Commun. Korean Math. Soc. **18** (2003), No. 3, pp. 515–519

HARMONIC KÄHLER FORMS ON HYPERKÄHLER MANIFOLDS

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ABSTRACT. Let M be a hyperkähler manifold with the hyperkähler structure (g, I, J, K). In [5], D. Huybrechts suggests that it is an open and interesting question whether any Kähler class that stays Kähler in the twistor family can actually be represented by an harmonic Kähler form. In this paper we will consider both this problem and the set of all the primitive harmonic Kähler forms on M.

1. Introduction

Given a manifold M, to study it, we usually use its cohomology ring. Moreover, with the aids of Hodge theory if we have some information about its harmonic space, then we can do much more things about the manifold M.

DEFINITION 1.1. Let M be a $4n$-dimensional manifold. M is said to be *hyperkähler* if there is a metric g on M such that for some complex structures I, J, and K on M with the properties $I \circ J = -J \circ I = K$, the metric g is Kähler with respect to each complex structure R for $R \in \{I, J, K\}$.

We call (g, I, J, K) the hyperkähler structure on M. Let $S := \{aI + bJ + cK \mid a^2 + b^2 + c^2 = 1\}$. For each $R \in S$, it is easy to show that R is also a complex structure on M. We call R the induced complex structure on M and S the twistor family of all the induced complex structures on M.

Received October 31, 2002.
2000 Mathematics Subject Classification: 53C26.
Key words and phrases: hyperkähler, harmonic, Kähler forms.

2. Kähler classes

For a 1-dimensional complex manifold $(\mathbb{C}, I)$, there is a holomorphic coordinate $z = x + \sqrt{-1}y$ on $\mathbb{C}$ which is compatible with the complex structure I. Then from the equation $I(dz) = \sqrt{-1}dz$, we have

$$I(dx) = -dy$$
$$I(dy) = dx.$$

Furthermore, by the relation $I(dz)(\frac{\partial}{\partial z}) = dz(I(\frac{\partial}{\partial z}))$, where

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - \sqrt{-1}\frac{\partial}{\partial y})$$
$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + \sqrt{-1}\frac{\partial}{\partial y}),$$

we get

$$I(\frac{\partial}{\partial x}) = \frac{\partial}{\partial y}$$
$$I(\frac{\partial}{\partial y}) = -\frac{\partial}{\partial x}.$$

Consider the complex manifold $(\mathbb{C}, -I)$. From the above, we have

$$(-I)(d\overline{z}) = \sqrt{-1}d\overline{z} \qquad (-I)(dx) = dy \qquad \qquad (-I)(\frac{\partial}{\partial x}) = -\frac{\partial}{\partial y}$$
$$(-I)(dz) = -\sqrt{-1}dz, \qquad (-I)(dy) = -dx, \qquad \text{and} \qquad (-I)(\frac{\partial}{\partial y}) = \frac{\partial}{\partial x}.$$

Thus we see that $\overline{z} = x - \sqrt{-1}y$ is the holomorphic coordinate on the complex manifold $(\mathbb{C}, -I)$.

PROPOSITION 2.1. *Let (X, I) be a Kähler manifold. If α is a Kähler form on the complex manifold (X, I), then the form $-\alpha$ is Kähler on the complex manifold $(X, -I)$.*

PROOF. This is an immediate result from the definition of a Kähler form on the complex manifold (X, I). □

Now, we will do the main theorem.

THEOREM 2.1. *Let X be a $4n$-dimensional compact hyperkähler manifold with the hyperkähler structure (g, I, J, K). Let $S := \{aI+bJ+cK \mid a^2 + b^2 + c^2 = 1\}$. Then there are no Kähler classes α on the complex* *manifold (X, I) such that the class α is Kähler on each complex manifold (X, R) for $R \in S$.*

PROOF. Assume that α is a Kähler class on the complex manifold (X, I) such that the class α is Kähler on each complex manifold (X, R) for $R \in S$. Then by the Hodge theory, there is a harmonic form $\widetilde{\alpha} \in \mathcal{H}^{1,1}(X, I)$ such that its cohomology class $[\widetilde{\alpha}]$ is equal to the class α. Throughout this paper, we will denote by $[\beta]$ the cohomology class of a closed form β. By the assumption and Hodge theory, the harmonic form $\widetilde{\alpha}$ is of type $(1,1)$ on the complex manifold (X, R) for each $R \in S$, since the harmonic forms depend only on the metric g and not on the complex structures R. Thus, by our assumption and the definition of the Kähler class, there is a 1-form $\beta_R \in A^1(X, R)$ such that the 2-form $\widetilde{\alpha} + d\beta_R$ is Kähler on (X, R) for each $R \in S$.

Since the 2-form $\widetilde{\alpha}+d\beta_I$ is Kähler on (X, I), by Proposition 2.1, the 2-form $-(\widetilde{\alpha}+d\beta_I)$ is also Kähler on $(X, -I)$. Thus the 2-forms $-(\widetilde{\alpha}+d\beta_I)$ and $\widetilde{\alpha} + d\beta_{-I}$ are Kähler on $(X, -I)$. Let $\mathcal{K}_{(X,-I)}$ be the set of all Kähler classes on the complex manifold $(X, -I)$. Then we already know that $\mathcal{K}_{(X,-I)}$ is an open convex cone in $H^{1,1}(X, -I) \cap H^2(X, \mathbb{R})$. But $[-(\widetilde{\alpha} + d\beta_I)] = [-\widetilde{\alpha}] = -[\widetilde{\alpha}] \in \mathcal{K}_{(X,-I)}$ and $[\widetilde{\alpha} + d\beta_{-I}] = [\widetilde{\alpha}] \in \mathcal{K}_{(X,-I)}$, contradiction. Therefore, we complete our proof. $\square$

REMARK 2.1. In [5], D. Huybrechts says that it is an open and interesting question whether any Kähler class that stays Kähler in the twistor family can actually be represented by an harmonic Kähler form. But by Theorem 2.1, this question is absurd.

3. Harmonic primitive Kähler forms

Let X be a $4n$-dimensional compact hyperkähler manifold with the hyperkähler structure (g, I, J, K). Let A be the set of all harmonic primitive Kähler forms on (X, I) , $\mathcal{K}_X$ the set of all Kähler forms on (X, I), and $\widetilde{\mathcal{K}}_X$ the set of all Kähler classes on (X, I). And let

$$\widetilde{\mathcal{K}}^i_X := \{\omega \in \mathcal{K}_X \mid \omega^{2n-i} \cdot \omega_I^i = c \cdot \omega_I^{2n} \text{ for some scalar constant } c\}$$

for $0 \le i \le 2n - 1$.

Then the followings are well-known:

1. $\widetilde{\mathcal{K}}_X$ is an open convex cone in $H^{1,1}(X, \mathbb{R})$.

2. the set $\widetilde{\mathcal{K}}^0_X$ is isomorphic to the Kähler cone $\widetilde{\mathcal{K}}_X$ ([1], [2], [3], [5], [6]).

3. the canonical projection $\widetilde{\mathcal{K}}_X^i \mapsto \widetilde{\mathcal{K}}_X$ is injective for $1 \leq i \leq 2n-1$ [5].

PROPOSITION 3.1. ([5]) *With the above notations, the set* $\widetilde{\mathcal{K}}_X^{2n-1}$ *is an open subset of the harmonic space* $\mathcal{H}^{1,1}(X,\mathbb{R})$*, where* $\mathcal{H}^{1,1}(X,\mathbb{R})$ *is the space of all harmonic real* $(1,1)$*-forms on* (X,I).

PROOF. For any $\omega \in \widetilde{\mathcal{K}}_X^{2n-1}$, we have

$$\omega \cdot \omega_I^{2n-1} = c \cdot \omega_I^{2n} \quad \text{for some scalar constant } c.$$

Since $0 = \omega_I^{2n-1}(\omega - c\cdot\omega_I) = L_{\omega_I}^{2n-1}(\omega - c\cdot\omega_I)$, the form $\omega - c\cdot\omega_I$ is closed and ω_I-primitive. Then it is not hard to show that $\omega - c\cdot\omega_I$ is ω_I-harmonic. By the above facts 1 and 3, $\widetilde{\mathcal{K}}_X^{2n-1}$ is an open subset of $\mathcal{H}^{1,1}(X,\mathbb{R})$. □

Furthermore, we know

$$A = \mathcal{H}^{1,1}(X,\mathbb{R})_\perp \cap \mathcal{K}_X$$

for the primitive decomposition $\mathcal{H}^{1,1}(X,\mathbb{R}) = \mathbb{R}\cdot\omega_I \oplus \mathcal{H}^{1,1}(X,\mathbb{R})_\perp$ with respect to the Kähler form ω_I. Thus, for any $\alpha \in A$ we have that the form α is of type $(1,1)$ on the complex manifold (X,R) for each $R \in S$, where $S := \{aI + bJ + cK \mid a^2+b^2+c^2 = 1\}$. Let $\mathcal{K}_X^\perp := \widetilde{\mathcal{K}}_X^{2n-1} \cap \mathcal{H}^{1,1}(X,\mathbb{R})_\perp$. Then by Proposition 3.1, $\mathcal{K}_X^\perp$ is an open subset in the space $\mathcal{H}^{1,1}(X,\mathbb{R})_\perp$. Since $\mathcal{K}_X^\perp \subset A \subset \mathcal{H}^{1,1}(X,\mathbb{R})_\perp$, we obtain

$$\dim A = h^{1,1}(X) - 1, \quad \text{where } h^{1,1}(X) := \dim H^{1,1}(X).$$

Moreover, the set A is an open convex cone in the space $\mathcal{H}^{1,1}(X,\mathbb{R})_\perp$.

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