1$, $z=-1$, and $\gamma'=1+\beta+\beta'-\alpha$ in (2.2) and using (1.6), we obtain

$$\begin{aligned} &H_A\left(\alpha,\beta,\beta';\gamma,1+\beta+\beta'-\alpha;x,1,-1\right) \\ &= \frac{\Gamma(1+\frac{1}{2}\beta')\Gamma(1+\beta+\beta'-\alpha)}{\Gamma(1+\beta')\Gamma(1+\beta-\alpha)}\sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta)_m\Gamma(1-\alpha-m)}{(\gamma)_m\;m!\;\Gamma(1+\frac{1}{2}\beta'-\alpha-m)}x^m, \end{aligned}$$

which, upon using the following identity:

$$\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}=\frac{(-1)^n}{(1-\alpha)_n}\quad(\alpha\neq 0,\pm1,\,\pm2,\dots),$$

immediately yields

$$\begin{aligned} &= \frac{\Gamma(1+\frac{1}{2}\beta')\Gamma(1+\beta+\beta'-\alpha)\Gamma(1-\alpha)}{\Gamma(1+\beta')\Gamma(1+\beta-\alpha)\Gamma(1+\frac{1}{2}\beta'-\alpha)}\sum_{m=0}^{\infty}\frac{(\beta)_m(\alpha-\frac{1}{2}\beta')_m}{(\gamma)_m\;m!}x^m \\ &= \frac{\Gamma(1-\alpha)\Gamma(1+\frac{1}{2}\beta')\Gamma(1+\beta+\beta'-\alpha)}{\Gamma(1+\beta')\Gamma(1+\beta-\alpha)\Gamma(1+\frac{1}{2}\beta'-\alpha)}\,{}_2F_1\left[\begin{matrix}\beta,\alpha-\frac{1}{2}\beta' \\ \gamma\end{matrix};x\right]. \end{aligned}$$

This completes the proof of (2.1).

Furthermore, we will consider some interesting special cases of (2.1). In (2.1), setting $x = 1$ with (1.2), $x = -1$ and $\gamma = 1 + \beta - \alpha + \frac{1}{2}\beta'$ with (1.3), $x = \frac{1}{2}$ and $\gamma = \frac{1}{2}(1 + \alpha - \beta - \frac{1}{2}\beta')$ with (1.4), and $x = \frac{1}{2}$ and $\beta' = 2\alpha + 2\beta - 2$ with (1.5), after some simplification, we, respectively, obtain the following interesting special cases of (2.1):

(2.3)
$$H_A\,(\alpha, \beta, \beta'; \gamma, 1 + \beta + \beta' - \alpha; 1, 1, -1) = \frac{\Gamma(1-\alpha)\Gamma(1+\frac{1}{2}\beta')\Gamma(1+\beta+\beta'-\alpha)\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta+\frac{1}{2}\beta')}{\Gamma(1+\beta')\Gamma(1+\beta-\alpha)\Gamma(1+\frac{1}{2}\beta'-\alpha)\Gamma(\gamma-\beta)\Gamma(\gamma-\alpha+\frac{1}{2}\beta')};$$

(2.4)
$$H_A\left(\alpha, \beta, \beta'; 1 + \beta - \alpha + \frac{1}{2}\beta', 1 + \beta + \beta' - \alpha; -1, 1, -1\right) = \frac{\Gamma(1-\alpha)\Gamma(1+\frac{1}{2}\beta')\Gamma(1+\beta+\beta'-\alpha)\Gamma(1+\frac{1}{2}\beta)\Gamma(1-\alpha+\beta+\frac{1}{2}\beta')}{\Gamma(1+\beta')\Gamma(1+\beta-\alpha)\Gamma(1+\frac{1}{2}\beta'-\alpha)\Gamma(1+\beta)\Gamma(1+\frac{1}{2}\beta-\alpha+\frac{1}{2}\beta')};$$

(2.5)
$$H_A\left(\alpha, \beta, \beta'; \frac{1}{2}(1 + \alpha + \beta - \frac{1}{2}\beta'), 1 + \beta + \beta' - \alpha; \frac{1}{2}, 1, -1\right) = \frac{\Gamma(1-\alpha)\Gamma(1+\frac{1}{2}\beta')\Gamma(1+\beta+\beta'-\alpha)\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\frac{1}{2}\alpha+\frac{1}{2}\beta-\frac{1}{4}\beta')}{\Gamma(1+\beta')\Gamma(1+\beta-\alpha)\Gamma(1+\frac{1}{2}\beta'-\alpha)\Gamma(\frac{1}{2}\beta+\frac{1}{2})\Gamma(\frac{1}{2}+\frac{1}{2}\alpha-\frac{1}{4}\beta')};$$

(2.6)
$$H_A\left(\alpha, \beta, 2\alpha + 2\beta - 2; \gamma, 3\beta + \alpha - 1; \frac{1}{2}, 1, -1\right) = \frac{\Gamma(1-\alpha)\Gamma(\alpha+\beta)\Gamma(3\beta+\alpha-1)\Gamma(\frac{1}{2}\gamma)\Gamma(\frac{1}{2}\gamma+\frac{1}{2})}{\Gamma(\beta)\Gamma(1+\beta-\alpha)\Gamma(2\alpha+2\beta-1)\Gamma(\frac{1}{2}\gamma+\frac{1}{2}\beta)\Gamma(\frac{1}{2}\gamma-\frac{1}{2}\beta+\frac{1}{2})}.$$

By using the known identity (2.10) and (2.2), we find that H_A is reducible to a generalized hypergeometric function:

(2.7)
$$H_A\,(\alpha, \beta, \beta'; x, 1, 1) = \frac{\Gamma(\gamma')\,\Gamma(\gamma'-\alpha-\beta-\beta')}{\Gamma(\gamma'-\beta')\,\Gamma(\gamma'-\alpha-\beta)} \times {}_4F_3\left[\begin{matrix}\alpha, \beta, 1+\frac{1}{2}\alpha+\frac{1}{2}\beta-\frac{1}{2}\gamma', \frac{1}{2}+\frac{1}{2}\alpha+\frac{1}{2}\beta-\frac{1}{2}\gamma' \\ \gamma, 1+\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2}\beta'-\frac{1}{2}\gamma', \frac{1}{2}+\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2}\beta'-\frac{1}{2}\gamma'\end{matrix}; x\right].$$

Similarly, considering Srivastava's H_C function (see [6, p. 69, Eq. (38)]) defined by

(2.8)
$$H_C\,(\alpha,\beta,\beta';\gamma;x,y,z) = \sum_{m,\,n,\,p=0}^{\infty} \frac{(\alpha)_{m+p}(\beta)_{m+n}(\beta')_{n+p}}{(\gamma)_{m+n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}$$
$$(|x|<1,\ \ |y|<1,\ \ |z|<1),$$

we obtain

$$\begin{aligned} &H_C\,(\alpha,\beta,\beta';\gamma;x,y,z) \\ &= \sum_{m=0}^{\infty} \frac{(\alpha)_m(\beta)_m}{(\gamma)_m} F_1[\beta',\alpha+m,\beta+m;\gamma+m;z,y]\,\frac{x^m}{m!}, \end{aligned} \tag{2.9}$$

which, upon using the known identity [6, p. 55, Eq.(15)]:

$$\begin{aligned} F_1[a,b,b';c;1,1] &= \frac{\Gamma(c)\,\Gamma(c-a-b-b')}{\Gamma(c-a)\,\Gamma(c-b-b')} \\ &(\Re(c-a-b-b')>0;\ \ c\neq 0,-1,-2,\ldots), \end{aligned} \tag{2.10}$$

yields

(2.11)
$$\begin{aligned} &H_C\,(\alpha,\beta,\beta';\gamma;x,1,1) \\ &= \frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta-\beta')}{\Gamma(\gamma-\beta')\,\Gamma(\gamma-\alpha-\beta)}\,{}_3F_2\left[\begin{matrix}\alpha,\beta,1+\alpha+\beta-\gamma \\ \gamma-\beta',\,1+\alpha+\beta+\beta'-\gamma\end{matrix};x\right]. \end{aligned}$$

If we take, in (2.11), $x=1$, $\beta=\beta'$, and $\gamma=3\beta$ with Watson's theorem [4, p. 245], and $x=1$, $1+\alpha=4\beta-\beta'$, and $\gamma=3\beta$ with Dixon' theorem [4, p. 243], we, respectively, obtain

(2.12)
$$\begin{aligned} &H_C\,(\alpha,\beta,\beta;3\beta;1,1,1) \\ &= \frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}+\beta\right)\,\Gamma(3\beta)\,\Gamma(\beta-\alpha)\Gamma(1+\alpha-\beta)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\alpha\right)\,\Gamma(2\beta)\,\Gamma\left(2\beta-\frac{\alpha}{2}\right)\,\Gamma\left(1+\frac{1}{2}\alpha-\beta\right)\,\Gamma\left(\frac{1}{2}-\frac{1}{2}\alpha+\beta\right)} \end{aligned}$$

and

$$\begin{aligned} &H_C\,(\alpha,\beta,4\beta-\alpha-1;3\beta;1,1,1) \\ &= \frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}+\beta\right)\,\Gamma(3\beta)\,\Gamma(1-2\beta)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\alpha\right)\Gamma\left(1+\frac{1}{2}\alpha-\beta\right)\,\Gamma\left(2\beta-\frac{1}{2}\alpha\right)\,\Gamma\left(\frac{1}{2}-\frac{1}{2}\alpha+\beta\right)}. \end{aligned} \tag{2.13}$$

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