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L_P SPACES STRUCTURE OF THE BANACH ENVELOPE OF $WEAKL^1$

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ABSTRACT. The Banach envelope of $WeakL_1$ contains a complemented Banach sublattice that is isometrically isomorphic to $L_p(\mu)$ space.

1. Introduction

In this paper we will show that the Banach envelope of $WeakL_1$ (denoted $wL_{\hat{1}}$) contains a complemented Banach sublattice that is isometrically isomorphic to $L_p(\mu)$ space where μ is a separable probability measure. For $1 < p < \infty$, we can find a lattice isometry $T : L_p(\mu) \rightarrow wL_{\hat{1}}$ such that the range of T is a complemented subspace of the Banach envelope of $WeakL_1$. In [7, Theorem 3.7], J. Kupka and T. Peck proved that there exists a lattice isometry T from L_1 into $wL_{\hat{1}}$ such that the range of T is a complemented subspace of $wL_{\hat{1}}$. In [11, Theorem 1], T. Peck and M. Talagrand proved that if E is a separable Banach lattice with order continuous norm, then there is a lattice isometry of E into $wL_{\hat{1}}$. Since L_p $(1 < p < \infty)$ is also a sublattice of $wL_{\hat{1}}$, naturally we can ask that L_p is also a complemented sublattice of $wL_{\hat{1}}$. We will give answer for this question. The main result of this paper is the extension of J. Kupka and T. Peck's theorem 3.9 in [7].

The space $WeakL_1$, as a Lorentz space $L(1, \infty)$, was introduced in analysis when key operators of harmonic analysis did not map L_1 into L_1. As examples of such operators, one can give the Hardy-Littlewood maximal function and the Hilbert transform. It became natural to investigate $WeakL_1$, the space of measurable functions f satisfying $\mu(\{x \in \Omega : |f(x)| > y\}) \leq \frac{c}{y}$, from these important operators in analysis.

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It is known that (except for some trivial measure space), $WeakL_1$ is not normable (see [1]). The question therefore arose as to whether any nontrivial continuous linear functionals on $WeakL_1$ exists. In [1, Theorem 6], the answer for this question was observed. This implies $WeakL_1$ has a nontrivial dual space. In [7], J. Kupka and T. Peck studied the structure of $WeakL_1$. They showed that the space L_∞ is dense in the dual of $WeakL_1$ with $weak^*$-topology and showed lattice embeddings of L_1, $l_1[0,1]$, l_∞ and $c_0[0,1]$ into $wL_{\hat{1}}$ where $wL_{\hat{1}}$ is the Banach envelope of $WeakL_1$. Later on, T. Peck and M. Talagrand proved that every separable order continuous Banach lattice is lattice isometric to a sublattice of $wL_{\hat{1}}$ in [11, Theorem 1]. Finally, H. Lotz and T. Peck removed the hypothesis of order continuity in the separable case, in [10, Theorem 2].

As a Lorentz space, we'll study the space $L(1,\infty)$ which is called $WeakL_1$ (denote wL_1).

$$wL_1 = \{f \in L_0 : \mu(\{x \in \Omega : |f(x)| > y\}) < \frac{c}{y}\}, \tag{1.1}$$

where $c > 0$ is independent of $y > 0$. As we mentioned , wL_1 is not normable, but we can find nontrivial linear functionals on wL_1. This was first observed by M. Cwikel and Y. Sagher in [1, Theorem 6].

In [3], if μ is nonatomic, then we can get an equivalent integral-like seminorm

$$\|f\|_{wL_{\hat{1}}} = \lim_{n\to\infty} \sup_{\frac{q}{p}\geq n} \frac{1}{\ln\frac{q}{p}} \int_{\{p\leq|f|\leq q\}} |f| d\mu. \tag{1.2}$$

Later on, in [4] actually the Banach envelope seminorm on wL_1 was calculated to be exactly as above. Note that the seminorm on wL_1 defined in (1.2) is a lattice seminorm. This is not quite obvious, but using integration by parts one can readily show that the seminorm $\|\cdot\|_{wL_{\hat{1}}}$ is exactly same as (see [7, 1.5])

$$\lim_{n\to\infty} \sup_{\frac{q}{p}\geq n} \frac{1}{\ln\frac{q}{p}} \int_p^q \mu(\{x \in \mu : |f(x)| > t\}) dt. \tag{1.3}$$

Even though wL_1 is complete with respect to the quasinorm $q(f) = \sup_{a>0} a\ \mu\{x \in \Omega : |f(x)| > a\}$, it is not complete with respect to the seminorm $\|\cdot\|_{wL_{\hat{1}}}$. This is due to M. Cwikel and C. Feffreman in [3] and also we can see this in [7, 1.4]. Let $\mathcal{N} = \{f \in wL_1 : \|f\|_{wL_{\hat{1}}} = 0\}$. Then we obtain the quotient space $wL_1/\mathcal{N}$. We define $wL_{\hat{1}}$ as the *normed envelope* (and its completion as the *Banach envelope*) of wL_1.

2. L_p space structure in $wL_{\hat{1}}$

To study this subject, we need some basic facts about the dual of $wL_{\hat{1}}$. We would like to change nonlinear limit superior expression (1.4) for $\|\cdot\|_{wL_{\hat{1}}}$ into a linear expression by directing the number $I_a^b(f) = \frac{1}{\ln\frac{b}{a}}\int_{\{a\leq|f|\leq b\}}|f|d\mu$ in some fashion. By [4, Section 1], we can define (1.4) as

$$\|f\|_{wL_{\hat{1}}} = \lim_{n\to\infty}(\sup\{I_a^b(f) : b/a \geq n\}). \tag{2.1}$$

For this, we introduce an ultrafilter $\mathcal{U}$ so that the limit of the I_a^b along $\mathcal{U}$ will determine a canonical integral-like linear functional $I_{\mathcal{U}} \in wL_{\hat{1}}^*$.

We now construct an ultrafilter $\mathcal{U}$ (see [7, Section 2.1]). For $n = 1, 2, \cdots$, let $F_n = \{(a,b) : 1 \leq a < b, \frac{b}{a} \geq n\}$ and then define $\mathcal{F} = \{F_n : n \geq 1\}$. Treating $\mathcal{F}$ as a filter of subsets of the set $S = [1,\infty) \times [1,\infty)$, we obtain from Zorn's lemma an ultrafilter $\mathcal{U}$ of subsets of S such that $\mathcal{F} \subset \mathcal{U}$. From now, we'll fix the ultrafilter $\mathcal{F} \subset \mathcal{U}$. The significance of the ultrafilter property lies in the fact that for every function $f \in wL_1$, and for every integer n sufficiently large $\{I_a^b(f) : (a,b) \in F_n\}$ is bounded, so that the limit $l = \lim_{\mathcal{U}} I_a^b(f)$ always exists(for every $\epsilon > 0$, there is a set $U \in \mathcal{U}$ such that $|I_a^b(f) - l| < \epsilon$ whenever $(a,b) \in U$).

Define the "ersatz integral" $I_{\mathcal{U}}$ for every nonnegative function $f \in wL_1$ by $I_{\mathcal{U}}(f) = \lim_{\mathcal{U}} I_a^b(f)$. For more properties of $I_{\mathcal{U}}(f)$, refer to [7, 2.3 key lemma]. We define for an arbitrary function $f \in wL_1$ by $I_{\mathcal{U}}(f) = I_{\mathcal{U}}(f^+) - I_{\mathcal{U}}(f^-)$. Then we have $|I_{\mathcal{U}}(f)| \leq \|f\|_{wL_{\hat{1}}}$. Define $\|f\|_{\mathcal{U}} = I_{\mathcal{U}}(|f|)$. Note that (see [7, 2.12])

$$\|f\|_{\mathcal{U}} \leq \|f\|_{wL_{\hat{1}}}. \tag{2.2}$$

For the dual of wL_1(or $wL_{\hat{1}}$), we state the theorem which is due to J. Kupka and T. Peck in [7, 2.8].

THEOREM 2.1. *Define a linear operator $T_{\mathcal{U}} : L_\infty(\mu) \to wL_{\hat{1}}^*$ by $T_{\mathcal{U}}(m)(f) = I_{\mathcal{U}}(mf)$ for all $m \in L_\infty$, and for all $f \in wL_1$. Then $T_{\mathcal{U}}$ constitutes an isometric order isomorphism of $L_\infty(\mu)$ into $wL_{\hat{1}}^*$. Moreover, the linear span of the subspaces $T_{\mathcal{U}}(L_\infty(\mu))$, as $\mathcal{U}$ ranges over the collection of ultrafilter (of subset of S) which contain $\mathcal{F}$ constitutes a norming, and hence a weak* dense, subspace of $wL_{\hat{1}}^*$.*

The operator $T_{\mathcal{U}}$ of Theorem 2.1 determines an isometric order isomorphic embedding of $L_\infty(\mu)$ into $wL_1(\mathcal{U})^*$ where $wL_1(\mathcal{U}) = wL_1(\mathcal{U})/$

$N_{\mathcal{U}}$ and $N_{\mathcal{U}} = \{f \in wL_1 : \|f\|_{\mathcal{U}} = 0\}$. Moreover, the range of this embedding is norming, and hence *weak** dense in $wL_1(\mathcal{U})^*$.

Let $L(\mathcal{U}) = \{f \in wL_1 | \|f\|_{wL_{\hat{1}}} = \|f\|_{\mathcal{U}}\}$. Then $L(\mathcal{U})$ is a closed subset of $wL_{\hat{1}}$(see [6]) and if f is a $\frac{1}{x}$-like function, then $\|f\|_{wL_{\hat{1}}} = \|f\|_{\mathcal{U}} = I_{\mathcal{U}}(f)$.

LEMMA 2.2. *If $\phi \neq 0$ is a linear functional on $wL_1(\mathcal{U})$, then ϕ is a linear functional on $wL_{\hat{1}}$ with $\|\phi\| \neq 0$.*

PROOF. Let $\phi \neq 0$ be a linear functional on $wL_1(\mathcal{U})$. Then for any $f \in wL_{\hat{1}}$ with $\|f\|_{\mathcal{U}} > 0$ (since $f \in wL_{\hat{1}}$ is also regarded as $f \in wL_1(\mathcal{U})$).

$$\begin{aligned} 0 < |\phi(f)| &\leq \|\phi\| \|f\|_{\mathcal{U}} \\ &\leq \|\phi\| \|f\|_{wL_{\hat{1}}} \quad \text{by (2.2)}. \end{aligned}$$

Hence, $\|\phi\| \neq 0$ on $wL_{\hat{1}}$. This implies $\phi \neq 0$ is a linear functional on $wL_{\hat{1}}$. □

We now give a lemma about linear functionals on $wL_{\hat{1}}$ which is actually due to J. Kupka and T. Peck (see [7, 2.20]).

LEMMA 2.3. *For a ultrafilter $\mathcal{U}$ defined as above, let $f \in wL_1$ be a nonnegative function with $\|f\|_{\mathcal{U}} = 1$. Then for any $g \in wL_1$, disjointly supported from f, we can find a positive $\phi \in wL_{\hat{1}}^*$ such that $\|\phi\| = 1$, $\phi(f) = 1$ and $\phi(g) = 0$.*

Let $(f_n)_{n=1}^{\infty}$ be a sequence of nonnegative elements in $wL_{\hat{1}}$ with $\|f_n\|_{\mathcal{U}} = 1$ for all $n = 1, 2, 3, \cdots$ and such that the f_n have pairwise disjoint supports. Applying the inductive argument to Lemma 2.3, for each f_n, we can find a linear functional ϕ_n on $wL_{\hat{1}}$ such that $\phi_n(f_n) = 1$, $\|\phi_n\| = 1$ and $\phi_n(f_m) = 0$ if $n \neq m$.

LEMMA 2.4. *Let $(f_n)_{n=1}^{\infty}$ be a sequence of nonnegative elements in wL_1 such that the f_n have pairwise disjoint supports with $\|f_n\|_{\mathcal{U}} = 1$, for all $n = 1, 2, \cdots$ and let $(\phi_n)_{n=1}^{\infty}$ be a sequence of positive linear functionals on $wL_{\hat{1}}$ selected as above. Then for any $f \in wL_{\hat{1}}$, we have $\sum_{n=1}^{\infty} |\phi_n(f)| \leq \|f\|_{wL_{\hat{1}}}$.*

PROOF. For an arbitrary element $f \in wL_{\hat{1}}$, the number $\phi_n(f)$ is the limit of a subnet of the sequence $\{I_{\mathcal{U}}(\chi_{E_{n,k}} \cdot f)\}$ where $(E_{n,k})_{k=1}^{\infty}$ is a decreasing sequence of subsets of $E_n = supp(f_n)$, and f_n is bounded on $E_{n,k}^c$ for all k (see [7, 2.20]). Fix $n \neq m$, let $(E_{n,k})_{k=1}^{\infty}$ be the decreasing sequence of measurable sets for f_n and $(E_{m,k})_{k=1}^{\infty}$ the corresponding sequence for f_m. Let $r = sgnI_{\mathcal{U}}(\chi_{E_{n,k}} \cdot f)$, $s = sgnI_{\mathcal{U}}(\chi_{E_{m,k}} \cdot f)$. Put $m = r\chi_{E_{n,k}} + s\chi_{E_{m,k}}$ so that $\|m\|_\infty = 1$. By Theorem 2.1 and Lemma 2.3, we can identify $T_{\mathcal{U}}(m) = \widehat{m}$ as a linear functional on $wL_{\hat{1}}$. Then we have

$$\begin{aligned}
\widehat{m}(f) &= |I_{\mathcal{U}}(\chi_{E_{n,k}} \cdot f)| + |I_{\mathcal{U}}(\chi_{E_{m,k}} \cdot f)| \\
&= I_{\mathcal{U}}(m \cdot f) \\
&\leq \|m\|_\infty \|f\|_{\mathcal{U}} \quad \text{since } \|m\|_\infty = 1 \\
&= \|f\|_{\mathcal{U}} \quad \text{by (2.2)} \\
&\leq \|f\|_{wL_{\hat{1}}}.
\end{aligned}$$

By the additive rule for nets [5, Lemma 6, p.28], we can say that in the limit

$$\begin{aligned}
|\phi_n(f)| + |\phi_m(f)| &\leq \|f\|_{\mathcal{U}} \quad \text{by (2.2)} \\
&\leq \|f\|_{wL_{\hat{1}}}.
\end{aligned}$$

To show $\sum_{n=1}^{\infty} |\phi_n(f)| \leq \|f\|_{wL_{\hat{1}}}$, it suffices to show that for any $N \in \mathbf{N}$, $\sum_{n=1}^{N} |\phi_n(f)| \leq \|f\|_{wL_{\hat{1}}}$. For $n = 1, 2, \cdots$, let $(E_{n,k})_{k=1}^{\infty}$ be the decreasing sequence of measurable sets for f_n and $E_n = supp(f_n)$. Let $r_n = sgn(\chi_{E_{n,k}} \cdot f)$. Put $m = \sum_{n=1}^{N} r_n \chi_{E_{n,k}}$. Then we have $\|m\|_\infty = 1$. By the same argument as above, one can get

$$\begin{aligned}
\widehat{m}(f) &= \sum_{n=1}^{N} |I_{\mathcal{U}}(\chi_{E_{n,k}} \cdot f)| \\
&= I_{\mathcal{U}}(m \cdot f) \\
&\leq \|m\|_\infty \|f\|_{\mathcal{U}} \quad \text{since } \|m\|_\infty = 1 \text{ and by (2.2)} \\
&\leq \|f\|_{wL_{\hat{1}}}.
\end{aligned}$$

By the additive rule for nets [5, Lemma 6, p.28], we can say that in the limit

$$\begin{aligned}
\sum_{n=1}^{N} |\phi_n(f)| &\leq \|f\|_{\mathcal{U}} \qquad \text{since } \|m\|_\infty = 1 \\
&\leq \|f\|_{wL_{\hat{1}}}.
\end{aligned}$$

We can therefore say that $\sum_{n=1}^{\infty} |\phi_n(f)| \leq \|f\|_{wL_{\hat{1}}}$. This proves the lemma. □

We now need to recall the T. Peck and M. Talagrand's theorem. In [11, Theorem 1], one can see the following theorem; Let Ω be a set and $\Omega_{i,n}$, $n \geq 0$, $1 \leq i \leq 2^n$ be a set of Ω such that $\Omega_{1,0} = \Omega$, $\Omega_{i,n} \cap \Omega_{j,n} = \emptyset$, if $i \neq j$ and $\Omega_{i,n} = \Omega_{2i-1,n+1} \cup \Omega_{2i,n+1}$. Let $\chi_{i,n}$ be the characteristic function of $\Omega_{i,n}$, $n > 0$, $1 \leq i \leq 2^n$ and let Y be the linear span of the functions $\chi_{i,n}$, $n > 0$, $1 \leq i \leq 2^n$.

THEOREM 2.5 [**11, T. Peck and M. Talagrand**]. *Let X be the completion of Y under some lattice norm on Y where Y is given the usual pointwise order. Then there is a lattice isometry of X into $wL_{\hat{1}}$.*

T. Peck and M. Talagrand constructed for $n > 0$, $1 \leq i \leq 2^n$ under lattice isometry T, $T\chi_{i,n}=f_{i,n}$, where $f_{i,n}=\sum_{m\geq n}\sum_{j=1}^{2^{m-n}} e_{2^{m-n}(i-1)+j,m}$ and each $e_{i,n}(x) = \frac{b_{i,n}}{x-u_{i,n}}$, $x \in [v_{i,n}, w_{i,n}]$ is a $\frac{1}{x}$-like function. Note that $f_{i,n}$ are all nonnegative and pairwise disjointly supported in wL_1 and $f_{i,n} = f_{2i,n+1} + f_{2i+1,n+1}$, for all n, and $1 \leq i \leq 2^n$ (see [11, proof of Theorem 1]).

THEOREM 2.6. *For $1 < p < \infty$, the Banach envelope of $WeakL_1$ contains a complemented sublattice that is isometrically isomorphic to $L_p(\Omega, \Sigma, \mu)$ where μ is a separable probability measure.*

PROOF. As an immediate corollary of Theorem 2.5 (see [11, Corollary 2]), we can see that if E is a separable order continuous Banach lattice then there is a lattice isometry of E into $wL_{\hat{1}}$. Since for $1 < p < \infty$, $L_p(\mu)$ space is also an order continuous Banach lattice, we can find a lattice isometry T of L_p into $wL_{\hat{1}}$ where μ is a separable probability measure. Also for $1 < p < \infty$, L_p space is a reflexive Banach lattice, $T(L_p)$ is a reflexive sublattice $wL_{\hat{1}}$. This implies that the unit ball B_{TL_p} is weakly compact. Since every separable reflexive Banach lattice has an order continuous norm, L_p has an order continuous norm. Hence we can apply the construction of T in Theorem 2.5. Let $(\chi_{i,n})_{i=1}^{2^n}$ be the subset of L_p defined in Theorem 2.5. Without loss of generality, one can assume $\|\chi_{i,n}\| = 1$ for all $1 \leq i \leq 2^n$. Then we have $\overline{span}(\chi_{i,n})_{i=1}^{2^n} \subset L_p$.

Define $T\chi_{i,n} = f_{i,n}$, then $\overline{span}(f_{i,n}) \simeq \overline{span}(\chi_{i,n})$. Since $\{\chi_{i,n}\}$ form a dense subset of L_p, $\{f_{i,n}\}$ form a dense subset of TL_p. Moreover, for fixed n, the $f_{i,n}$ are pairwise disjointly supported nonnegative elements in TL_p with $\|f_{i,n}\|_{wL_{\hat{1}}} = 1$. Hence by Lemma 2.3, we can find linear functionals $\phi_{i,n}$ on $wL_{\hat{1}}$ such that $\phi_{i,n}(f_{j,n}) = \delta_{i,j}$ and $\|\phi_{i,n}\| = 1$, for all $i = 1, 2, \cdots$. For each n, let $B_n = \{f_{i,n}\}_{i=1}^{2^n}$ and define $P_{B_n} : wL_{\hat{1}} \longrightarrow$

$\overline{span}(f_{i,n})_{i=1}^{2^n} \subset TL_p$ by

$$P_{B_n}(f) = \sum_{i=1}^{2^n} \phi_{i,n}(f) f_{i,n}. \tag{2.3}$$

Since, for all $f \in wL_{\hat{1}}$

$$\begin{aligned} \|P_{B_n}(f)\|_{wL_{\hat{1}}} &= \|\sum_{i=1}^{2^n} \phi_{i,n}(f) f_{i,n}\|_{wL_{\hat{1}}} \\ &\leq \sum_{i=1}^{2^n} |\phi_{i,n}(f)| \|f_{i,n}\|_{wL_{\hat{1}}} \\ &\leq \sum_{i=1}^{2^n} |\phi_{i,n}(f)| \quad \text{by Lemma 2.4 and } \|f_{i,n}\|_{wL_{\hat{1}}} = 1 \\ &\leq \|f\|_{wL_{\hat{1}}}. \end{aligned} \tag{2.4}$$

This implies $\|P_{B_n}\| \leq 1$, and P_{B_n} is a well defined linear map. Moreover, $f_{j,n} \in TL_p \subset wL_{\hat{1}}$,

$$\begin{aligned} P_{B_n}(f_{j,n}) &= \sum_{i=1}^{2^n} \phi_{i,n}(f_{j,n}) f_{i,n} \\ &= \phi_{j,n}(f_{j,n}) f_{j,n} \\ &= f_{j,n}. \end{aligned} \tag{2.5}$$

Hence $\|P_{B_n}(f_{j,n})\|_{wL_{\hat{1}}} = \|f_{j,n}\|_{wL_{\hat{1}}} = 1$, and $P_{B_n}^2 = P_{B_n}$. Hence P_{B_n} is a projection $wL_{\hat{1}}$ onto $\overline{span}(f_{i,n})_{i=1}^{2^n} \subset TL_p$. From this, we want to find a projection P from $wL_{\hat{1}}$ onto TL_p. We define a partial order on $\{B_n\}_{n=1}^{\infty}$ by $B_n \prec B_m$ if $\overline{span}(f_{i,n}) \subset \overline{span}(f_{i,m})$. Then for each B_n, we have $\|P_{B_n}(f)\|_{wL_{\hat{1}}} \leq \|f\|_{wL_{\hat{1}}}$, for all $f \in wL_{\hat{1}}$ by (2.4). Hence the vector $P_{B_n}(f)$ belongs to $\{g \in TL_p : \|g\|_{wL_{\hat{1}}} \leq \|f\|_{wL_{\hat{1}}}\}$ which is a weakly compact subset in TL_p. Now consider the following product space;

$$\prod_{f \in wL_{\hat{1}}} \{g \in TL_p : \|g\|_{wL_{\hat{1}}} \leq \|f\|_{wL_{\hat{1}}}\}. \tag{2.6}$$

Note that by Tychonoff's theorem, $\prod_{f \in wL_{\hat{1}}} \{g \in TL_p : \|g\|_{wL_{\hat{1}}} \leq \|f\|_{wL_{\hat{1}}}\}$ is compact for the weak topology. Hence the net $\{P_{B_n}\}$ of projections from $wL_{\hat{1}}$ to TL_p has a subnet which converges to some limit point P, in the topology of pointwise convergence on $wL_{\hat{1}}$, taking the weak topology on TL_p. Let $\{P_{B_{n_\alpha}}\}$ be a subnet of $\{P_{B_n}\}$ which converges to P. Then we have the weak limit $P(f) = \lim_\alpha P_{B_{n_\alpha}}(f)$, for all $f \in wL_{\hat{1}}$. Since each P_{B_n} is contractive, positive, and norm one, P is contractive, positive, and norm one.

Finally, we need to show that for all $f \in TL_p$, $P(f) = f$. Since $(f_{i,n})$ are dense, given $\epsilon > 0$ one can find $B_n = \{f_{i,n}\}$ such that $\| \sum_{i=1}^{2^n} a_i f_{i,n} - f\|_{wL_{\hat{1}}} < \epsilon/2$ for some $(a_i)_{i=1}^{2^n}$. Let $g = \sum_{i=1}^{2^n} a_i f_{i,n}$. Then

$$\begin{aligned}
\|P(f) - f\|_{wL_{\hat{1}}} &\leq \|P(f) - P(g)\|_{wL_{\hat{1}}} + \|P(g) - g\|_{wL_{\hat{1}}} + \|g - f\|_{wL_{\hat{1}}} \\
&\leq \|P(f - g)\|_{wL_{\hat{1}}} + \|g - f\|_{wL_{\hat{1}}} \\
&\qquad \text{since } \|P(g) - g\|_{wL_{\hat{1}}} = 0 \\
&\leq \|f - g\|_{wL_{\hat{1}}} + \|g - f\|_{wL_{\hat{1}}} \\
&< \epsilon.
\end{aligned}$$

Hence $P(f) = f$ for all $f \in TL_p$. Therefore P is a positive norm one projection from $wL_{\hat{1}}$ onto TL_p. This proves the theorem. □

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