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ON THE VOLUME MEAN VALUE PROPERTY FOR M-HARMONIC FUNCTIONS

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ABSTRACT. We will show that if U is a Bergman ball $E(a;\delta)$ in $B_n \subset \mathbb{C}^n$ and if U has M–harmonic volume mean value property at a for all M–harmonic functions f on $\overline{U}$, then U must be a ball centered at the origin with $a = 0$.

1. Introduction

There are two types of the converses to the mean value property of M-harmonic functions in a region Ω in the unit ball B_n of $\mathbb{C}^n$. One is on the dimension n and the other is on the region over which M-harmonic functions are integrated. See [1] and [3] for the first and the second type, respectively. For the case of harmonic functions in the Euclidean space, see [2, 4, 5].

Let $\phi_w(z)$ be an automorphism on B_n defined by

$$\phi_w(z) = \frac{w - P_w(z) - sQ_w(z)}{1- < z, w >}, \qquad s = \sqrt{1 - |w|^2}. \tag{1}$$

For a region $\Omega \subset B_n$, each function $f \in C^2(\Omega)$ is said to be M-harmonic in Ω if

$$\widetilde{\Delta} f = 0 \tag{2}$$

in Ω, or equivalently, if $\Delta(f \circ \phi_z)(0) = 0$ for each $z \in \Omega$, where Δ is the usual Euclidean Laplacian. See [6] for details.

Let U be an open connected, relatively compact set in $B_n \subset \mathbb{C}^n$ containing a with $\partial U = \partial \overline{U}$. As proved in [3] by J. Bruna and J.

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Detraz, if $a \in U$ satisfies

$$f(a) = \frac{1}{\lambda(U)} \int_U f(z)\, d\lambda(z) \tag{3}$$

for all M-harmonic functions f in a neighborhood of $\overline{U}$, then U is a Bergman ball centered at $a \in U$; i.e., U is the image $E(a;r)$ of a ball $B(0;r)$ centered at 0 with radius r under ϕ_a for some $0 < r < 1$. Here the measure

$$d\lambda = \frac{d\nu}{(1-|z|^2)^{n+1}} \tag{4}$$

is invariant under the action of $Aut(B_n)$.

On the other hand, they raised a question: what pairs (a, U) $a \in U$ have the property (2) with respect to $d\nu$? Does it follow that $a = 0$ (and hence U is a ball)?

In this short paper, we will show that if U is a Bergman ball $E(a;\delta)$ in $B_n \subset \mathbb{C}^n$ and if U has M-harmonic volume mean value property at a for all M-harmonic functions f on $\overline{U}$, then U must be a ball centered at the origin with $a = 0$.

The letter ν and σ will always denote the normalized Lebesgue measure on B_n and the regular probability Borel measure on the unit sphere S, respectively.

Any unexplained notations or terminologies are as in [6, 7].

2. The main result

We first begin with our definition.

DEFINITION 1. We will say that U as above has M-harmonic volume mean value property at $a \in U$ if it satisfies (3) with respect to $d\nu$ for all M-harmonic functions f on $\overline{U}$.

Now we will consider the case when U is a Bergman ball centered at a with radius δ. Recall that the involution ϕ_a interchanges 0 and a, and that the binomial series

$$(1-x)^{-\lambda} = \sum_{k=0}^{\infty} \frac{\Gamma(k+\lambda)}{k!\Gamma(\lambda)} x^k \tag{5}$$

holds for $|x| < 1$ when λ is not a negative integer. Here Γ denotes the Gamma function.

THEOREM 1. *If a Bergman ball $E(a;\delta) \subset B_n$ satisfies M-harmonic volume mean value property at a, then a is the origin, and hence $E(a;\delta) = B(0;\delta)$.*

PROOF. Let $\phi_a(z)_1$ be the first component of $\phi_a(z)$. Then clearly $\phi_a(z)_1$ is M–harmonic on $\overline{E}(a;\delta)$ since $\phi_a(z)$ is an automorphism on B_n. Hence by the change of variable $z = \phi_a(w)$ followed by the integration in polar coordinates, we get

$$\int_{E(a;\delta)} \phi_a(z)_1 \, d\nu(z) = \int_{B(0;\delta)} w_1 J_R(\phi_a(w)) \, d\nu(w) \tag{6}$$

$$= 2n \int_0^\delta r^{2n-1} \, dr \int_S r\zeta_1 J_R\phi_a(r\zeta) \, d\sigma(\zeta), \tag{7}$$

where $J_R(\phi_a(w))$ is the Jacobian of $\phi_a(w)$. Now we will calculate the inner integral. By [6], it is evident that

$$J_R\phi_a(z) = \left(\frac{1-|a|^2}{|1- < z, a > |^2} \right)^{n+1} \tag{8}$$

Let $e_1 = (1, 0, 0, \cdots, 0) \in \mathbb{C}^n$. Then by letting $a = |a|e_1$ by the unitary invariance of σ, it follows that

$$\begin{aligned} \int_S \zeta_1 J_R\phi_a(r\zeta) \, d\sigma(\zeta) &= (1-|a|^2)^{n+1} \int_S \zeta_1 |1- < a, r\zeta > |^{-2n-2} \, d\sigma(\zeta) \\ &= (1-|a|^2)^{n+1} \sum_{l,m} \frac{\Gamma(l+n+1)\Gamma(m+n+1)}{\Gamma(n+1)l!\Gamma(n+1)m!} \\ &\quad \times (r|a|)^{l+m} \int_S \zeta_1^{l+1} \overline{\zeta_1}^m \, d\sigma(\zeta). \end{aligned}$$

The last equality holds by the binomial series (5). Recall that the holomorphic monomials z^α are orthogonal to each other in $L^2(\sigma)$. Thus, the last integral is zero unless $m = l + 1$. Therefore, by Proposition 1.4.9 [6], we obtain

$$\int_S {\zeta_1}^{l+1} {\overline{\zeta}_1}^{l+1} \, d\sigma(\zeta) = \frac{(n-1)!(l+1)!}{(n+l)!}. \tag{9}$$

Thus it follows that

$$\begin{aligned}\int_S \zeta_1 J_R\phi_a(r\zeta)\, d\sigma(\zeta) &= (1-|a|^2)^{n+1} r|a| \sum_{l=0}^{\infty} \frac{\Gamma(l+n+2)}{n\Gamma(n+1)l!}(r|a|)^{2l} \\ &= \frac{n+1}{n}(1-|a|^2)^{n+1} r|a|(1-(r|a|)^2)^{-(n+2)}.\end{aligned}$$

Since the integral in the left hand side of (6) is $\nu(E(a;\delta))\phi_a(a)$, which is obviously zero, by our assumption, a must be 0. This completes the proof. □

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