Commun. Korean Math. Soc. **18** (2003), No. 3, pp. 551–558

ON THE MAXIMAL INEQUALITY FOR AANA RANDOM VARIABLES AND A STRONG LAW OF LARGE NUMBERS

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ABSTRACT. In this paper we obtain the Hájeck-Rènyi type inequality for the asymptotically almost negatively associated(AANA) random variables and extend some results for negatively associated random variables to the AANA case by applying this inequality.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and $\{X_n, n \geq 1\}$ be a sequence of random variables defined on $(\Omega, \mathcal{F}, \mathcal{P})$. A finite family $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated(NA) if for any disjoint subsets $A, B \subset \{1, \cdots, n\}$ and any real coordinatewise nondecreasing functions $f : R^A \to R$ and $g : R^B \to R$,

$$Cov(f(X_i; i \in A), g(X_j; j \in B)) \leq 0.$$

Infinite family of random variables is NA if every finite subfamily is NA. This concept was introduced by Joag-Dev and Proschan [7]. By inspecting the proof of Matula's [9] maximal inequality for the NA random variables, we see that one can also allow positive correlations provided they are small. Primarily motivated by this, Chandra and Ghosal ([1], [2]) introduced the following dependence: A sequence $\{X_n, n \geq 1\}$ of random variables is called asymptotically almost negatively associated(AANA)

Received October 23, 2001.
2000 Mathematics Subject Classification: 60F15.
Key words and phrases: Hájeck-Rènyi inequality, negatively associated random variables, asymptotically almost negatively associated, strong law of large numbers.
This paper was supported by Joosan Fellowship in Wonkwang University Foundation in 2001.

if there is a nonnegative sequence $q(m) \to 0$ such that

$$\begin{aligned} & Cov(f(X_m), g(X_{m+1}, \cdots, X_{m+k})) \\ (1) \qquad \leq\ & q(m)(Var(f(X_m))Var(g(X_{m+1}, \cdots, X_{m+k})))^{\frac{1}{2}} \end{aligned}$$

for all $m, k \geq 1$ and for all coordinatewise increasing continuous functions f and g whenever the right-hand side of (1) is finite. This definition was introduced by Chandra and Ghosal ([1], [2]).

The family of AANA sequences contains negatively associated (in particular, independent) sequences (with $q(m) = 0, \forall m \geq 1$) and some more sequences of random variables which are not much deviated from being negatively associated.

The condition means roughly that asymptotically the future is almost negatively associated with the present. For an example of AANA which is not negatively associated, consider $X_n = Y_n + \alpha_n Y_{n+1}$, where $Y_1, Y_2, \cdots$ are i.i.d. $N(0,1)$ and $\alpha_n \to 0, \alpha_n > 0$. (See Chandra and Ghosal [2].)

Hájeck-Rènyi [6] proved the following important inequality: If $\{X_n, n \geq 1\}$ is a sequence of independent random variables with $EX_n = 0$, and $EX_n^2 < \infty, n \geq 1$, and $\{b_n, n \geq 1\}$ is a positive nondecreasing real sequence, then for any $\epsilon > 0$ and any positive integer $m < n$

$$(2) \qquad P\left(\max_{m \leq j \leq n} \left|\frac{\sum_{i=1}^{j} X_i}{b_j}\right| \geq \epsilon\right) \leq \epsilon^{-2}\left(\sum_{j=m+1}^{n} \frac{EX_j^2}{b_j^2} + \sum_{j=1}^{m} \frac{EX_j^2}{b_m^2}\right).$$

Since then, this inequality has been studied by many authors (e.g. Chow [3], Gan [5], Liu, Gan and Chen [8], Christofides [4]). Especially, Liu, Gan and Chen [8] extended (2) to NA random variables.

In this paper, we derive the Hájeck-Rènyi type inequality for AANA random variables and discuss some results for AANA random variables which have not been established previously in the literature. In other words, we show that some results for NA random variables remain true if the assumption of NA random variables is relaxed to AANA random variables with $A^2 = \sum_{k=1}^{\infty} q^2(k) < \infty$.

2. Results

From the definition of AANA we easily obtain the following lemma.

LEMMA 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of asymptotically almost negatively associated*(AANA) *random variables and $\{f_n, n \geq 1\}$ a* *sequence of increasing continuous functions. Then* $\{f_n(X_n), n \geq 1\}$ *is also a sequence of* AANA *random variables.*

LEMMA 2.2. (Chandra and Ghosal ([1], [2]) *Let* $\{X_1, \cdots, X_n\}$ *be a sequence of mean zero, square integrable random variables such that* (1) *holds for* $1 \leq m < k+m \leq n$ *and for all coordinatewise increasing continuous functions* f *and* g *whenever the right-hand side of* (1) *is finite. Let* $A_n^2 = \sum_{m=1}^{n-1} q^2(m)$ *and* $\sigma_k^2 = EX_k^2, k \geq 1$. *Then, for* $\epsilon > 0$

$$(3) \qquad P\left\{\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right| \geq \epsilon\right\} \leq 2\epsilon^{-2}(A_n + (1+A_n^2)^{\frac{1}{2}})^2 \sum_{k=1}^{n} \sigma_k^2.$$

THEOREM 2.3. *Let* $\{b_n, n \geq 1\}$ *be a sequence of positive nondecreasing real numbers and* $\{X_1, \cdots, X_n\}$ *a sequence of mean zero, square integrable random variables such that* (1) *holds for* $1 \leq m < k+m \leq n$ *and for all coordinatewise increasing continuous functions* f *and* g *whenever the right-hand side of* (1) *is finite. Let* $A_n^2 = \sum_{m=1}^{n-1} q^2(m)$ *and* $\sigma_k^2 = EX_k^2$, $k \geq 1$. *Then, for* $\epsilon > 0$

$$(4) \qquad P\left\{\max_{1\leq k\leq n}\left|\frac{\sum_{i=1}^{k} X_i}{b_k}\right| \geq \epsilon\right\} \leq 8\epsilon^{-2}(A_n + (1+A_n^2)^{\frac{1}{2}})^2 \sum_{k=1}^{n} \frac{\sigma_k^2}{b_k^2}.$$

PROOF. First note that $\{\frac{X_1}{b_1}, \frac{X_2}{b_2}, \cdots, \frac{X_n}{b_n}\}$ are AANA by Lemma 2.1 and that $\frac{X_i}{b_i}$'s are mean zero and square integrable since X_i's are mean zero and square integrable. Next, the nonnegative sequence $q(m) \to 0$ satisfies

$$\begin{aligned} & Cov(f(b_m^{-1}X_m),\ g(b_{m+1}^{-1}X_{m+1}, \cdots, b_{m+k}^{-1}X_{m+k})) \\ (5) \quad \leq \ & q(m)[Var(f(b_m^{-1}X_m))Var(g(b_{m+1}^{-1}X_{m+1}, \cdots, b_{m+k}^{-1}X_{m+k})]^{\frac{1}{2}} \end{aligned}$$

for $1 \leq m < k+m \leq n$ and for all coordinatewise increasing continuous functions f and g whenever the right-hand side of (5) is finite.

Now, we follow the argument given in the proof of Theorem 2.1 of Liu, Gan and Chen [8]. For the sake of completeness we repeat it here. Without loss of generality letting $b_0 = 0$, we have

$$\begin{aligned} \sum_{j=1}^{k} X_j = \sum_{j=1}^{k}\left(b_j \cdot \frac{X_j}{b_j}\right) &= \sum_{j=1}^{k}\left(\sum_{i=1}^{j}(b_i - b_{i-1})\frac{X_j}{b_j}\right) \\ &= \sum_{i=1}^{k}(b_i - b_{i-1}) \sum_{i\leq j\leq k} \frac{X_j}{b_j} \end{aligned}$$

and thus since $b_k^{-1}\sum_{j=1}^k(b_j-b_{j-1})=1$, we have

$$\left\{\left|\frac{\sum_{j=1}^k X_j}{b_k}\right|\geq\epsilon\right\}\subset\left\{\max_{1\leq i\leq k}\left|\sum_{i\leq j\leq k}\frac{X_j}{b_j}\right|\geq\epsilon\right\}.$$

Therefore,

$$\begin{aligned}\left\{\max_{1\leq k\leq n}\left|\frac{\sum_{j=1}^k X_j}{b_k}\right|\geq\epsilon\right\} &\subset \left\{\max_{1\leq k\leq n}\max_{1\leq i\leq k}\left|\sum_{i\leq j\leq k}\frac{X_j}{b_j}\right|\geq\epsilon\right\}\\ &= \left\{\max_{1\leq i\leq k\leq n}\left|\sum_{j\leq k}\frac{X_j}{b_j}-\sum_{j<i}\frac{X_j}{b_j}\right|\geq\epsilon\right\}\\ &\subset \left\{\max_{1\leq i\leq n}\left|\sum_{j=1}^{i}\frac{X_j}{b_j}\right|\geq\frac{\epsilon}{2}\right\}. \qquad (6)\end{aligned}$$

By the Kolmogorov type inequality of AANA random variables (see Lemma 2.2) from (5) and (6) the desired result (4) follows. □

REMARK. Especially, taking $b_n=1$, the result of Chandra and Ghosal ([1], Theorem 1) is a simple corollary.

From Theorem 2.3 we can get the following more generalized inequality.

THEOREM 2.4 *Let $\{b_n, n\geq 1\}$ be a sequence of positive nondecreasing real numbers and $\{X_n, n\geq 1\}$ a sequence of mean zero, and square integrable* AANA *random variables satisfying* (1) *for* $1\leq m<k+m\leq n$ *and for all coordinatewise continuous functions f and g whenever the right-hand side of* (1) *is finite. Let $A_n^2=\sum_{m=1}^{n-1}q^2(m)$ and $\sigma_k^2=EX_k^2, k\geq 1$. Then, for $\epsilon>0$ and for any positive integer $m<n$*

$$\begin{aligned}&P\{\max_{m\leq k\leq n}|b_k^{-1}\sum_{j=1}^k X_j|\geq\epsilon\}\\ &\leq 32\epsilon^{-2}(A_n+(1+A_n^2)^{\frac{1}{2}})^2\left(\sum_{j=m+1}^{n}\frac{\sigma_j^2}{b_j}+\sum_{j=1}^{m}\frac{\sigma_j^2}{b_m^2}\right). \qquad (7)\end{aligned}$$

PROOF. By Theorem 2.3 we have

$$
\begin{aligned}
&P\left\{\max_{m\le k\le n}\left|\frac{\sum_{j=1}^{k}X_j}{b_k}\right|\ge\epsilon\right\}\\
&\le P\left\{\left|\frac{\sum_{j=1}^{m}X_j}{b_m}\right|\ge\frac{\epsilon}{2}\right\}+P\left\{\max_{m+1\le k\le n}\left|\frac{\sum_{j=m+1}^{k}X_j}{b_k}\right|\ge\frac{\epsilon}{2}\right\}\\
&\le P\left\{\frac{1}{b_m}\max_{1\le k\le m}\left|\sum_{j=1}^{k}X_j\right|\ge\frac{\epsilon}{2}\right\}+P\left\{\max_{m+1\le k\le n}\left|\frac{\sum_{j=m+1}^{k}X_j}{b_k}\right|\ge\frac{\epsilon}{2}\right\}\\
&\le 32\epsilon^{-2}(A_n+(1+A_n^2)^{\frac{1}{2}})^2\left(\sum_{j=1}^{m}\frac{\sigma_j^2}{b_m^2}+\sum_{j=m+1}^{n}\frac{\sigma_j^2}{b_j^2}\right).
\end{aligned}
$$

Now we show that Theorem 3.1 of Liu et al. [8] still holds. □

THEOREM 2.5. *Let $\{b_n, n\ge 1\}$ be a sequence of positive nondecreasing real numbers and $\{X_n, n\ge 1\}$ a sequence of mean zero, square integrable random variables such that* (1) *holds for $1\le m<k+m\le n$ and for all coordinatewise increasing continuous functions f and g whenever the right-hand side of* (1) *is finite. Let $\sigma_k^2=EX_k^2,\ k\ge 1$.*

Assume

$$A^2=\sum_{k=1}^{\infty}q^2(k)<\infty, \tag{8}$$

$$\sum_{k=1}^{\infty}\frac{\sigma_k^2}{b_k^2}<\infty. \tag{9}$$

Then, for any $0<r<2$,
(A) $E(\sup_n(S_n/b_n)^r)<\infty$,
(B) $0<b_n\uparrow\infty$ implies $S_n/b_n\to 0$ a.s. as $n\to\infty$, where $S_n=\sum_{i=1}^{n}X_i$.

PROOF. (A): Note that

$$E\sup_n\left(\frac{|S_n|}{b_n}\right)^r<\infty\Leftrightarrow\int_1^{\infty}P\left(\sup_n\frac{|S_n|}{b_n}>t^{\frac{1}{r}}\right)dt<\infty.$$

By Theorem 2.3, it follows from (8) and (9) that

$$\begin{aligned} &\int_1^\infty P\left(\sup_n \frac{|S_n|}{b_n} > t^{\frac{1}{r}}\right) dt \\ \le\ & 8\int_1^\infty t^{-\frac{2}{r}}(A + (1+A^2)^{\frac{1}{2}})^2 \sum_{n=1}^\infty \frac{\sigma_n^2}{b_n^2} dt \\ \le\ & 8(A + (1+A^2)^{\frac{1}{2}})^2 \sum_{n=1}^\infty \frac{\sigma_n^2}{b_n^2} \int_1^\infty t^{-\frac{2}{r}} dt < \infty. \end{aligned}$$

(B): We need to show that for any $\epsilon > 0$,

$$P(\cup_{k=m}^\infty \{\left|\frac{S_k}{b_k}\right| \ge \epsilon\}) \to 0 \ as\ m \to \infty.$$

Note that

$$\begin{aligned} & P(\cup_{k=m}^\infty \{\left|\frac{S_k}{b_k}\right| \ge \epsilon\}) \\ =\ & \lim_{n\to\infty} P(\max_{m\le k\le n} \frac{|S_k|}{b_k} \ge \epsilon) \\ \le\ & 32\epsilon^{-2}(A + (1+A^2)^{\frac{1}{2}})^2 \left(\sum_{j=m+1}^\infty \frac{\sigma_j^2}{b_j^2} + \sum_{j=1}^m \frac{\sigma_j^2}{b_m^2}\right). \end{aligned}$$

By Kronecker Lemma, we have

$$\sum_{j=1}^m \frac{\sigma_j^2}{b_m^2} \to 0 \ as\ m \to \infty,$$

which proves the assertion by (9). □

REMARK. By taking $b_n = n$ Theorem 2.5(B) shows that Corollary of Theorem 3 of Matula [9] remains true if the assumption of NA random variables is relaxed to AANA random variables with $A^2 = \sum_{k=1}^\infty q^2(k) < \infty$.

COROLLARY 2.6. *Let $\{X_n, n \ge 1\}$ be a sequence of mean zero, square integrable AANA random variables satisfying (1) for $1 \le m < k+m \le n$ and for all coordinatewise increasing continuous functions f and g whenever the right-hand side of (1) is finite. Let $A^2 = \sum_{m=1}^\infty q^2(m)$ and $\sigma_k^2 = EX_k^2, k \ge 1$. Then, for $0 < t < 2$, $\epsilon > 0$, and $l \ge 1$,*

$$P\left\{\sup_{j\ge l} \frac{|S_j|}{j^{\frac{1}{t}}} \ge \epsilon\right\} \le 32\epsilon^{-2}(A + (1+A^2)^{\frac{1}{2}})^2 \frac{t}{2-t} l^{\frac{t-2}{t}} \sup_n \sigma_n^2$$

where $S_n = \sum_{j=1}^{n} X_j$.

The following corollary shows that Corollary 3.2 of Liu et al. [8] remains true if the assumption of NA random variables is relaxed to AANA random variables with $A^2 = \sum_{k=1}^{\infty} q^2(k) < \infty$.

COROLLARY 2.7. *Let* $\{X_n, n \geq 1\}$ *be a sequence of mean zero, square integrable AANA random variables satisfying* (1). *Assume that*

$$\sum_{k=1}^{\infty} q^2(k) < \infty \text{ and } \sup_n \sigma_n^2 < \infty.$$

Then, for $0 < t < 2$
(A) $S_n/n^{\frac{1}{t}} \to 0$ a.s. *as* $n \to \infty$,
(B) $E\sup_n(|S_n|/n^{\frac{1}{t}})^r < \infty$ *for any* $0 < r < 2$,
where $S_n = \sum_{j=1}^{n} X_j$.

ACKNOWLEDGMENTS. The authors wish to thank the referee for a very thorough review of this paper and the first author thanks Wonkwang University for providing the research assistant. Part of this work appears in the Wonkwang University doctorial dissertation of the second author.

References

[1] T. K. Chandra and S. Ghosal, *Extensions of the strong law of large numbers of Marcinkiewicz and Zygmund for dependent variables,* Acta. Math. Hungar. **71** (1996a), 327–336.

[2] ———, *The strong law of large numbers for weighted averages under dependence assumption,* J. Theor. Probab. **9** (1996b), 797–809.

[3] Y. S. Chow, *A martingale inequality and the law of large numbers,* Proc. Amer. Math. Soc. **11** (1960), 107–111.

[4] T. C. Christofides, *Maximal inequalities for demimartingales and a strong law of large number,* Statist. Probab. Lett. **50** (2000), 357–363.

[5] S. Gan, *The Hajeck-Renyi inequality for Banach space valued martingales and the p smoothness of Banach space,* Statist. Probab. Lett. **32** (1997), 245–248.

[6] J. Hájeck and A. Rènyi, *Generalization of an inequality of Kolmogorov,* Acta. Math. Acad. Sci. Hungar. **6** (1955), 281–283.

[7] K. Joag-Dev and F. Proschan, *Negative association of random variables with applications,* Ann. Statist. **11** (1983), 286–295.

[8] J. Liu, S. Gan, and P. Chen, *The Hajeck-Renyi inequality for the NA random variables and its application,* Statist. Probab. Lett. **43** (1999), 99–105.

[9] P. Matula, *A note on the almost sure convergence of sums of negatively dependent random variables,* Statist. Probab. Lett. **15** (1992), 209–213.

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