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# TWO-SCALE CONVERGENCE FOR PARTIAL DIFFERENTIAL EQUATIONS WITH RANDOM COEFFICIENTS

Hee Chul Pak

Abstract. We introduce the notion of two-scale convergence for partial differential equations with random coefficients that gives a very efficient way of finding homogenized differential equations with random coefficients. For an application, we find the homogenized matrices for linear second order elliptic equations with random coefficients. We suggest a natural way of finding the two-scale limit of second order equations by considering the flux term.

## 1. Introduction

It is well known that the modeling of physical processes in strongly inhomogeneous media leads to the study of differential equations with rapidly varying coefficients. Regarding coefficients as periodic functions, many attempts for getting approximate solutions have been made and some of successful ones are G-convergence by Spagnolo, H-convergence by Tartar and $\Gamma$-convergence by De Giorgi. Another common way is to use formal asymptotic expansions - we first guess by a formal expansion what the limit should be and then justify it by energy method. The two-scale convergence is an efficient way of combining these two procedures. But it is restricted to periodic cases, which was pointed out by G. Allaire in [3]. In this note, we present a new approach which makes use of the two-scale convergence technique for differential equations with random coefficients.

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For an application, we find the homogenized matrix of linear second-order elliptic equations with random coefficients:

$$\begin{aligned} -div\mathbf{A}_\varepsilon \nabla u_\varepsilon &= f \quad \text{in } \Omega \\ u_\varepsilon &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

The first rigorous trial for finding homogenized matrices for periodic coefficients was accomplished by De Giorgi and Spagnolo in 1972. Their proof was rather complicated and relied on the results of Spagnolo concerning G-convergence. Shortly afterwards direct arguments were suggested. Among them are the method of compensated compactness by Tartar, and the method of asymptotic expansions by Bachvalov, Bensoussan, Lions and Papanicolaou. By the probabilistic approach, Freidlin obtained earlier the homogenized matrix for a non-divergent equation. In this note we use the two-scale convergence technique to find the homogenized matrix by considering the flux term. It should be noted that the consideration of the two-scale limit of flux is more natural than applying the test function $\varphi(x)+\varepsilon\phi(x,\frac{x}{\varepsilon})$ on the weak-formulation. Proposition 5 deals with a theory to find the limit of flux term.

## 2. Two-scale convergence for random variables

Before we define the two-scale convergence for random variables, we briefly review the two-scale convergence for periodic functions.

A common (or, maybe, the quickest) way for homogenization for partial differential equations with periodic coefficients is to use the formal asymptotic expansion method. That is to say, in order to find the precise form of the homogenized differential equation, we postulate the following series expansion for the solution $v^\varepsilon$:

$$v^\varepsilon = V_0(x,\frac{x}{\varepsilon}) + \varepsilon V_1(x,\frac{x}{\varepsilon}) + \varepsilon^2 V_2(x,\frac{x}{\varepsilon}) + \cdots , \tag{1}$$

where each term $V_i(x,y)$ is periodic in $Y \equiv [0,1]^n$.

Once we find $V_0(x,y)$, $V_1(x,y)$, $V_2(x,y),\cdots$, we let $y=\frac{x}{\varepsilon}$. This will give us an expansion for $v^\varepsilon$:

$$v^\varepsilon = V_0(x,y) + \varepsilon V_1(x,y) + \varepsilon^2 V_2(x,y) + \cdots \big|_{y=\frac{x}{\varepsilon}} ,$$

which is a series solution of the given differential equation. Unfortunately, we may not be able to find all of $V_i(x,y)$'s, and they may not exist at all. Even if they exist, we can not guarantee the convergence of the series. In the case when we get the first term $V_0(x,y)$ or the second term $V_1(x,y)$, we can regard $V_0(x,\frac{x}{\varepsilon})$ (or $V_0(x,\frac{x}{\varepsilon})+\varepsilon V_1(x,\frac{x}{\varepsilon})$) as the solution $v^\varepsilon$ if $\varepsilon > 0$ is small enough.

We substitute the expansion (1) into the original differential equation to get the homogenized differential equations. Then in order to justify this formal procedure we find an energy estimate which is usually very complicated. That is, this technique involves with two steps: the *formal* derivation of the cell and the homogenized equation, and justification by the energy method. Because this procedure can be repeated as long as we use a formal expansion and in some cases it is not easy to work out the energy method, it is meaningful to have a machinery to combine these two steps in one - the two-scale convergence method is designed to perform these steps at once. In fact, the two-scale convergence is a (special kind of) weak-formulation. Formally, as $\varepsilon \to 0$ in (1),

$$\begin{aligned} v^\varepsilon &= V_0(x,\frac{x}{\varepsilon}) + \varepsilon V_1(x,\frac{x}{\varepsilon}) + \varepsilon^2 V_2(x,\frac{x}{\varepsilon}) + \cdots \\ &\to V_0(x,y) + 0 + 0 + \cdots = V_0(x,y). \end{aligned}$$

We justify this formal convergence as follows. Multiply both sides of (1) by a test function $\psi(x,y)$ (which makes the boundary term zero when we apply integration by parts) and integrate it to have

$$\left\langle v^\varepsilon, \psi(x,\frac{x}{\varepsilon})\right\rangle_{L^2(S)}.$$

Suppose this converges to the one we expected:

$$\lim_{\varepsilon\to 0}\left\langle v^\varepsilon, \psi(x,\frac{x}{\varepsilon})\right\rangle_{L^2(S)} = \langle V_0, \psi\rangle_{L^2(S\times Y)}. \tag{2}$$

We call this convergence the two-scale convergence. Hence the two-scale convergence method is in some sense a machinery to find a hidden variable. By finding it, we want to understand the solution $v^\varepsilon$ as a special case of the limit.

Fortunately, the two-scale convergence is not far from our purpose. It is clear from the definition that the two-scale convergence is stronger than the weak convergence

$$\lim_{\varepsilon\to 0}\langle v^\varepsilon, \phi\rangle_{L^2(S)} = \langle v, \phi\rangle_{L^2(S)}, \quad \forall \phi \in L^2(S)$$

(note that the second factor of each inner product is fixed), but weaker than the strong convergence

$$\lim_{\varepsilon\to 0}\langle v^\varepsilon, w^\varepsilon\rangle_{L^2(S)} = \langle v, w\rangle_{L^2(S)}, \quad \forall w^\varepsilon \rightharpoonup w \in L^2(S)$$

(note that the second factors of the inner products have been changed).

From the fact that periodic functions can be regarded as functions on tori, tori may be replaced by any compact Riemannian manifolds.

Hereafter, $S$ represents an open domain in $\mathbb{R}^n$ containing the origin and $Y$ is a compact oriented n-dimensional Riemannian manifold with volume 1 - this restriction is just for the sake of Stokes' theorem. We fix $y_0 \in Y$, and we identify $\mathbb{R}^n \overset{i}{\cong} T_{y_0}Y$, where $T_{y_0}Y$ is the tangent space of $Y$ at $y_0$. For any $x \in S$, we denote $\exp_{y_0}(\frac{1}{\varepsilon}i(x))$ by $e^{x/\varepsilon}$, i.e., $e^{x/\varepsilon} \equiv \exp_{y_0}(\frac{1}{\varepsilon}i(x))$, where $\exp_{y_0}$ denotes the exponential function at the point $y_0$.

DEFINITION 1. A sequence $\{u_\varepsilon\} \in L^2(S)$ is said to be two-scale convergent to a limit $U_0(x,y) \in L^2(S \times Y)$ (we denote it by $u_\varepsilon \xrightarrow{2} U_0$) if for any test function $\psi(x,y) \in C_0^\infty(S \times Y)$, we have

$$\left\langle u_\varepsilon, \psi(x, e^{x/\varepsilon})\right\rangle_{L^2(S)} \longrightarrow \langle U_0, \psi\rangle_{L^2(S\times Y)}.$$

PROPOSITION 2. *For $\psi(x,y) \in L^\infty(S; C(Y))$, we have*

$$\lim_{\varepsilon\to 0}\left\|\psi\left(x, e^{x/\varepsilon}\right)\right\|_{L^2(S)} = \| \psi \|_{L^2(S\times Y)} \cdot$$

PROOF. We quote some arguments from [1] in which the measurability of $\psi(x, e^{x/\varepsilon})$ ($\psi$ is a Caratheodory-type function) is justified. From the fact that the difference can be considered as follows

$$\begin{aligned} &\left\|\psi\left(x, e^{x/\varepsilon}\right)\right\|_{L^2(S)} - \|\psi\|_{L^2(S\times Y)} \\ = &\left(\left\|\psi\left(x, e^{x/\varepsilon}\right)\right\| - \left\|\psi_n\left(x, e^{x/\varepsilon}\right)\right\|_{L^2(S)}\right) \\ &+ \left(\left\|\psi_n\left(x, e^{x/\varepsilon}\right)\right\| - \|\psi_n\|_{L^2(S\times Y)}\right) \\ &+ \left(\|\psi_n\| - \|\psi\|_{L^2(S\times Y)}\right), \end{aligned}$$

it suffices to show that the step functions of the type $\psi_n(x,y) = \sum \psi(x, y_i)\chi_i(y)$ satisfy the assertion and they converge strongly to $\psi$ in $L^2(S; C(Y))$. In fact, by Birkhoff Ergodic Theorem, we have

$$\begin{aligned} \lim_{\varepsilon\to 0}\left\|\psi_n\left(x, e^{x/\varepsilon}\right)\right\|_{L^2(S)} &= \left\|\psi_n\left(x, \exp_{y_0}(\frac{1}{\varepsilon}i(x))\right)\right\|_{L^2(S)} \\ &\to \| \psi_n \|_{L^2(S\times Y)} \cdot \end{aligned}$$

From an elementary measure theory, we know that for each fixed $x$, $\psi(x,\cdot)$ is approximated by $\psi_n(x,\cdot)$ with respect to sup-norm in the compact manifold $Y$. Also, by Lebesgue Dominated Convergence Theorem, we can show that $\psi_n$ converges strongly to $\psi$ in $L^2(S;C(Y))$. □

The following theorem is a corner stone which is useful for the two-scale convergence of the parabolic equations on perforated domain and implies the compactness theorem.

THEOREM 3. (Two-scale convergence with parameter) *Suppose that* $\{u_\varepsilon\}$ *is a bounded sequence in* $L^\infty([0,\infty);L^2(S))$. *Then there is a subsequence* $\{u_{\varepsilon_j}\}$ *of* $\{u_\varepsilon\}$ *and a function* $U_0(t,x,y)\in L^\infty([0,\infty);L^2(S\times Y))$ *such that for any* $\psi(t,x,y)\in L^1([0,\infty);C_0^\infty(S\times Y))$

$$\lim_{\varepsilon_j\to 0}\int_0^\infty \left\langle u_{\varepsilon_j}(t,x),\psi(t,x,e^{x/\varepsilon_j})\right\rangle_{L^2(S)} dt=\int_0^\infty \langle U_0(t),\psi(t)\rangle_{L^2(S\times Y)}\,dt.$$

PROOF. We define $\mathbf{F}_\varepsilon(\psi)\equiv\int_0^\infty\langle u_\varepsilon,\psi(x,e^{x/\varepsilon})\rangle_{L^2(S)}dt$ and $\mathbf{D}\equiv L^2(S;C(Y))$. There is a $C>0$ such that $\|u^\varepsilon\|_{L^\infty([0,\infty);L^2(S))}\le C$. Since $\psi\in L^1([0,\infty);\mathbf{D})$ and

$$\begin{aligned}|\,\mathbf{F}_\varepsilon(\psi)\,|\le&\int_0^\infty \|u_\varepsilon(t)\|_{L^2(S)}\left\|\psi(t,x,e^{x/\varepsilon})\right\|_{L^2(S)}dt\\ \le& C\int_0^\infty \max_{y\in Y}\|\psi(t,\cdot,y)\|_{L^2(S)}\,dt=C\int_0^\infty\|\psi(t)\|_{\mathbf{D}}\,dt,\end{aligned}$$

we have $\mathbf{F}_\varepsilon\in\left(L^1([0,\infty);\mathbf{D})\right)'$. So a subsequence $\{\mathbf{F}_{\varepsilon_j}\}$ is *weak**-convergent to some $\Upsilon_0\in\left(L^1([0,\infty);\mathbf{D})\right)'$. Hence for any $\psi$ in $L^1([0,\infty);\mathbf{D})$, we have

$$\left\|\psi\left(t,x,e^{x/\varepsilon_j}\right)\right\|_{L^2(S)}\le\|\,\psi(t)\,\|_{\mathbf{D}},$$

and we get the following by the Dominated Convergence theorem

$$\begin{aligned}|\,\Upsilon_0(\psi)\,|=\lim_{\varepsilon_j\to 0}|\,\mathbf{F}_{\varepsilon_j}(\psi)\,|\le C\int_0^\infty \limsup_{\varepsilon_j\to 0}\left\|\psi\left(t,x,e^{x/\varepsilon_j}\right)\right\|_{L^2(S)}dt\\ =C\int_0^\infty\|\,\psi(t)\,\|_{L^2(S\times Y)}\,dt.\end{aligned}$$

Since $L^1([0,\infty);\mathbf{D})$ is dense in $L^1([0,\infty);L^2(S\times Y))$, it follows that

$$\Upsilon_0\in\left(L^1([0,\infty);L^2(S\times Y))\right)'.$$

By the Riesz Representation Theorem, we have

$$\Upsilon_0(\psi) = \int_0^\infty \langle U_0(t), \psi(t)\rangle_{L^2(S\times Y)}\, dt$$

for some $U_0 \in L^\infty\left([0,\infty); L^2(S\times Y)\right)$. Therefore

$$\begin{aligned}\lim_{\varepsilon_j\to 0}\int_0^\infty \left\langle u_{\varepsilon_j}(t,x), \psi(t,x,e^{x/\varepsilon_j})\right\rangle_{L^2(S)} dt &\equiv \lim_{\varepsilon_j\to 0}\mathbf{F}_{\varepsilon_j}(\psi) = \Upsilon_0(\psi)\\ &= \int_0^\infty \langle U_0(t), \psi(t)\rangle_{L^2(S\times Y)} dt.\end{aligned}$$

□

COROLLARY 4. (Compactness Theorem) *Any bounded sequence* $\{u_\varepsilon\}$ *in* $L^2(S)$ *has a two-scale convergent subsequence.*

We define the divergence operator and the curl operator on $L^2(S)$. For a vector field $\vec{p} \equiv (p_1, p_2, \cdots, p_n) \in L^2(S)$, we define $div\,\vec{p}$ and $curl\,\vec{p}$ as following:

$$div\,\vec{p} \equiv \partial_1 p_1 + \partial_2 p_2 + \cdots + \partial_n p_n$$
$$(curl\,\vec{p})_{ij} \equiv \partial_j p_i - \partial_i p_j \quad \text{for } i,j = 1,\ldots,n,$$

where each $\partial_i$ represents the i-th partial derivative in the distributional sense. Define

$$\begin{aligned} H^1_{div}(S) &\equiv \left\{\vec{p}\in L^2(S)^n : div\,\vec{p}\in L^2(S)\right\},\\ H^1_{curl}(S) &\equiv \left\{\vec{p}\in L^2(S)^n : curl\,\vec{p}\in L^2(S)^{n(n-1)/2}\right\}.\end{aligned}$$

PROPOSITION 5. (i) *Let* $\{u_\varepsilon\}$ *be a bounded sequence in* $H^1(S)$. *Then there exist* $U_0 \in H^1(S)$ *and* $U_1(x,y) \in L^2(S; H^1(Y))$ *such that, up to a subsequence,* $\{u_\varepsilon\}$ *two-scale converge to* $U_0(x)$ *and* $\{\nabla u_\varepsilon\}$ *two-scale converge to* $\nabla U_0(x) + \nabla_y U_1(x,y)$.

(ii) *Suppose* $\{\vec{p}_\varepsilon\}$ *is bounded in* $H^1_{div}(S)$, *that is,* $\{\vec{p}_\varepsilon\}$ *and* $\{div\ \vec{p}_\varepsilon\}$ *are bounded sequences in* $L^2(S)^n$, *in* $L^2(S)$, *respectively. Then there is a subsequence* $\{\vec{p}_{\varepsilon_j}\}$ *of* $\{\vec{p}_\varepsilon\}$ *such that* $\{\vec{p}_{\varepsilon_j}\}$ *two-scale converges to* $\vec{P}_0(x,y)$ *and* $div\ \vec{p}_{\varepsilon_j} \xrightarrow{2} div\ \vec{P}_0(x,y) + div_y\ \vec{P}_1(x,y)$, *for some* $\vec{P}_0(x,y)$, $\vec{P}_1(x,y) \in L^2(S\times Y)^n$ *with* $div_y\ \vec{P}_0 = 0$.

(iii) *Suppose* $\{\vec{p}_\varepsilon\}$ *is bounded in* $H^1_{curl}(S)$, *that is,* $\{\vec{p}_\varepsilon\}$ *is* $L^2(S)^n$-*bounded and* $\{curl\,\vec{p}_\varepsilon\}$ *is* $L^2(S)^{n(n-1)/2}$-*bounded, then there exist* $\vec{P}_0(x,y)$, $\vec{P}_1(x,y) \in L^2(S\times Y)^n$ *and a subsequence* $\{\vec{p}_{\varepsilon_j}\}$ *such that* $\{\vec{p}_{\varepsilon_j}\}$ *two-scale converge to* $\vec{P}_0$ *and* $\{curl\ \vec{p}_{\varepsilon_j}\}$ *two-scale converges to* $curl\ \vec{P}_0 + curl_y \vec{P}_1$. *Moreover,* $\vec{P}_0 \in ker(curl_y)$.

PROOF. We prove the second fact, and the proof of the others can be carried out similarly. By the assumptions, we have a subsequence, say $\{\vec{p}_\varepsilon\}$ again, such that for any form $\Psi \in C_0^\infty(S \times Y)^n$, $\phi \in C_0^\infty(S \times Y)$,

$$\left\langle \vec{p}_\varepsilon, \Psi(x, e^{x/\varepsilon}) \right\rangle_{L^2(S)^n} \longrightarrow \langle \vec{P}_0, \Psi \rangle_{L^2(S\times Y)^n}$$
$$\left\langle div\, \vec{p}_\varepsilon, \phi(x, e^{x/\varepsilon}) \right\rangle_{L^2(S)} \longrightarrow \langle \zeta, \phi \rangle_{L^2(S\times Y)},$$

$\vec{P}_0(x,y) \in L^2(S \times Y)^n$, $\zeta(x,y) \in L^2(S \times Y)$. Since $-\, div$ is the adjoint operator of $\nabla$, we have

$$\varepsilon\left\langle div\, \vec{p}_\varepsilon, \phi(x, e^{x/\varepsilon}) \right\rangle_{L^2(S)} = -\left\langle \vec{p}_\varepsilon, \varepsilon\nabla_x \phi(x, e^{x/\varepsilon}) + \nabla_y \phi(x, e^{x/\varepsilon}) \right\rangle_{L^2(S)}.$$

By passing to the limit in the equation above we get $div_y\, \vec{P}_0 = 0$. Next, for $\phi \in C_0^\infty(S \times Y)$ such that $\nabla_y\, \phi(x,y) = 0$, we have

$$\begin{aligned} \left\langle \vec{p}_\varepsilon, \nabla_x \phi(x, e^{x/\varepsilon}) \right\rangle_{L^2(S)^n} &\longrightarrow \langle \vec{P}_0, \nabla\phi \rangle_{L^2(S\times Y)^n} \\ &= -\, \langle \zeta, \phi(x,y) \rangle_{L^2(S\times Y)}. \end{aligned}$$

It follows from this fact that $\langle \zeta - div\, \vec{P}_0, \phi(x,y) \rangle_{L^2(S\times Y)} = 0$ for any $\phi \in C_0^\infty(S\times Y)$ such that $\nabla_y\, \phi(x,y) = 0$. Hence $\zeta - div\, \vec{P}_0 \in (ker\, \nabla_y)^\perp$. We observe that $(ker\, \nabla_y)^\perp \subseteq Im\, (div_y)$. Indeed, for any $f \in (Im\, (div_y))^\perp$, $\nabla_y f(\phi) \equiv -\, \langle f, div_y\, \phi \rangle = 0$, for all $\phi \in C_0^\infty(Y)$. So $f \in ker\, \nabla_y$. Hence there exists $\vec{P}_1(x,y) \in L^2(S \times Y)^n$ such that $\zeta - div\, \vec{P}_0 = div_y\, \vec{P}_1$. That is, $\zeta = div\, \vec{P}_0 + div_y\, \vec{P}_1$. This completes the proof. □

LEMMA 6. *Let $\{A^\varepsilon(x)\}$ be a sequence of $n \times n$ matrices such that*

$$\lim_{\varepsilon\to 0} \|A^\varepsilon\|_{L^2(S)^{n^2}} = \|\, A \,\|_{L^2(S\times Y)^{n^2}}$$

*for some $n \times n$-matrix $A(x,y)$. Then, for any sequence $\{\vec{p}_\varepsilon\}$ in $L^2(S)^n$ that two-scale convergent to a limit $\vec{P}_0(x,y) \in L^2(S \times Y)^n$, we have*

$$A^\varepsilon \vec{p}_\varepsilon \xrightarrow{\;2\;} A(x,y)\vec{P}_0(x,y).$$

PROOF. Let $\{B_n(x,y)\}$ be a sequence of $n \times n$ matrices such that $B_n(x,y)$ are smooth with respect to x,y-variables and converge strongly to $A(x,y)$ in $L^2(S \times Y)^{n^2}$. Then by Proposition 2 and the assumption, we have

$$\lim_{\varepsilon\to 0} \left\| A^\varepsilon - B_n(x, e^{x/\varepsilon}) \right\|^2_{L^2(S)^{n^2}} = \|\, A - B_n \,\|^2_{L^2(S\times Y)^{n^2}}\ .$$

Hence as $n$ goes to infinity, we obtain $\lim_{n\to\infty}\lim_{\varepsilon\to 0}\left\|A^\varepsilon - B_n(x,e^{x/\varepsilon})\right\|_{L^2(S)^{n^2}}$ $= 0$. Let $\{\vec{p}_\varepsilon\}$ be a sequence in $L^2(S)^n$ that two-scale converge to a limit $\vec{P}_0(x,y)\in L^2(S\times Y)^n$. For any $\psi(x,y)\in C_0^\infty(S\times Y)^n$, we have

$$\left\langle A^\varepsilon(x)\vec{p}_\varepsilon(x),\psi(x,e^{x/\varepsilon})\right\rangle_{L^2(S)^n} - \left\langle B_n(x,e^{x/\varepsilon})\vec{p}_\varepsilon(x),\psi(x,e^{x/\varepsilon})\right\rangle_{L^2(S)^n}$$
$$= \left\langle\left\{A^\varepsilon(x) - B_n(x,e^{x/\varepsilon})\right\}\vec{p}_\varepsilon(x),\psi(x,e^{x/\varepsilon})\right\rangle_{L^2(S)^n}.$$

Passing to the limit as $\varepsilon$ goes to zero yields

$$\left|\lim_{\varepsilon\to 0}\left\langle A^\varepsilon(x)\vec{p}_\varepsilon(x),\psi(x,e^{x/\varepsilon})\right\rangle_{L^2(S)^n} - \left\langle B_n\vec{P}_0,\psi\right\rangle_{L^2(S\times Y)^n}\right|$$
$$\le Constant\lim_{\varepsilon\to 0}\left\|A^\varepsilon - B_n(x,e^{x/\varepsilon})\right\|_{L^2(S)^{n^2}}.$$

Letting $n$ go to infinity we get the desired result. □

## 3. Linear second order elliptic equations

Let $S$ be a bounded open set of $\mathbf{R}^n$. Let $f$ be a given function in $L^2(S)$. We consider the following linear second order elliptic equation

$$-div\left(\mathbf{A}\left(e^{x/\varepsilon}\right)\nabla u_\varepsilon\right) = f \quad \text{in } S \tag{3}$$
$$u_\varepsilon = 0 \quad \text{on } \partial S,$$

where $\mathbf{A}(y)$ is an $n\times n$ matrix defined on $Y$ and $Y$ is a compact oriented Riemannian manifold, such that there exist two positive constants $\alpha$ and $\beta$ with $0<\alpha\le\beta$ satisfying

$$\alpha\mid\xi\mid^2\le\sum_{i,j=1}^{N}\mathbf{A}_{ij}(y)\xi_i\xi_j\le\beta\mid\xi\mid^2 \qquad \text{for any } \xi\in\mathbf{R}^n,$$

and each $\mathbf{A}_{ij}(\cdot)$ is measurable with

$$\lim_{\varepsilon\to 0}\|\mathbf{A}_{ij}(e^{x/\varepsilon})\|_{L^2(S)} = \|\mathbf{A}_{ij}(y)\|_{L^2(S\times Y)}.$$

We define the flux vector field $\vec{p}_\varepsilon$ by $\vec{p}_\varepsilon(x)\equiv\mathbf{A}(e^{x/\varepsilon})\nabla u_\varepsilon(x)$, so we have

$$-div\,\vec{p}_\varepsilon = f. \tag{4}$$

From energy estimates on (3),

$$\alpha\|\nabla u_\varepsilon\|^2_{L^2(S)}\le\left\langle\mathbf{A}\left(e^{\frac{x}{\varepsilon}}\right)\nabla u_\varepsilon,\nabla u_\varepsilon\right\rangle_{L^2(S)}$$
$$= \langle f,\nabla u_\varepsilon\rangle_{L^2(S)}\le\|f\|_{L^2(S)}\|\nabla u_\varepsilon\|_{L^2(S)},$$

we observe that $\{u_\varepsilon\}$ is bounded in $H^1(S)$, and so $\{\vec{p_\varepsilon}\}$ and $div\{\vec{p_\varepsilon}\}$ are bounded in $L^2(S)^n$, in $L^2(S)$, respectively. Hence applying Lemma 6 and Proposition 5, we have the two-scale limits:

$$- \, div\, \vec{p}_0 - div_y\, \vec{p}_1 = f \tag{5}$$

$$\vec{p}_0 = \mathbf{A}(y)(\nabla u_0 + \nabla_y u_1) \tag{6}$$

$$- \, div_y \vec{p}_0 = 0. \tag{7}$$

We integrate (5) over $Y$ to get

$$- \, div \int_Y \mathbf{A}(y)(\nabla u_0 + \nabla_y u_1)\, dy = f. \tag{8}$$

THEOREM 7. (Existence and Uniqueness) *There are the unique solutions $u_0 \in H_0^1(S)$ and $u_1 \in L^2(S; H^1(Y)/\mathbb{R})$ of the two-scale homogenized system* (8) *and* (7).

PROOF. A variational formulation associated to (8) and (7) is given by

$$\int_S \int_Y \mathbf{A}(y)(\nabla u_0(x) + \nabla_y u_1(x,y)) \cdot (\nabla \phi_0(x) + \nabla_y \phi_1(x,y))\, dxdy$$
$$= \int_S f(x)\phi(x)dx.$$

We define a bilinear form $\mathbf{B}$ on the Hilbert space $H_0^1(S) \times L^2(S; H^1(Y)/\mathbb{R})$ with the norm $\|\nabla u_0\|_{L^2(S)} + \|\nabla_y u_1\|_{L^2(S\times Y)}$;

$$\mathbf{B}[(u_0, u_1), (\phi_0, \phi_1)]$$
$$\equiv \int_S \int_Y \mathbf{A}(y)(\nabla u_0(x) + \nabla_y u_1(x,y)) \cdot (\nabla \phi_0(x) + \nabla_y \phi_1(x,y))\, dxdy.$$

Then it is clear that $\mathbf{B}$ is coercive since

$$\begin{aligned} \mathbf{B}[(\phi_0, \phi_1), (\phi_0, \phi_1)] &\geq \alpha \int_S \int_Y |(\nabla \phi_0(x) + \nabla_y \phi_1(x,y))|^2\, dxdy \\ &= \alpha \int_S |(\nabla \phi_0(x)|^2\, dx + \int_S \int_Y |\nabla_y u_1(x,y))|^2\, dxdy. \end{aligned}$$

Hence, by virtue of Lax-Milgram lemma, there exists a unique solution of the two-scale homogenized system (8) and (7). □

We apply separation of variables $u_1(x,y) \equiv \vec{\omega}(y) \cdot \nabla u_0(x)$, for some $\vec{\omega}(y) \in H^1(Y)^n$ and so (6) reads $\vec{p}_0 = \mathbf{A}(y) \left(I + (\nabla_y \vec{\omega})^T (y)\right) \nabla u_0$. Define $X_A(y) \equiv \mathbf{A}(y) \left(I + (\nabla_y \vec{\omega})^T (y)\right)$ and $\bar{\mathbf{A}} \equiv \int_Y X_A(y) dy = \langle X_A \rangle_Y$.

Then from (8), we get the homogenized equation;

$$-\,div\ \bar{\mathbf{A}}\nabla u_0 = f. \tag{9}$$

Also we have

$$-div_y X_A = \vec{\mathbf{0}}, \tag{10}$$

which means that each column of $X_A$ is divergence free.

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Department of Mathematics
Yonsei University
Seoul 120-749, Korea
*E-mail*: hpak@yonsei.ac.kr