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AGE-TIME DISCONTINUOUS GALERKIN METHOD FOR THE LOTKA-MCKENDRICK EQUATION

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ABSTRACT. The Lotka-McKendrick equation which describes the evolution of a single population under the phenomenological conditions is developed from the well-known Malthus' model. In this paper, we introduce the Lotka-McKendrick equation for the description of the dynamics of a population. We apply a discontinuous Galerkin finite element method in age-time domain to approximate the solution of the system. We provide some numerical results. It is experimentally shown that, when the mortality function is bounded, the scheme converges at the rate of h^2 in the case of piecewise linear polynomial space. It is also shown that the scheme converges at the rate of $h^{3/2}$ when the mortality function is unbounded.

1. The Lotka-McKendrick equation

The Lotka-McKendrick model is a strict analogue of the Malthus' model. We consider a single population living isolated in an invariant habitat. We assume that all of its individuals are perfectly equal to their properties but not for their age. In particular, we assume that there are no sex differences. According to this setting, fertility and mortality are intrinsic parameters of the population growth and do not depend on time, nor on the population size. They are functions of age only. The evolution of the population is then described by its age density function at time t:

$$u(a,t), \qquad a \in [0, a_\dagger], \quad t \geq 0.$$

Here $a_\dagger$ is the maximum age which may reach an individual of the population and we assume $a_\dagger < +\infty$. Thus the integral $\int_{a_1}^{a_2} u(a,t)da$ gives

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the number of individuals at time t with age in the interval $[a_1, a_2]$. Age specific fertility $\beta(a)$ can be defined as the number of newborns in one time unit coming from a single individual whose age is a. Thus $\int_{a_1}^{a_2} \beta(a)u(a,t)da$ gives the number of newborns in one time unit coming from individuals with age in $[a_1, a_2]$. The age specific mortality $\mu(a)$ is the death rate of people having age a. The Lotka-McKendrick system is thus given by the following system:

$$
\begin{aligned}
&u_t(a,t) + u_a(a,t) + \mu(a)u(a,t) = 0, \quad 0 < a < a_\dagger, \quad t > 0,\\
&u(0,t) = \int_0^{a_\dagger} \beta(a)u(a,t)da, \quad t > 0, \qquad (1.1)\\
&u(a,0) = u_0(a), \quad 0 \le a < a_\dagger.
\end{aligned}
$$

Here $u_0(a)$ is the initial age distribution. In order for the system to be biologically meaningful and/or also for the mathematical treatment, we assume that the basic functions $\beta(\cdot)$ and $\mu(\cdot)$ satisfy the following assumptions:

$$
\begin{aligned}
&\beta(\cdot) \text{ is nonnegative and belongs to } \mathrm{L}^\infty(0, a_\dagger),\\
&\mu(\cdot) \text{ is nonnegative and belongs to } \mathrm{L}^1_{\mathrm{loc}}(0, a_\dagger),\\
&\int_0^{a_\dagger} \mu(\sigma)d\sigma = +\infty, \qquad (1.2)\\
&u_0 \in \mathrm{L}^1(0, a_\dagger), \ u_0(a) \ge 0, \ a \in [0, a_\dagger].
\end{aligned}
$$

Condition (1.2) is necessary for the survival probability $\Pi(a)$ to vanish at the maximum age $a_\dagger < +\infty$. Here $\Pi(a)$ is given by

$$\Pi(a) = e^{-\int_0^a \mu(\sigma)d\sigma}, \quad a \in [0, a_\dagger]$$

which means the probability for an individual to survive to age a. Thus the mortality function μ typically blows up near the maximum age $a_\dagger$. In the next section, we consider a discontinuous Galerkin finite element method to approximate the solution of the system (1.1). Here we note that, since the mortality function μ is unbounded, most numerical methods would not guarantee the optimal convergence rates and some methods would not even converge [6].

2. Discontinuous Galerkin finite element method

In this section we introduce in detail the discontinuous Galerkin finite element method to approximate the solution of a linear hyperbolic problem. We then apply the discontinuous Galerkin finite element method to the Lotka-McKendrick system (1.1). Problems with mainly hyperbolic character such as convection-diffusion problems with small or vanishing diffusion typically occur in fluid mechanics, gas dynamics, or wave propagation. In contrast to the case of elliptic and parabolic problems, standard applications of the finite element method to hyperbolic problems frequently do not give reasonable results. More precisely, it was observed that standard finite element method for hyperbolic problem does not work well in cases where the exact solution is not smooth. If the exact solution has a jump discontinuity, the finite element solution will in general exhibit large spurious oscillations even from the jump and will then not be close to the exact solution anywhere. This is of particular concern since in many interesting hyperbolic equations, the exact solution is not smooth. Only recently it has been possible to overcome these difficulties and construct modified non-standard finite element methods for hyperbolic problems with satisfactory convergence properties.

In this section, we first introduce the discontinuous Galerkin finite element method to a linear hyperbolic equation. Let Ω be a bounded convex polygonal domain in $\mathbb{R}^2$ with boundary Γ and let $\gamma = (\gamma_1, \gamma_2)$ be a constant vector with $|\gamma| := \sqrt{\gamma_1^2 + \gamma_2^2} = 1$. We shall consider the following boundary value problem:

$$\begin{aligned} u_\gamma + u &= f \quad \text{in} \quad \Omega, \\ u &= g \quad \text{on} \quad \Gamma_-, \end{aligned} \tag{2.1}$$

where Γ_- is the inflow boundary defined by

$$\Gamma_- = \{x \in \Gamma : n(x) \cdot \gamma < 0\}.$$

Here $n(x)$ denotes the outward unit normal to Γ at the point $x \in \Gamma$, and $v_\gamma = \gamma \cdot \nabla v$ is the derivative in the γ−direction. To introduce the discontinuous Galerkin finite element method let us first introduce some notations. Let $\mathcal{T}_h$ be an admissible triangulation of Ω with mesh size h which satisfies as usual the minimum angle condition. For $K \in \mathcal{T}_h$ we split the boundary ∂K of the triangle K into an inflow part ∂K_- and an outflow part ∂K_+ defined by

$$\begin{aligned} \partial K_- &= \{x \in \partial K : n(x) \cdot \gamma < 0\}, \\ \partial K_+ &= \{x \in \partial K : n(x) \cdot \gamma \geq 0\}. \end{aligned}$$

Consider a function v which may have a jump discontinuity across interelement boundaries. We define the left and right hand limits v_- and

v_+ by

$$v_-(x) = \lim_{s\to 0^-} v(x+s\gamma),$$
$$v_+(x) = \lim_{s\to 0^+} v(x+s\gamma),$$

and we also define the jump $[v]$ across interelement boundaries by

$$[v] = v_+ - v_-.$$

We shall use the following notations: $(v,w) = \int_\Omega vw\,dx$, $\langle v,w\rangle = \int_\Gamma vwn\cdot\gamma\,ds$,

$$\langle v,w\rangle_- = \int_{\Gamma_-} vwn\cdot\gamma\,ds, \quad \langle v,w\rangle_+ = \int_{\Gamma_+} vwn\cdot\gamma\,ds,$$

where $\Gamma_+ = \Gamma\backslash\Gamma_- = \{x\in\Gamma : n(x)\cdot\gamma \geq 0\}$, dx denotes the element of area in $\mathbb{R}^2$ and ds the element of arc length along the boundary.

In order to obtain a variational formulation of problem (2.1), we multiply the differential equation of (2.1) with an arbitrary test function $v\in H^1(\Omega)$ and then integrate it over Ω. According to the Green's formula and the boundary condition, we thus have the following variational formulation: Find $u\in L^2(\Omega)$ such that

$$-(u,v_\gamma) + (u,v) + \langle u,v\rangle_+ = (f,v) - \langle g,v\rangle_-, \quad \forall v\in H^1(\Omega). \tag{2.2}$$

We shall now consider the discontinuous Galerkin finite element method for (2.2). It is based on using the following finite element space:

$$W_h = \{v\in L^2(\Omega) : v|_K \in P_r(K), \forall K\in\mathcal{T}_h\}.$$

That is, the space of piecewise polynomials of degree $r\geq 0$ with no continuity requirement across interelement boundaries. The discontinuous Galerkin finite element method is then formulated to find $u^h\in W_h$ such that

$$\begin{aligned}&\sum_{K\in\mathcal{T}_h}\int_K(-u^h v_\gamma + u^h v)\,dx + \langle u^h, v\rangle_+ \\ &= \sum_{K\in\mathcal{T}_h}\int_K fv\,dx - \langle g,v\rangle_-, \quad \forall v\in P_r(K).\end{aligned} \tag{2.3}$$

We also use Green's formula for $\int_K u^h v_\gamma\,dx$ to obtain

$$\int_K u^h v_\gamma\,dx = \int_{\partial K_-} u_+^h v_+ n\cdot\gamma\,ds + \int_{\partial K_+} u_-^h v_+ n\cdot\gamma\,ds - \int_K u_\gamma^h v\,dx. \tag{2.4}$$

Therefore, we have

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}_h}\int_K u^h v_\gamma\,dx\\
&=\sum_{K\in\mathcal{T}_h}\left\{\int_{\partial K_-} u_+^h v_+ n\cdot\gamma\,ds+\int_{\partial K_+} u_-^h v_+ n\cdot\gamma\,ds-\int_K u_\gamma^h v\,dx\right\}\\
&=\sum_{\partial K_-\subset\Gamma_-}\int_{\partial K_-} u_+^h v_+ n\cdot\gamma\,ds+\sum_{\partial K_+\subset\Gamma_+}\int_{\partial K_+} u_-^h v_+ n\cdot\gamma\,ds\\
&\quad+\left\{\sum_{\partial K_-\not\subset\Gamma_-}\int_{\partial K_-} u_+^h v_+ n\cdot\gamma\,ds+\sum_{\partial K_+\not\subset\Gamma_+}\int_{\partial K_+} u_-^h v_+ n\cdot\gamma\,ds\right\}\\
&\quad-\sum_{K\in\mathcal{T}_h}\int_K u_\gamma^h v\,dx.
\end{aligned}
\tag{2.5}
$$

Since $\mathcal{T}_h$ is an admissible triangulation and u^h is not required to be continuous across interelement boundaries, we see that

$$
\begin{aligned}
&\sum_{\partial K_-\not\subset\Gamma_-}\int_{\partial K_-} u_+^h v_+ n\cdot\gamma\,ds+\sum_{\partial K_+\not\subset\Gamma_+}\int_{\partial K_+} u_-^h v_+ n\cdot\gamma\,ds\\
&=\sum_{\partial K_-\not\subset\Gamma_-}\int_{\partial K_-}[u^h]v_+ n\cdot\gamma\,ds.
\end{aligned}
\tag{2.6}
$$

Thus, the equation (2.5) is written in the following form:

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}_h}\int_K u^h v_\gamma\,dx\\
&=-\sum_{K\in\mathcal{T}_h}\int_K u_\gamma^h v\,dx+\sum_{\partial K_-\subset\Gamma_-}\int_{\partial K_-} u_+^h v_+ n\cdot\gamma\,ds\\
&\quad+\sum_{\partial K_+\subset\Gamma_+}\int_{\partial K_+} u_-^h v_+ n\cdot\gamma\,ds+\sum_{\partial K_-\not\subset\Gamma_-}\int_{\partial K_-}[u^h]v_+ n\cdot\gamma\,ds.
\end{aligned}
\tag{2.7}
$$

Since

$$\sum_{\partial K_+\subset\Gamma_+}\int_{\partial K_+} u_-^h v_+ n\cdot\gamma\,ds=\langle u^h,v\rangle_+,$$

from (2.7), we finally obtain the following compact form of the discontinuous Galerkin finite element method for (2.3): Find $u^h\in W_h$ such that

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}_h}\int_K (u^h_\gamma + u^h)v dx - \sum_{\partial K_-\subset\Gamma_-}\int_{\partial K_-} u^h_+ v_+ n\cdot\gamma\, ds \\
&- \sum_{\partial K_-\not\subset\Gamma_-}\int_{\partial K_-} [u^h] v_+ n\cdot\gamma\, ds \qquad (2.8)\\
&= \sum_{K\in\mathcal{T}_h}\int_K fv\,dx - \int_{\Gamma_-} g v_+ n\cdot\gamma\, ds, \quad \forall v\in P_r(K).
\end{aligned}
$$

Since the function v in W_h varies independently on each K, we may alternatively formulate (2.8) independently as follows: For $K \in \mathcal{T}_h$, given u^h_- on ∂K_- find $u^h \equiv u^h|_K \in P_r(K)$ such that

$$
\begin{aligned}
&\int_K (u^h_\gamma + u^h)v dx - \int_{\partial K_-} u^h_+ v_+ n\cdot\gamma\, ds \qquad (2.9)\\
&= \int_K fv\,dx - \int_{\partial K_-} u^h_- v_+ n\cdot\gamma\, ds, \quad \forall v\in P_r(K).
\end{aligned}
$$

Here we note that if $\partial K_- \subset \Gamma_-$, then $u^h_-|_{\partial K_-} = g$ is provided by (2.1). Thus, if u^h_- is given on ∂K_-, then $u^h|_K$ is uniquely determined by (2.9). Now, we may start to determine u^h on the triangles K with $\partial K_- \subset \Gamma_-$. We then determine u^h on the triangles K next to Γ_- , and we may continue this process until u^h is determined on the whole domain.

We now apply the discontinuous Galerkin finite element method (2.9) to the Lotka-McKendrick equation (1.1). Let $T > 0$ be the final time and let $\Omega = [0, a_\dagger] \times [0, T]$. The derivative direction γ is then given by $\gamma = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. The inflow boundary Γ_- consists of $\Gamma_1 = \{(0,t) : t > 0\}$ and $\Gamma_2 = \{(a, 0) : 0 \le a < a_\dagger\}$, and $\Gamma_- = \Gamma_1 \bigcup \Gamma_2$. Thus, equation (1.1) may also be written as

$$
\begin{aligned}
&u_\gamma(a,t) + \frac{1}{\sqrt{2}}\mu(a)u(a,t) = 0, \quad \text{in} \quad \Omega, \qquad (2.10)\\
&u(a,t) = g(a,t), \quad \text{on} \quad \Gamma_-,
\end{aligned}
$$

where

$$
g(a,t) = \begin{cases} \int_0^{a_\dagger} \beta(\sigma)u(\sigma,t)d\sigma, & \text{if } (a,t)\in\Gamma_1,\\ u_0(a), & \text{if } (a,t)\in\Gamma_2. \end{cases} \qquad (2.11)
$$

Our variational problem for (2.10) is then to find $u \in L^2(\Omega)$ such that

$$
-(u, v_\gamma) + (\frac{1}{\sqrt{2}}\mu u, v) + \langle u, v\rangle_+ = -\langle g, v\rangle_-, \quad \forall v \in H^1(\Omega). \qquad (2.12)
$$

We now consider the discontinuous Galerkin finite element method using a triangulation in age-time domain for (2.12). For the convenience of numerical computation, we shall consider the piecewise linear polynomial space as an approximate space and let

$$W_h = \{v \in L^2(\Omega) : v|_K \in P_1(K), \forall K \in \mathcal{T}_h\}$$

be the space of piecewise linear polynomials with no continuity requirements across interelement boundaries. Here, we consider the uniform triangulation $\mathcal{T}_h$ according to Figure 1. Let us denote by $h = \triangle a = \triangle t$ the age-time mesh size chosen so that $M = \frac{a_\dagger}{h}$ is a positive even integer and let $N = [\frac{T}{h}]$ be a positive integer. Let $a_i = ih, \quad 0 \le i \le M$, $t^j = jh, \quad 0 \le j \le N$ be the nodal coordinates. The triangles of $\mathcal{T}_h$ are denoted by K_i^j or $\widetilde{K}_i^j$ depending on whether the inflow boundary is parallel to the a-axis, or to the t-axis. Triangles K_i^j and $\widetilde{K}_i^j$ are shown in Figure 2.

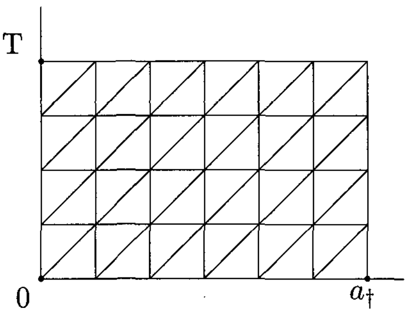

Figure 1.

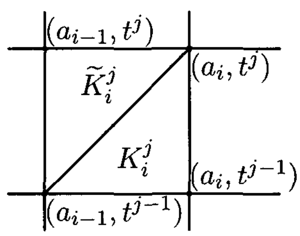

Figure 2.

The discontinuous Galerkin method then reads: For $K \in \mathcal{T}_h$, given u_-^h on ∂K_- find $u^h \equiv u^h|_K \in P_1(K)$ such that

$$\begin{aligned} &\int_K (u_\gamma^h + \frac{1}{\sqrt{2}}\mu u^h) v\, dx - \int_{\partial K_-} u_+^h v_+ n \cdot \gamma\, ds \\ &= - \int_{\partial K_-} u_-^h v_+ n \cdot \gamma\, ds, \quad \forall v \in P_1(K). \end{aligned} \tag{2.13}$$

Since $n \cdot \gamma = -\frac{1}{\sqrt{2}}$ on ∂K_-, (2.13) may also be written

$$\int_K (u_a^h + u_t^h + \mu u^h) v\, dx + \int_{\partial K_-} u_+^h v_+\, ds = \int_{\partial K_-} u_-^h v_+\, ds, \quad \forall v \in P_1(K). \tag{2.14}$$

We compute the discrete solution u^h successively on one strip after another starting for each strip on the left and moving triangle by triangle to the right. We note that a function $u^h \in P_1(K)$ has the representation

$$u^h = \sum_{i=1}^{3} \eta_i T_i, \tag{2.15}$$

where the T_i are the linear local shape functions appropriate to a three-noded triangle element. Hence, substituting (2.15) into (2.14) with $v = T_j$, $j = 1, 2, 3$, we obtain the following linear system of equations

$$A\eta = B, \tag{2.16}$$

where

$$A = (a_{ij})_{3\times 3}, \qquad a_{ij} = \int_K (\frac{\partial T_i}{\partial a} + \frac{\partial T_i}{\partial t}) T_j \, da \, dt + A_i^j(K),$$

with

$$A_k^s(K) = \int_K \mu(a) T_k(a,t) T_s(a,t) \, da \, dt,$$

and

$$B = (b_j)_{3\times 1}, \qquad b_j = \int_{\partial K_-} u_-^h T_j \, ds, \qquad \eta = (\eta_j)_{3\times 1}.$$

We are frequently concerned with integrals taken over elements such as

$$\int_K \frac{\partial T_i}{\partial a} T_j \, da \, dt, \quad \int_K \frac{\partial T_i}{\partial t} T_j \, da \, dt, \quad \int_K \mu T_i T_j \, da \, dt. \tag{2.17}$$

Those double integrals are easily computed with standard local shape function $\widehat{T}_i$ defined over the reference triangle $\widehat{K}$, with vertices (-1,-1), (1,-1), (-1,1); that is, for $(\xi, \eta) \in \widehat{K}$,

$$\widehat{T}_1 = -\frac{1}{2}(\xi + \eta), \quad \widehat{T}_2 = \frac{1}{2}(1 + \xi), \quad \widehat{T}_3 = \frac{1}{2}(1 + \eta). \tag{2.18}$$

Thus, we easily obtain, from (2.17), (2.18), that, for $K = K_i^j$,

$$A = \begin{pmatrix} \frac{h}{3} + A_1^1(K) & \frac{h}{6} + A_1^2(K) & A_1^3(K) \\ A_2^1(K) & \frac{h}{6} + A_2^2(K) & -\frac{h}{6} + A_2^3(K) \\ \frac{h}{6} + A_3^1(K) & \frac{h}{6} + A_3^2(K) & \frac{h}{6} + A_3^3(K) \end{pmatrix}.$$

Similarly we see that, for $K = \widetilde{K}_i^j$,

$$A = \begin{pmatrix} \frac{h}{3} + A_1^1(K) & A_1^2(K) & \frac{h}{6} + A_1^3(K) \\ \frac{h}{6} + A_2^1(K) & \frac{h}{6} + A_2^2(K) & \frac{h}{6} + A_2^3(K) \\ A_3^1(K) & -\frac{h}{6} + A_3^2(K) & \frac{h}{6} + A_3^3(K) \end{pmatrix}.$$

Due to the different type of inflow boundary condition (2.11), vector B is computed in different way depending on K. For $\partial K_- \subset \Gamma_2$, we have

$$B = \begin{pmatrix} \int_{\partial K_-} u_0(a) T_1 \, da \\ 0 \\ \int_{\partial K_-} u_0(a) T_3 \, da \end{pmatrix}.$$

For $\partial K_- \subset \Gamma_1$, the boundary condition involves the unknown function u_-^h. We approximately compute $u_-^h(0, t^j)$ using Simpson's rule and we let

$$s(j) = \frac{h}{3 - h\beta(0)} \Big\{ \beta(a_\dagger) u^h|_{K_M^j}(a_\dagger, t^j) + 4 \sum_{i=1}^{M/2} \beta(a_{2i-1}) u^h|_{K_{2i-1}^j}(a_{2i-1}, t^j) + 2 \sum_{i=1}^{(M-2)/2} \beta(a_{2i}) u^h|_{K_{2i}^j}(a_{2i}, t^j) \Big\}.$$

Then we have, for $\partial K_- \subset \Gamma_1$,

$$B = \begin{pmatrix} \frac{h}{6}(2s(j) + s(j-1)) \\ \frac{h}{6}(2s(j-1) + s(j)) \\ 0 \end{pmatrix}.$$

We note that, for $K = K_i^j$, $j \neq 1$, B is obtained from the previous strip and we have that

$$B = \begin{pmatrix} \frac{h}{6}(2\eta + \delta) \\ 0 \\ \frac{h}{6}(2\delta + \eta) \end{pmatrix},$$

where $\delta = u^h|_{\widetilde{K}_i^{j-1}}(a_{i-1}, t^{j-1})$ and $\eta = u^h|_{\widetilde{K}_i^{j-1}}(a_i, t^{j-1})$. Similarly, we see that, for $K = \widetilde{K}_i^j$, $i \neq 1$, B is obtained from the previous step of this strip. Thus, we have

$$B = \begin{pmatrix} \frac{h}{6}(2s + l) \\ \frac{h}{6}(2l + s) \\ 0 \end{pmatrix},$$

where $s = u^h|_{K_{i-1}^j}(a_{i-1}, t^j)$ and $l = u^h|_{K_{i-1}^j}(a_{i-1}, t^{j-1})$.

We compute the discrete solution u^h successively on one strip after another starting for each strip on the left and moving triangle by triangle to the right. More precisely, we first compute u^h on K_i^j for $i = 1, ..., M$, over the strip $[0, a_\dagger] \times [t^{j-1}, t^j]$. Second, we compute u^h on $\widetilde{K}_i^j$ for $i = 1, ..., M$, over the same strip. We then march to the next strip $[0, a_\dagger] \times [t^j, t^{j+1}]$.

3. Numerical results

In this section we present numerical results. In the test we computed the order of convergence of the algorithms by the well-known formula:

$$r_i(h) = \frac{\log \frac{E_i(h)}{E_i(\frac{h}{2})}}{\log 2}, \quad i = 1, 2,$$

where $E_1(h)$, $E_2(h)$ are the approximation errors defined by

$$E_1(h) = \max_{j \geq 1, i \geq 0} |u^h(a_i, t^j) - u(a_i, t^j)|,$$
$$E_2(h) = \|u^h(\cdot, \cdot) - u(\cdot, \cdot)\|_2,$$

respectively, for population density, where $\| \cdot \|_2$ denotes L^2 norm and the notation $u(\cdot, \cdot)$ denotes exact solution.

EXAMPLE 3.1. We solve problem (2.10) with the following data: $a_\dagger = 1$, $\beta(a) = 20a(1-a)$, $\mu(a) = 10\exp(-100(1-a))$ and $u(a,0) = u_0(a)$ where $u_0(a)$ is to be specified later.

We then find the exact solution of separable type $u(a,t) = \omega_0 \omega(a) p(t)$, where $p(t)$ is the solution of

$$p' = \alpha^* p, \ p(0) = 1,$$

so that the total population p is given as

$$p(t) = \exp(\alpha^* t)$$

and

$$\omega(a) = \exp(-\int_0^a \mu(\xi)\, d\xi - \alpha^* a).$$

Here α^* is given by the relation

$$1 = \int_0^1 \beta(a)\omega(a)\, da$$

and is computed as

$$\alpha^* \approx 2.78576939.$$

ω_0 is also given by the relation

$$1 = \int_0^1 \omega_0 \omega(a)\, da,$$

and is computed as

$$\omega_0 \approx 2.969447356.$$

We then note that the compatibility condition at (0,0) is satisfied, which guarantees the continuity of the solution $u(a,t)$. We take $u_0(a) = \omega_0\omega(a)$ as the initial data.

Convergence results are shown to the Table 1.

h	$E_1(h)$	$r_1(h)$	$E_2(h)$	$r_2(h)$
1/8	0.400262	1.938250	0.0905811	2.060904
1/16	0.102512	1.965158	0.0225618	2.005334
1/32	0.028436	1.933149	0.0057869	1.963026
1/64	0.00708297	1.922158	0.00149810	1.949663
1/128	0.00183668	1.947260	0.00038371	1.965067

TABLE 1. Convergence estimates for Example 3.1.

From Table 1, we see that the discontinuous Galerkin finite element method converges with convergent rate of order 2 when the piecewise linear polynomial space is used as an approximate space. In the next example we consider the case that the mortality function μ is unbounded.

EXAMPLE 3.2. We solve problem (2.10) with the following data: $a_\dagger = 1$, $\beta(a) = e$, $\mu(a) = \frac{1}{1-a}$, and $u_0(a) = \omega(a)$ where $\omega(a)$ is given below. The exact solution is $u(a,t) = \omega(a)p(t)$, where $p(t) = e^t$ and $\omega(a) = (1-a)e^{-a}$.

h	$E_2(h)$	$r_2(h)$
1/8	0.00233010	1.219569
1/16	0.000862253	1.434213
1/32	0.000305325	1.497770
1/64	0.000107350	1.508028
1/128	0.000037815	1.505308

TABLE 2. Convergence estimates for Example 3.2.

Table 2 shows the convergent rate is of order 3/2.

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