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PROJECTIVELY FLAT FINSLER SPACE WITH AN APPROXIMATE MATSUMOTO METRIC

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Abstract. The Matsumoto metric is an (α,β)-metric which is an exact formulation of the model of Finsler space. Lately, this metric was expressed as an infinite series form for $|\beta| < |\alpha|$ by the first author. He introduced an approximate Matsumoto metric as the (α,β)-metric of finite series form and investigated it in [11]. The purpose of the present paper is devoted to finding the condition for a Finsler space with an approximate Matsumoto metric to be projectively flat.

1. Introduction

A Finsler metric function L in a differentiable manifold M is called an (α,β)-metric, if L is a positively homogeneous function of degree one of a Riemannian metric $\alpha = (a_{ij}y^iy^j)^{1/2}$ and a non-vanishing 1-form $\beta = b_iy^i$ on M. The Matsumoto metric is an interesting (α,β)-metric which is an exact formulation of the model of Finsler space. This metric was introduced physically by using the gradient of slope, speed and gravity [5]. But this metric is expressed as an infinite series form for $|\beta| < |\alpha|$. The first author introduced an approximate Matsumoto metric as the r-th finite series (α,β)-metric form and investigated it in [11].

A change $L \longrightarrow \overline{L}$ of a Finsler metric on a same underlying manifold M is called *projective*, if any geodesic in (M, L) remains to be a geodesic in $(M, \overline{L})$ and vice versa. A Finsler space is called *projectively flat* if it is projective to a locally Minkowski space. The condition for a Finsler

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space with (α, β)-metric to be projectively flat was studied by M. Matsumoto [6]. Aikou, Hashiguchi and Yamauchi [2] give interesting results on the projective flatness of Matsumoto space.

The purpose of the present paper is to consider the projective flatness of Finsler space with an approximate Matsumoto metric.

2. Preliminaries

In a Finsler space (M, L), the metric

$$L(\alpha,\beta) = \alpha\left\{\sum_{k=0}^{r}\left(\frac{\beta}{\alpha}\right)^k\right\} \tag{2.1}$$

is called an *approximate Matsumoto metric.* The Matsumoto metric is expressed as

$$\lim_{r\to\infty} \alpha\left\{\sum_{k=0}^{r}\left(\frac{\beta}{\alpha}\right)^k\right\} = \frac{\alpha^2}{\alpha-\beta}$$

for $|\beta| < |\alpha|$ in (2.1). If $r = 0$, then $L = \alpha$ is a Riemannian metric. If $r = 1$, then $L = \alpha + \beta$ is a Randers metric. The condition for a Randers space to be projectively flat was given by Hashiguchi-Ichijō [3], and M. Matsumoto [6]. Therefore in this paper, we suppose that $r > 1$.

Let $\gamma_j{}^i{}_k$ be the Christoffel symbols with respect to α and denote by (;) the covariant differentiation with respect to $\gamma_j{}^i{}_k$. From the differential 1-form $\beta(x, y) = b_i(x)y^i$ we define

$$2r_{ij} = b_{i;j} + b_{j;i}, \quad 2s_{ij} = b_{i;j} - b_{j;i} = (\partial_j b_i - \partial_i b_j),$$
$$s^i_j = a^{ir}s_{rj}, \quad b^i = a^{ir}b_r, \quad b^2 = a^{rs}b_r b_s.$$

We shall denote the homogeneous polynomials in (y^i) of degree r by $hp(r)$ for brevity and the subscription 0 means contraction by y^i, for instance, $t_0 = t_i y^i$. In the following we denote $L_\alpha = \partial_\alpha L$, $L_\beta = \partial_\beta L$, $L_{\alpha\alpha} = \partial_\alpha\partial_\alpha L$.

Now the following Matsumoto's theorem [6] is well-known.

THEOREM M. *A Finsler space (M, L) with an (α, β)-metric $L(\alpha, \beta)$ is projectively flat if and only if for any point of space M there exist local coordinate neighborhoods containing the point such that $\gamma_j{}^i{}_k$ satisfies:*

$$\begin{aligned}(\gamma_0{}^i{}_0 - \gamma_{000}y^i/\alpha^2)/2 &+ (\alpha L_\beta/L_\alpha)s^i_0 \\ &+ (L_{\alpha\alpha}/L_\alpha)(C + \alpha r_{00}/2\beta)(\alpha^2 b^i/\beta - y^i) = 0,\end{aligned} \tag{2.2}$$

where C is given by

$$C + (\alpha^2 L_\beta/\beta L_\alpha)s_0 + (\alpha L_{\alpha\alpha}/\beta^2 L_\alpha)(\alpha^2 b^2 - \beta^2)(C + \alpha r_{00}/2\beta) = 0. \tag{2.3}$$

The equation (2.3) is rewritten in the form

$$(C + \alpha r_{00}/2\beta)\{1 + (\alpha L_{\alpha\alpha}/\beta^2 L_\alpha)(\alpha^2 b^2 - \beta^2)\} - (\alpha/2\beta)\{r_{00} - (2\alpha L_\beta/L_\alpha)s_0\} = 0, \tag{2.4}$$

that is,

$$C + \alpha r_{00}/2\beta = \frac{\alpha\beta(r_{00}L_\alpha - 2\alpha L_\beta s_0)}{2\{\beta^2 L_\alpha + \alpha L_{\alpha\alpha}(\alpha^2 b^2 - \beta^2)\}}.$$

Therefore (2.2) leads us to

$$\{L_\alpha(\alpha^2 \gamma_0{}^i{}_0 - \gamma_{000} y^i) + 2\alpha^3 L_\beta s^i{}_0\}\{\beta^2 L_\alpha + \alpha L_{\alpha\alpha}(\alpha^2 b^2 - \beta^2)\} + \alpha^3 L_{\alpha\alpha}(r_{00}L_\alpha - 2\alpha L_\beta s_0)(\alpha^2 b^i - \beta y^i) = 0. \tag{2.5}$$

3. Projectively flat space

In the n-dimensional Finsler space $F^n = (M, L)$ with the r-th $(r > 1)$ approximate Matsumoto metric (2.1), we have

$$L_\alpha = -\sum_{k=0}^{r}(k-1)\left(\frac{\beta}{\alpha}\right)^k, \quad L_\beta = \sum_{k=0}^{r} k\left(\frac{\beta}{\alpha}\right)^{k-1}, \quad L_{\alpha\alpha} = \frac{1}{\alpha}\sum_{k=0}^{r}(k-1)k\left(\frac{\beta}{\alpha}\right)^k. \tag{3.1}$$

Substituting (3.1) into (2.5), we have

$$\begin{aligned}&\left\{\sum_{k=0}^{r}(k-1)\left(\frac{\beta}{\alpha}\right)^k(\alpha^2\gamma_0{}^i{}_0 - \gamma_{000}y^i) - 2\alpha^3 s^i{}_0\sum_{k=0}^{r}k\left(\frac{\beta}{\alpha}\right)^{k-1}\right\}\\ &\times\left\{\beta^2\sum_{k=0}^{r}(k-1)\left(\frac{\beta}{\alpha}\right)^k - (\alpha^2 b^2 - \beta^2)\sum_{k=0}^{r}(k-1)k\left(\frac{\beta}{\alpha}\right)^k\right\}\\ &-\alpha^2\sum_{k=0}^{r}(k-1)k\left(\frac{\beta}{\alpha}\right)^k\left\{r_{00}\sum_{k=0}^{r}(k-1)\left(\frac{\beta}{\alpha}\right)^k + 2\alpha s_0\sum_{k=0}^{r}k\left(\frac{\beta}{\alpha}\right)^{k-1}\right\}\\ &\times(\alpha^2 b^i - \beta y^i) = 0.\end{aligned} \tag{3.2}$$

We shall divide our consideration in two cases of which r is even or odd

(i) Case of $r = 2h$ (h is a positive integer).
When $r = 2h$, we have

$$
\begin{aligned}
\sum_{k=0}^{r}(k-1)\left(\frac{\beta}{\alpha}\right)^{k} &= \frac{1}{\alpha^{2h}}\sum_{k=0}^{2h}(2h-k-1)\alpha^{k}\beta^{2h-k}, \\
\sum_{k=0}^{r}k\left(\frac{\beta}{\alpha}\right)^{k-1} &= \frac{\alpha}{\alpha^{2h}}\sum_{k=0}^{2h}(2h-k)\alpha^{k}\beta^{2h-k-1}, \\
\sum_{k=0}^{r}(k-1)k\left(\frac{\beta}{\alpha}\right)^{k} &= \frac{1}{\alpha^{2h}}\sum_{k=0}^{2h}(2h-k-1)(2h-k)\alpha^{k}\beta^{2h-k}.
\end{aligned}
\tag{3.3}
$$

Separating the rational and irrational parts in y^i, we have
(3.4)

$$
\begin{aligned}
&\sum_{k=0}^{2h}(2h-k-1)\alpha^{k}\beta^{2h-k} \\
&= \sum_{k=0}^{h}(2h-2k-1)\alpha^{2k}\beta^{2h-2k} + \alpha\sum_{k=0}^{h-1}(2h-2k-2)\alpha^{2k}\beta^{2h-2k-1} \\
&= A + \alpha B, \\
&\sum_{k=0}^{2h}(2h-k-1)(2h-k)\alpha^{k}\beta^{2h-k} = D + \alpha E, \\
&\sum_{k=0}^{2h}(2h-k)\alpha^{k}\beta^{2h-k-1} = F + \alpha G,
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \sum_{k=0}^{h}(2h-2k-1)\alpha^{2k}\beta^{2h-2k}, \\
B &= \sum_{k=0}^{h-1}(2h-2k-2)\alpha^{2k}\beta^{2h-2k-1}, \\
D &= \sum_{k=0}^{h}(2h-2k-1)(2h-2k)\alpha^{2k}\beta^{2h-2k},
\end{aligned}
$$

$$E=\sum_{k=0}^{h-1}(2h-2k-2)(2h-2k-1)\alpha^{2k}\beta^{2h-2k-1},$$
$$F=\sum_{k=0}^{h}(2h-2k)\alpha^{2k}\beta^{2h-2k-1},$$
$$G=\sum_{k=0}^{h-1}(2h-2k-1)\alpha^{2k}\beta^{2h-2k-2}.$$

Substituting (3.3) and (3.4) into (3.2), we have

$$\begin{aligned}&(\alpha^2\gamma_0{}^i{}_0-\gamma_{000}y^i)\{\beta^2(A^2+2\alpha AB+\alpha^2B^2)\\&-(\alpha^2b^2-\beta^2)(AD+\alpha(AE+BD)+\alpha^2BE)\}\\&-2\alpha^4s^i{}_0\{\beta^2(AF+\alpha(BF+AG)+\alpha^2BG)\\&-(\alpha^2b^2-\beta^2)(FD+\alpha(FE+GD)+\alpha^2GE)\}\\&-\alpha^2(\alpha^2b^i-\beta y^i)\{r_{00}(AD+\alpha(AE+BD)+\alpha^2BE)\\&+2\alpha^2s_0(FD+\alpha(FE+GD)+\alpha^2GE)\}=0,\end{aligned}$$

that is,

$$P+\alpha Q=0,$$

where

$$\begin{aligned}P&=(\alpha^2\gamma_0{}^i{}_0-\gamma_{000}y^i)\{\beta^2(A^2+\alpha^2B^2)-(\alpha^2b^2-\beta^2)(AD+\alpha^2BE)\}\\&\quad-2\alpha^4s^i{}_0\{\beta^2(AF+\alpha^2BG)-(\alpha^2b^2-\beta^2)(FD+\alpha^2GE)\}\\&\quad-\alpha^2(\alpha^2b^i-\beta y^i)\{r_{00}(AD+\alpha^2BE)+2\alpha^2s_0(FD+\alpha^2GE)\},\\Q&=(\alpha^2\gamma_0{}^i{}_0-\gamma_{000}y^i)\{2\beta^2AB-(\alpha^2b^2-\beta^2)(AE+BD)\}\\&\quad-2\alpha^4s^i{}_0\{\beta^2(BF+AG)-(\alpha^2b^2-\beta^2)(FE+GD)\}\\&\quad-\alpha^2(\alpha^2b^i-\beta y^i)\{r_{00}(AE+BD)+2\alpha^2s_0(FE+GD)\}.\end{aligned}$$

Since P, Q are rational parts and α is an irrational part in y^i, $P=0$ and $Q=0$, that is,

(3.5)
$$\begin{aligned}&(\alpha^2\gamma_0{}^i{}_0-\gamma_{000}y^i)\{\beta^2(A^2+\alpha^2B^2)-(\alpha^2b^2-\beta^2)(AD+\alpha^2BE)\}\\&-2\alpha^4s^i{}_0\{\beta^2(AF+\alpha^2BG)-(\alpha^2b^2-\beta^2)(FD+\alpha^2GE)\}\\&-\alpha^2(\alpha^2b^i-\beta y^i)\{r_{00}(AD+\alpha^2BE)+2\alpha^2s_0(FD+\alpha^2GE)\}=0,\end{aligned}$$

$$
\begin{aligned}
&(\alpha^2\gamma_0{}^i{}_0-\gamma_{000}y^i)\{2\beta^2AB-(\alpha^2b^2-\beta^2)(AE+BD)\}\\
&-2\alpha^4s^i{}_0\{\beta^2(BF+AG)-(\alpha^2b^2-\beta^2)(FE+GD)\}\\
&-\alpha^2(\alpha^2b^i-\beta y^i)\{r_{00}(AE+BD)+2\alpha^2s_0(FE+GD)\}=0.
\end{aligned}
\tag{3.6}
$$

Eliminating $(\alpha^2\gamma_0{}^i{}_0-\gamma_{000}y^i)$ from (3.5) and (3.6), we have

$$
\begin{aligned}
&2\alpha^2s^i{}_0[-\{\beta^2(AF+\alpha^2BG)+(\alpha^2b^2-\beta^2)(FD+\alpha^2GE)\}\\
&\times\{2\beta^2AB-(\alpha^2b^2-\beta^2)(AE+BD)\}\\
&+\{\beta^2(BF+AG)-(\alpha^2b^2-\beta^2)(FE+GD)\}\\
&\times\{\beta^2(A^2+\alpha^2B^2)-(\alpha^2b^2-\beta^2)(AD+\alpha^2BE)\}]\\
&-(\alpha^2b^i-\beta y^i)[\{r_{00}(AD+\alpha^2BE)+2\alpha^2s_0(FD+\alpha^2GE)\}\\
&\times\{2\beta^2AB-(\alpha^2b^2-\beta^2)(AE+BD)\}\\
&-\{r_{00}(AE+BD)+2\alpha^2s_0(FE+GD)\}\\
&\times\{\beta^2(A^2+\alpha^2B^2)-(\alpha^2b^2-\beta^2)(AD+\alpha^2BE)\}]=0.
\end{aligned}
\tag{3.7}
$$

Transvecting (3.7) by b_i, we have

(3.8)

$$
\begin{aligned}
&2\alpha^2s_0[-(AF+\alpha^2BG)\{2\beta^2AB-(\alpha^2b^2-\beta^2)(AE+BD)\}\\
&+(BF+AG)\{\beta^2(A^2+\alpha^2B^2)-(\alpha^2b^2-\beta^2)(AD+\alpha^2BE)\}]\\
&-r_{00}(\alpha^2b^2-\beta^2)\{2(AD+\alpha^2BE)AB-(AE+BD)(A^2+\alpha^2B^2)\}=0.
\end{aligned}
$$

The term of (3.8) which does not contain α^2 is $2(2h-1)^4r_{00}\beta^{8h+1}$. Therefore there exists $hp(8h+1):V_{8h+1}$ such that

$$
2(2h-1)^4r_{00}\beta^{8h+1}=\alpha^2V_{8h+1}. \tag{3.9}
$$

We suppose that $\alpha^2\not\equiv 0 \pmod{\beta}$. In this case, there exists from (3.9) a function $k=k(x)$ satisfying $V_{8h+1}=k\beta^{8h+1}$, and hence

$$
r_{00}=\lambda\alpha^2, \tag{3.10}
$$

where $\lambda=k/2(2h-1)^4$. Substituting (3.10) into (3.8), we have

(3.11)

$$
\begin{aligned}
&2s_0[-(AF+\alpha^2BG)\{2\beta^2AB-(\alpha^2b^2-\beta^2)(AE+BD)\}\\
&+(BF+AG)\{\beta^2(A^2+\alpha^2B^2)-(\alpha^2b^2-\beta^2)(AD+\alpha^2BE)\}]\\
&-\lambda(\alpha^2b^2-\beta^2)\{2(AD+\alpha^2BE)AB-(AE+BD)(A^2+\alpha^2B^2)\}=0.
\end{aligned}
$$

It is observed from (3.11) that $(c_1 s_0 + c_2\lambda\beta)\beta^{8h+2}$ must have a factor α^2, that is,

$$(c_1 s_0 + c_2\lambda\beta)\beta^{8h+2} = \alpha^2 W_{8h+1},$$

where $c_1 = (2h-1)^2(8h^2-6h+1)$, $c_2 = (2h-1)^2(2h-2)$. Since $\alpha^2 \not\equiv 0 \pmod{\beta}$, $c_1 s_0 + c_2\lambda\beta = 0$, that is, $c_1 s_i + c_2\lambda b_i = 0$. Transvecting this by b^i, we have $c_2\lambda b^2 = 0$.

(a) If $c_2 = 0$, that is, $h = 1$, then

$$A = \beta^2 - \alpha^2,\ \ B = 0,\ \ D = 2\beta^2,\ \ E = 0,\ \ F = 2\beta,\ \ G = 1.$$

Hence (3.5) and (3.6) is written as

$$\begin{aligned}(3.12)\qquad &(\alpha^2\gamma_0{}^i{}_0 - \gamma_{000}y^i)\{(1+2b^2)\alpha^4 - 2(2+b^2)\alpha^2\beta^2 + 3\beta^4\}\\ &+ 4\alpha^4 s^i{}_0\{(1+2b^2)\alpha^2\beta - 3\beta^3\}\\ &+ 2\alpha^2(\alpha^2 b^i - \beta y^i)(r_{00}\alpha^2 - 4\alpha^2\beta s_0 - r_{00}\beta^2) = 0,\end{aligned}$$

$$(3.13)\qquad s^i{}_0(3\beta^2 - (1+2b^2)\alpha^2) + 2s_0(\alpha^2 b^i - \beta y^i) = 0.$$

Transvecting (3.13) by b_i, we have $s_0(\beta^2 - \alpha^2) = 0$. Since $\beta^2 - \alpha^2 \neq 0$, we get $s_0 = 0$. Substituting this into (3.13), we have

$$s^i{}_0\{(\beta^2 - \alpha^2) - 2(\alpha^2 b^2 - \beta^2)\} = 0,$$

from which $s^i{}_0 = 0$, that is, $s_{ij} = 0$. The term which does not contain α^2 in (3.12) is $-3\gamma_{000}y^i\beta^4$. Therefore there exists $hp(1) : \mu_0 = \mu_i(x)y^i$ such that

$$(3.14)\qquad \gamma_{000} = \mu_0\alpha^2.$$

Substituting $s^i{}_0 = 0$, $s_0 = 0$ and (3.14) into (3.12), we have

$$(3.15)\qquad \{(1+2b^2)\alpha^2 - 3\beta^2\}(\gamma_0{}^i{}_0 - \mu_0 y^i) + 2r_{00}(\alpha^2 b^i - \beta y^i) = 0.$$

The terms of $-3\beta^2(\gamma_0{}^i{}_0 - \mu_0 y^i) - 2r_{00}\beta y^i$ of (3.15) must contain the factor α^2. Hence there exists 1-forms $\nu^i{}_0 = \nu^i{}_j(x)y^j$ such that

$$(3.16)\qquad 3\beta(\gamma_0{}^i{}_0 - \mu_0 y^i) + 2r_{00}y^i = \nu^i{}_0\alpha^2.$$

Transvecting (3.16) by y_i, we have

$$2r_{00} = \nu^i{}_0 y_i. \tag{3.17}$$

On the other hand, (3.15) is rewritten as the form

$$\alpha^2\{(1+2b^2)(\gamma_0{}^i{}_0 - \mu_0 y^i) + 2r_{00}b^i\} = \beta\{3\beta(\gamma_0{}^i{}_0 - \mu_0 y^i) + 2r_{00}y^i\},$$

from which it is reduced to

$$(1+2b^2)(\gamma_0{}^i{}_0 - \mu_0 y^i) + 2r_{00}b^i = \beta\nu^i{}_0 \tag{3.18}$$

by virtue of (3.16).

Substituting (3.17) into (3.18), we get

$$(1+2b^2)(\gamma_0{}^i{}_0 - \mu_0 y^i) = \beta\nu^i{}_0 - \nu_{00}b^i, \tag{3.19}$$

where $\nu_{ij} = a_{ir}\nu^r{}_j$. From (3.15) and (3.19) we have

$$\nu_{i0}\{(1+2b^2)\alpha^2 - 3\beta^2\} = \nu_{00}\{(1+2b^2)y_i - 3\beta b_i\},$$

from which

$$\begin{aligned} &\nu_{ij}\{(1+2b^2)a_{kh} - 3b_k b_h\} + (jkh) \\ &= \nu_{jk}\{(1+2b^2)a_{ih} - 3b_i b_h\} + (jkh), \end{aligned} \tag{3.20}$$

where (jkh) denotes the terms obtained from preceding terms by cyclic permutation of indices j, k, h. It is easy to show that the tensor $(1 + 2b^2)a_{ij} - 3b_i b_j$ has reciprocal

$$M^{ij} = \{a^{ij} + 3b^i b^j/(1-b^2)\}/(1+2b^2).$$

Transvecting (3.20) by M^{hk}, we get

$$\nu_{ij} = M\{(1+2b^2)a_{ij} - 3b_i b_j\}, \tag{3.21}$$

where $M = M^{hk}\nu_{hk}/n$. Therefore, from (3.17) we have

$$r_{ij} = \frac{1}{2}M\{(1+2b^2)a_{ij} - 3b_i b_j\}. \tag{3.22}$$

Hence, we have

$$b_{i;j} = \frac{1}{2}M\{(1+2b^2)a_{ij} - 3b_ib_j\}. \tag{3.23}$$

Next, from (3.21) the equation (3.19) is reduced in the form

$$\gamma_0{}^i{}_0 - \mu_0 y^i = M(\beta y^i - \alpha^2 b^i), \tag{3.24}$$

that is,

$$\gamma_j{}^i{}_k = \frac{1}{2}\{(\mu_j + Mb_j)\delta^i_k + (\mu_k + Mb_k)\delta^i_j\} - Ma_{jk}b^i. \tag{3.25}$$

Conversely, it is easily verified that (3.12) and (3.13) are consequences of (3.23) and (3.25).

(b) For $h > 1$, $\lambda = 0$ or $b^2 = 0$.

First, if $\lambda = 0$, then $s_i = 0$ and $r_{00} = 0$ from (3.10). Therefore, from (3.7) we have

$$\begin{aligned} &2\alpha^2 s^i{}_0[-\{\beta^2(AF+\alpha^2BG) + (\alpha^2b^2-\beta^2)(FD+\alpha^2GE)\} \\ &\times\{2\beta^2AB - (\alpha^2b^2-\beta^2)(AE+BD)\} \\ &+\{\beta^2(BF+AG) - (\alpha^2b^2-\beta^2)(FE+GD)\} \\ &\times\{\beta^2(A^2+\alpha^2B^2) - (\alpha^2b^2-\beta^2)(AD+\alpha^2BE)\}] = 0. \end{aligned} \tag{3.26}$$

The term which does not contain α^2 is $(2h-1)^2(64h^4+48h^3-14h^2+4h+1)s^i{}_0\beta^{8h+2}$. Therefore there exists $hp(8h+1) : U_{8h+1}$ such that

$$(2h-1)^2(64h^4+48h^3-14h^2+4h+1)s^i{}_0\beta^{8h+2} = \alpha^2 U_{8h+1}.$$

Hence $s^i{}_0 = 0$, that is, $s_{ij} = 0$. From this and $r_{ij} = 0$ we have

$$b_{i;j} = 0. \tag{3.27}$$

Substituting $s^i{}_0 = 0$, $r_{ij} = 0$ and $s_0 = 0$ into (3.5), we must have $hp(1) : \sigma_0 = \sigma_i(x)y^i$ satisfying $\gamma_{000} = \sigma_0\alpha^2$. Therefore $\gamma_0{}^i{}_0 = \sigma_0 y^i$, that is,

$$2\gamma_j{}^i{}_k = \sigma_j\delta^i_k + \sigma_k\delta^i_j, \tag{3.28}$$

which shows that the associated Riemannian space is projectively flat.

Conversely it is easy to see that (3.2) is a consequence of (3.27) and (3.28).

Secondly, if $b^2 = 0$, then (3.11) is reduced to

$$
\begin{aligned}
&2s_0\{-(AF+\alpha^2BG)(2AB+AE+BD)\\
&+(BF+AG)(A^2+\alpha^2B^2+AD+\alpha^2BE)\}\\
&+\lambda\{2(AD+\alpha^2BE)AB-(AE+BD)(A^2+\alpha^2B^2)\}=0,
\end{aligned}
\tag{3.29}
$$

from which

$$2s_0\alpha^{8h-2} = \beta^3 U_{8h-4},$$

where U_{8h-4} is $hp(8h-4)$. Therefore $s_0 = 0$, and hence $\lambda = 0$. Thus we obtain (3.27) and (3.28).

(ii) Case of $r = 2h+1$ (h is a positive integer).

In this case, we have

(3.30)

$$
\begin{aligned}
\sum_{k=0}^{r}(k-1)\left(\frac{\beta}{\alpha}\right)^k &= \frac{1}{\alpha^{2h+1}}\sum_{k=0}^{2h+1}(2h-k)\alpha^k\beta^{2h-k+1}\\
&= \frac{1}{\alpha^{2h+1}}(H+\alpha A),\\
\sum_{k=0}^{r}k\left(\frac{\beta}{\alpha}\right)^{k-1} &= \frac{\alpha}{\alpha^{2h}}\sum_{k=0}^{2h+1}(2h-k+1)\alpha^k\beta^{2h-k}\\
&= \frac{\alpha}{\alpha^{2h}}(J+\alpha F),\\
\sum_{k=0}^{r}(k-1)k\left(\frac{\beta}{\alpha}\right)^k &= \frac{1}{\alpha^{2h+1}}\sum_{k=0}^{2h+1}(2h-k)(2h-k+1)\alpha^k\beta^{2h-k+1}\\
&= \frac{1}{\alpha^{2h+1}}(K+\alpha D),
\end{aligned}
$$

where

$$H=\sum_{k=0}^{h}(2h-2k)\alpha^{2k}\beta^{2h-2k+1},\quad J=\sum_{k=0}^{h}(2h-2k+1)\alpha^{2k}\beta^{2h-2k},$$

$$K=\sum_{k=0}^{h}(2h-2k)(2h-2k+1)\alpha^{2k}\beta^{2h-2k+1}.$$

Therefore (3.2) is written as

$$\begin{aligned}(3.31)\qquad &\{(\alpha^2\gamma_0{}^i{}_0-\gamma_{000}y^i)(H+\alpha A)-2\alpha^4 s^i{}_0(J+\alpha F)\}\\ &\times\{\beta^2(H+\alpha A)-(\alpha^2b^2-\beta^2)(K+\alpha D)\}\\ &-\alpha^2(K+\alpha D)\{r_{00}(H+\alpha A)+2\alpha^2 s_0(J+\alpha F)\}\\ &\times(\alpha^2b^i-\beta y^i)=0.\end{aligned}$$

Separating the rational and irrational parts in y^i, we have

$$P'+\alpha Q'=0,$$

where

(3.32)

$$\begin{aligned}P'=&(\alpha^2\gamma_0{}^i{}_0-\gamma_{000}y^i)\{\beta^2(H^2+\alpha^2A^2)+(HK+\alpha^2AD)(\alpha^2b^2-\beta^2)\}\\ &-2\alpha^4 s^i{}_0\{\beta^2(JH+\alpha^2FA)-(JK+\alpha^2FD)(\alpha^2b^2-\beta^2)\}\\ &-\alpha^2\{r_{00}(KH+\alpha^2DA)+2\alpha^2 s_0(KJ+\alpha^2DF)\}(\alpha^2b^i-\beta y^i)\\ =&0,\end{aligned}$$

(3.33)

$$\begin{aligned}Q'=&(\alpha^2\gamma_0{}^i{}_0-\gamma_{000}y^i)\{2\beta^2AH+(HD+AK)(\alpha^2b^2-\beta^2)\}\\ &-2\alpha^4 s^i{}_0\{\beta^2(FH+JA)-(JD+FK)(\alpha^2b^2-\beta^2)\}\\ &-\alpha^2\{r_{00}(KA+DH)+2\alpha^2 s_0(DJ+KF)\}(\alpha^2b^i-\beta y^i)=0.\end{aligned}$$

From (3.32) we have $8h^3\gamma_{000}y^i\beta^{4h+4}=\alpha^2V_{4h+6}$, where V_{4h+6} is a $hp(4h+6)$. Therefore there exists $hp(1): v_0$ satisfying

$$\gamma_{000}=v_0\alpha^2. \qquad (3.34)$$

Next, eliminating $(\alpha^2\gamma_0{}^i{}_0-\gamma_{000}y^i)$ from (3.32) and (3.33), we have

$$\begin{aligned}(3.35)\qquad &2\alpha^2 s^i{}_0[\{\beta^2(JH+\alpha^2FA)-(JK+\alpha^2FD)(\alpha^2b^2-\beta^2)\}\\ &\times\{2\beta^2AH+(HD+AK)(\alpha^2b^2-\beta^2)\}\\ &-\{\beta^2(FH+JA)-(JD+FK)(\alpha^2b^2-\beta^2)\}\\ &\times\{\beta^2(H^2+\alpha^2A^2)+(HK+\alpha^2AD)(\alpha^2b^2-\beta^2)\}\\ &+\{r_{00}(KH+\alpha^2DA)+2\alpha^2 s_0(KJ+\alpha^2DF)\}(\alpha^2b^i-\beta y^i)\\ &\times\{2\beta^2AH+(HD+AK)(\alpha^2b^2-\beta^2)\}\\ &-\{r_{00}(KA+DH)+2\alpha^2 s_0(DJ+KF)\}(\alpha^2b^i-\beta y^i)\\ &\times\{\beta^2(H^2+\alpha^2A^2)+(HK+\alpha^2AD)(\alpha^2b^2-\beta^2)\}]=0.\end{aligned}$$

The term of (3.35) which does not contain α^2 is $8h^3(2h-1)(16h^2+4h-1)r_{00}\beta^{8h+6}y^i$. Therefore there exists a function $\rho = \rho(x)$ such that

$$r_{00} = \rho\alpha^2. \tag{3.36}$$

Substituting (3.36) into (3.35) and transvecting it by b_i, we have

(3.37)
$$\begin{aligned}&2s_0[(JH+\alpha^2FA)\{2\beta^2AH+(HD+AK)(\alpha^2b^2-\beta^2)\}\\&-(FH+JA)\{\beta^2(H^2+\alpha^2A^2)+(HK+\alpha^2AD)(\alpha^2b^2-\beta^2)\}]\\&+\rho\{2(KH+\alpha^2DA)AH-(KA+DH)(H^2-\alpha^2A^2)\}(\alpha^2b^2-\beta^2)\\&=0.\end{aligned}$$

The term of (3.37) which does not contain α^2 is $(c'_1s_0 + c'_2\rho\beta)\beta^{8h+4}$, where $c'_1 = 8h^2(4h+1)(32h^4-12h^2+8h-1)$, $c'_2 = -8h^3(2h-1)$. Therefore $c'_1s_0 + c'_2\rho\beta = 0$, that is, $c'_1s_i + c'_2\rho b_i = 0$. Transvecting this equation by b_i, we have $c'_2\rho b^2 = 0$. Since $c'_2 \neq 0$ for a positive integer, $\rho = 0$ or $b^2 = 0$.

First, if $\rho = 0$, then $s_0 = 0$, that is, $s_i = 0$ and $r_{00} = 0$ from (3.36). Therefore we have from ${s^i}_0 = 0$, that is, $s_{ij} = 0$. Hence $b_{i;j} = 0$. Substituting $b_{i;j} = 0$ and (3.34) into (3.32) or (3.33), we have ${\gamma_0}^i{}_0 = v_0y^i$, that is, the associated Riemannian space is projectively flat. Secondly, if $b^2 = 0$, we have easily the above result by the same method of the case of $r = 2h$. Thus we have the following

THEOREM 3.1. *A Finsler space F^n with the approximate Matsumoto metric* (2.1) *provided $\alpha^2 \not\equiv 0$ (mod β) is projectively flat if and only if*

(i) *when $r = 2$, $b_{i;j}$ satisfies* (3.23) *and the Chrisroffel symbols of the associated Riemannian space are written in the form* (3.25),

(ii) *when $r > 2$, $b_{i;j} = 0$ and the associated Riemannian space is projectively flat.*

REMARK. In the case of $r = 2h+1$, if $h = 0$, then $H = 0$, $J = 1$, $A = -1$, $F = 0$, $D = 0$. Therefore (3.31) is reduced to

$$\alpha^2{\gamma_0}^i{}_0 - \gamma_{000}y^i + 2\alpha^3{s^i}_0 = 0.$$

This equation coincides with one in [6].

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