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LITTLE HANKEL OPERATORS ON WEIGHTED BLOCH SPACES IN $\mathbb{C}^n$

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ABSTRACT. Let B be the open unit ball in $\mathbb{C}^n$ and $\mu_q(q > -1)$ the Lebesgue measure such that $\mu_q(B) = 1$. Let $L^2_{a,q}$ be the subspace of $L^2(B, d\mu_q)$ consisting of analytic functions, and let $\overline{L^2_{a,q}}$ be the subspace of $L^2(B, d\mu_q)$ consisting of conjugate analytic functions. Let $\overline{P}$ be the orthogonal projection from $L^2(B, d\mu_q)$ into $\overline{L^2_{a,q}}$. The little Hankel operator $h^q_\varphi : L^2_{a,q} \to \overline{L^2_{a,q}}$ is defined by $h^q_\varphi(\cdot) = \overline{P}(\varphi\ \cdot)$. In this paper, we will find the necessary and sufficient condition that the little Hankel operator h^q_φ is bounded(or compact).

1. Introduction

Let D denote the open unit disk in the complex plane $\mathbb{C}$, and let $d\eta$ denote the usual normalized area measure on D. The Bergman space $L^2_a(D, d\eta)$ is the Hilbert space of analytic functions $g : D \to \mathbb{C}$ with inner product given by

$$< f, g >= \int_D f(z)\overline{g(z)}d\eta(z).$$

Let P denote the orthogonal projection on $L^2(D, d\eta)$ onto $L^2_a(D, d\eta)$, so $(I - P)$ is the orthogonal projection of $L^2(D, d\eta)$ onto $(L^2_a(D, d\eta))^\perp$.

For $f \in L^\infty(D, d\eta)$, the Hankel operator

$$H_f : L^2_a(D, d\eta) \to (L^2_a(D, d\eta))^\perp$$

is defined by

$$H_f(g) = (I - P)(fg).$$

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It is useful to consider Hankel operator H_f for $f \in L^2(D, d\eta)$. As usual, $H^\infty(D)$ denotes the set of bounded analytic functions on D. For $f \in L^2(D, d\eta)$, H_f maps $H^\infty(D)$ into $(L_a^2(D, d\eta))^\perp$ by the formula $H_f(g) = (I - P)(fg)$. The set $H^\infty(D)$ is dense in $L_a^2(D, d\eta)$. Thus if H_f is a bounded operator, then H_f extends to a bounded operator from $L_a^2(D, d\eta)$ to $(L_a^2(D, d\eta))^\perp$, also denoted by H_f.

The Bloch space of D consists of analytic functions f on D such that

$$\sup\{(1 - |z|^2)|f'(z)| : z \in D\} < +\infty.$$

The little Bloch space is the set of analytic functions f on D such that

$$(1 - |z|^2)|f'(z)| \to 0 \text{ as } |z| \to 1.$$

In [2], it was shown that for $f \in L_a^2(D, d\eta)$, the Hankel operator $H_{\overline{f}}$ is bounded if and only if f is in the Bloch space in D, and $H_{\overline{f}}$ is compact if and only if f is in the little Bloch space in D.

Let B be the open unit ball in $\mathbb{C}^n$ with normalized volume measure $d\nu$. The Bergman space $L_{a,\nu}^2 = L_a^2(B, d\nu)$ consists of those analytic functions which lie in $L^2(B, d\nu)$. Let P denote the orthogonal projection of $L^2(B, d\nu)$ onto $L_{a,\nu}^2$. As usual, $H^\infty(B)$ denotes the set of bounded analytic functions on B. The Hankel operator $H_f(\cdot) = (I - P)(\cdot)$ maps $H^\infty(B)$ into $(L_{a,\nu}^2)^\perp$ for $f \in L^2(B, d\nu)$. The set $H^\infty(B)$ is dense in $L_{a,\nu}^2$. Thus if $H_f : H^\infty \to (L_{a,\nu}^2)^\perp$ is a bounded operator, then H_f extends to a bounded operator from $L_{a,\nu}^2$ to $(L_{a,\nu}^2)^\perp$, also denoted by H_f.

For $z = (z_1, z_2, \cdots, z_n)$ and $w = (w_1, w_2, \cdots, w_n)$ in $\mathbb{C}^n$, the inner product is defined by $< z, w >= \sum_{j=1}^n z_j \overline{w_j}$ and the norm by $\| z \|^2 = < z, z >$. $\nabla f(z) = (\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n})$ is the holomorphic gradient of f. In [8], Timoney showed that the linear space of all analytic functions $f : B \to \mathbb{C}$ which satisfy

$$\sup_{z \in B}(1 - \| z \|^2) \| \nabla f(z) \| < \infty$$

is equivalent to the space $\mathcal{B}$ of Bloch functions on B. The little Bloch space $\mathcal{B}_0$ is the subspace of $\mathcal{B}$ consisting of those functions $f : B \to \mathbb{C}$ which satisfy

$$\lim_{\|z\| \to 1}(1 - \| z \|^2) \| \nabla f(z) \| = 0.$$

In [4], it was shown that for $f \in L_{a,\nu}^2$, the Hankel operator $H_{\overline{f}}$ is bounded if and only if $f \in \mathcal{B}$, and $H_{\overline{f}}$ is compact if and only if $f \in \mathcal{B}_0$.

For each $q > 0$, the weighted Bloch space of B, denoted by $\mathcal{B}_q$, consists of analytic functions $f : B \to \mathbb{C}$ which satisfy

$$\sup_{z \in B}(1 - \| z \|^2)^q \| \nabla f(z) \| < \infty .$$

The corresponding little Bloch space $\mathcal{B}_{q,0}$ is defined by the functions f in $\mathcal{B}_q$ such that

$$\lim_{\|z\| \to 1}(1 - \| z \|^2)^q \| \nabla f(z) \| = 0.$$

Let us define a norm on $\mathcal{B}_q$ as follows;

$$\| f \|_q = |f(0)| + \sup\{(1 - \| w \|^2)^q \| \nabla f(w) \| : w \in B\}.$$

For each $q > 0$, the space $\mathcal{B}_q$ is a Banach space with respect to the above norm (See [5]). It was also shown in [5] that the weighted little Bloch space $\mathcal{B}_{q,0}$ is the closure of the set of polynomials in the norm topology of $\mathcal{B}_q$ for each $q \geq 1$.

The measure $\mu_q (q > -1)$ is the weighted Lebesgue measure

$$d\mu_q = c_q(1 - \| z \|^2)^q d\nu(z),$$

where c_q is a normalization constant such that $\mu_q(B) = 1$. By $L^2_{a,q} = L^2_a(B, d\mu_q)$, we denote the Bergman subspace of $L^2(B, d\mu_q)$ consisting of analytic functions. We equip $L^2_{a,q}$ with the norm $\| f \|_{2,q} = (\int_B |f|^2 d\mu_q)^{1/2}$.

Let $\overline{L^2_{a,q}}$ be the closed space of $L^2(B, d\mu_q)$ consisting of conjugate analytic functions, and let

$$\overline{P} : L^2(B, d\mu_q) \to \overline{L^2_{a,q}}$$

be the orthogonal projection. When $q = 0$, $\overline{P}$ is the orthogonal projection from $L^2(B, d\nu)$ onto $\overline{L^2_{a,\nu}}$. If P_0 is the rank 1 projection (onto the constants) defined by $P_0 f = \int_B f(z) d\nu(z)$, it is easy to see (See [9]) that

$$\overline{P} - P_0 \leq I - P.$$

For any φ in $L^2(B, d\mu_q)$, the little Hankel operator

$$h^q_\varphi : L^2_{a,q} \to \overline{L^2_{a,q}}$$

is defined by

$$h^q_\varphi g = \overline{P}(\varphi g).$$

Suppose that $q > 0$. In Section 3, we will show that if φ is in $L^2_{a,q} \cap \mathcal{B}_{q+1}$, then the little Hankel operator h^q_φ is bounded on $L^2_{a,q}$. We will also show that if φ is in $L^2_{a,q} \cap \mathcal{B}_{q+1,0}$, then h^q_φ is compact on $L^2_{a,q}$. Conversely, we will show that if $h^q_{\overline{\varphi}}(\varphi \in L^2_{a,q})$ is bounded on $L^2_{a,q}$, then φ is in $\mathcal{B}_{q+1}$. We will also show that if $h^q_{\overline{\varphi}}$ is compact on $L^2_{a,q}$, then φ is in $\mathcal{B}_{q+1,0}$.

2. Some integral representation in weighted Bloch spaces

Fix a point $z \in B$. The functional e_z given by $e_z(f) = f(z), f \in L^2_{a,q}$, is continuous. By the Riesz representation theorem, there exists a function $k_{q,z} \in L^2_{a,q}$ such that

$$f(z) = \int_B f(w)\overline{k_{q,z}(w)}d\mu_q(w), z \in B.$$

Let us define the function $K_q(w,z)$ as $K_q(w,z) = k_{q,z}(w)$.

It is easily seen that $\{z^\alpha\}$, where α ranges over the set of multiindices, is an orthogonal basis in $L^2_{a,q}$. An integration shows that

$$\| z^\alpha \|_{2,q} = \frac{\alpha!\Gamma(q+n+1)}{\Gamma(|\alpha|+q+n+1)}$$

and thus the reproducing kernel is

$$\begin{aligned} K_q(z,w) &= \sum_{k=0}^{\infty}\sum_{|\alpha|=k} \frac{z^\alpha \overline{w}^\alpha}{\| z^\alpha \|_{2,q}} \\ &= \sum_{k=0}^{\infty} < z,w >^k \frac{\Gamma(k+q+n+1)}{\Gamma(q+n+1)} \\ &= \frac{1}{(1- < z,w >)^{q+n+1}} \end{aligned}$$

(See [1]). In this paper, S is the boundary of B and σ is the rotation invariant surface measure on S normalized by $\sigma(S) = 1$.

THEOREM 1. *If $f \in L^1(B, d\mu_q) \cap H(B), q > -1$, then*

$$f(z) = \int_B \frac{f(w)}{(1- < z, w >)^{n+q+1}} d\mu_q(w).$$

PROOF. See [5, Theorem 2]. □

THEOREM 2. *For $z \in B$, c is real, $t > -1$, define*

$$I_{c,t}(z) = \int_B \frac{(1 - \| w \|^2)^t}{|1- < z, w > |^{n+1+c+t}} d\nu(w), \quad z \in B.$$

Then,
(i) *$I_{c,t}(z)$ is bounded in B if $c < 0$;*
(ii) *$I_{0,t}(z) \sim -\log(1 - \| z \|^2)$ as $\| z \| \to 1^-$;*
(iii) *$I_{c,t}(z) \sim (1 - \| z \|^2)^{-c}$ as $\| z \| \to 1^-$ if $c > 0$.*

PROOF. See [7, Proposition 1.4.10]. □

LEMMA 3. *If $g \in L^\infty(B)$, then*

$$(\mathcal{R}_q g)(z) = \int_B \frac{g(w)}{(1- < z, w >)^{n+q+1}} d\nu(w)$$

is in $\mathcal{B}_{q+1} (q > -1)$.

PROOF. Differentiating

$$(\mathcal{R}_q g)(z) = \int_B \frac{\cdot\, g(w)}{(1- < z, w >)^{n+q+1}} d\nu(w)$$

under the integral sign, we obtain

$$\frac{\partial}{\partial z_j} (\mathcal{R}_q g)(z) = (n + q + 1) \int_B \frac{g(w)(-\overline{w}_j)}{(1- < z, w >)^{n+q+2}} d\nu(w),$$

for $j = 1, 2, \cdots, n$. This shows that

$$\| \nabla (\mathcal{R}_q g)(z) \| \leq (n + q + 1) \| g \|_\infty \int_B \frac{d\nu(w)}{|1- < z, w > |^{n+q+2}} .$$

By Theorem 2,

$$\| \nabla(\mathcal{R}_q g)(z) \| \leq (n+q) \| g \|_\infty (1- \| z \|^2)^{-(q+1)}.$$

Thus,

$$(1- \| z \|^2)^{q+1} \| \nabla(\mathcal{R}_q g)(z) \| \leq C \| g \|_\infty .$$

It is also clear that $|(\mathcal{R}_q g)(0)| \leq \| g \|_\infty$. Thus,

$$\begin{aligned} &\| (\mathcal{R}_q g) \|_{q+1} \\ &= |(\mathcal{R}_q g)(0)| + \sup\{(1- \| z \|^2)^{q+1} \| \nabla(\mathcal{R}_q g)(z) \| : z \in B\} \\ &\leq (C+1) \| g \|_\infty . \end{aligned}$$

Hence, $\mathcal{R}_q$ maps $L^\infty(B)$ boundedly into $\mathcal{B}_{q+1}$. □

Let N denote the set of natural numbers. A multi-index α is an ordered n-tuple $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ with $\alpha_j \in N, j = 1, 2, \cdots, n$. For a multi-index α and $z \in \mathbb{C}^n$, set

$$\begin{aligned} |\alpha| &= \alpha_1 + \alpha_2 + \cdots + \alpha_n, \\ \alpha! &= \alpha_1! \alpha_2! \cdots \alpha_n!, \\ z^\alpha &= z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}. \end{aligned}$$

Let $C(\overline{B})$ be the space of complex-valued continuous functions on the closed unit ball $\overline{B}$.

THEOREM 4. *If $f \in C(\overline{B})$, then $\mathcal{R}_q f$ is in $\mathcal{B}_{q+1,0}$.*

PROOF. Let $I = (i_1, i_2, \cdots, i_n)$. Since

$$\begin{aligned} < z, w >^m &= (z_1 \overline{w}_1 + z_2 \overline{w}_2 + \cdots + z_n \overline{w}_n)^m \\ &= \sum_{|I|=m} \frac{m!}{I!} (z_1 \overline{w}_1)^{i_1} (z_2 \overline{w}_2)^{i_2} \cdots (z_n \overline{w}_n)^{i_n}, \end{aligned}$$

$$\begin{aligned} &\frac{1}{(1- < z, w >)^{n+q+1}} \\ &= 1 + \sum_{m=1}^\infty \frac{(n+q+m)!}{m!(n+q)!} < z, w >^m \\ &= 1 + \sum_{m=1}^\infty \sum_{|I|=m} \frac{(n+q+m)!}{m!(n+q)!} \frac{m!}{I!} (z_1 \overline{w}_1)^{i_1} (z_2 \overline{w}_2)^{i_2} \cdots (z_n \overline{w}_n)^{i_n}. \end{aligned}$$

Hence,

$$\begin{aligned}
&\mathcal{R}_q(z^\alpha \overline{z}^\beta)\\
&= \int_B \frac{w^\alpha \overline{w}^\beta}{(1- < z, w >)^{n+q+1}} d\nu(w)\\
&= \int_B w^\alpha \overline{w}^\beta d\nu(w) + \sum_{m=1}^{\infty} \sum_{|I|=m} \frac{(n+q+m)!\, m!}{m!(n+q)!\; I!} z^I \int_B w^\alpha \overline{w}^\beta \overline{w}^I d\nu(w)\\
&= C_J z^J
\end{aligned}$$

for some J and some constant C_J [7, Proposition 1.4.8, Proposition 1.4.9].

By the Stone-Weierstrass approximation theorem, each function in $C(\overline{B})$ can be uniformly approximated by finite linear combinations of functions of the form $z^\alpha \overline{z}^\beta$, which are mapped by $\mathcal{R}_q$ to polynomials (finite linear combination of monomials). Since $\mathcal{R}_q$ maps $L^\infty(B)$ boundedly into $\mathcal{B}_{q+1}$ and $\mathcal{B}_{q+1,0}$ is closed in $\mathcal{B}_{q+1}$, $\mathcal{R}_q$ maps $C(\overline{B})$ boundedly into $\mathcal{B}_{q+1,0}$. □

For $z \in B$, let us define function $\mathcal{K}_z$ by

$$\mathcal{K}_z^2(w) = \frac{K(w,z)^2}{K(z,z)} = \frac{(1- \| z \|^2)^{n+1}}{(1- < w, z >)^{2(n+1)}},$$

where $K(z,w)$ is the Bergman reproducing kernel of $L^2(B, d\nu)$. Then the function $\mathcal{K}_z$ is a unit vector in $L_a^2(B, d\nu)$. For analytic function f, $\mathcal{V}_q f$ is the function on B defined by

$$\mathcal{V}_q f(z) = \int_B f(w) \overline{\mathcal{K}_z^2(w)} d\mu_q(w).$$

THEOREM 5. *If analytic function f is in $L^1(B, d\mu_q)$ and $\mathcal{V}_q f \in L^\infty(B)$, then $\mathcal{R}_q(\mathcal{V}_q f) = \frac{1}{c_{n+1}} f$ and $f \in \mathcal{B}_{q+1}$.*

PROOF. Applying Fubini's theorem and Theorem 1, we get

$$\begin{aligned}
&(\mathcal{R}_q(\mathcal{V}_q f))(z)\\
&= \int_B (\mathcal{V}_q f)(w) \overline{K_q(w,z)} d\nu(w)\\
&= \int_B \overline{K_q(w,z)} \int_B f(u) \frac{(1- \| w \|^2)^{n+1}}{(1- < w, u >)^{2(n+1)}} d\mu_q(u) d\nu(w)
\end{aligned}$$

$$= \int_B f(u) \int_B \frac{(1- \| w \|^2)^{(n+1)}}{(1- < w, u >)^{2(n+1)}} \overline{K_q(w,z)} d\nu(w) d\mu_q(u)$$
$$= \frac{1}{c_{n+1}} \int_B f(u)\overline{K_q(u,z)} d\mu_q(u)$$
$$= \frac{1}{c_{n+1}} f(z).$$

Thus $\mathcal{V}_q f \in L^\infty(B)$ implies that $f \in \mathcal{B}_{q+1}$ by Lemma 3. □

3. Little Hankel operator on weighted Bloch spaces

THEOREM 6. *Suppose $q > 1$. Then f is in $\mathcal{B}_q$ if and only if f is analytic and $(1 - \| z \|^2)^{q-1}|f(z)|$ is bounded on B.*

PROOF. See [5, Theorem 6]. □

THEOREM 7. *If φ is in $L^2_{a,q} \cap \mathcal{B}_{q+1} (q > 0)$, then h^q_φ is bounded on $L^2_{a,q}$.*

PROOF. Since $\overline{P}$ is given by

$$\overline{P}g(z) = \int g(w)\overline{K_q(z,w)} d\mu_q(w), \ g \in L^2(B, d\mu_q),$$

the little Hankel operator has the following integral form :

$$h^q_\varphi g(z) = \int_B \varphi(w) g(w) \overline{K_q(z,w)} d\mu_q(w)$$
$$= \int_B \frac{\varphi(w)g(w)}{(1- < w, z >)^{n+q+1}} d\mu_q(w).$$

Given f and g in H^∞, we can apply Fubini's Theorem and Theorem 1 to obtain

$$< h^q_\varphi f, \overline{g} > = \int_B g(z) d\mu_q(z) \int_B \varphi(w) f(w) \overline{K_q(z,w)} d\mu_q(w)$$
$$= \int_B \varphi(w) f(w) d\mu_q(w) \int_B g(z)\overline{K_q(z,w)} d\mu_q(z)$$
$$= \int_B \varphi(w) f(w) g(w) d\mu_q(w)$$
$$= c_q \int_B f(w) g(w) \varphi(w) (1- \| w \|^2)^q d\nu(w).$$

By Theorem 6, there exists a constant $C > 0$ such that

$$| < h^q_\varphi f, \overline{g} > | \leq C \parallel \varphi \parallel_{q+1} \parallel fg \parallel_{L^1} \leq C \parallel \varphi \parallel_{q+1} \parallel f \parallel_\infty \parallel g \parallel_\infty$$

for all f and g in H^∞. This shows that the operator h^q_φ is bounded on $L^2_{a,q}$ with $\parallel h^q_\varphi \parallel \leq C \parallel \varphi \parallel_{q+1}$. □

THEOREM 8. *Suppose that $q > 0$. If φ is in $L^2_{a,q} \cap \mathcal{B}_{q+1,0}$, then h^q_φ is compact on $L^2_{a,q}$.*

PROOF. If φ is a polynomial, then h^q_φ is a finite rank operator. Since finite rank operator is compact, h^q_φ is compact. For each φ in $\mathcal{B}_{q+1,0}$, there exists a sequence of polynomials $\{p_n\}$ such that

$$\parallel \varphi - p_n \parallel_{q+1} \rightarrow 0$$

as $n \rightarrow +\infty$. Since

$$\parallel h^q_\varphi - h^q_{p_n} \parallel \leq \parallel \varphi - p_n \parallel_{q+1} \rightarrow 0$$

as $n \rightarrow +\infty$ by Theorem 7, h^q_φ is compact. □

THEOREM 9. *Suppose that $q > 0$ and $\varphi \in L^2_{a,q}$. If $h^q_{\overline{\varphi}}$ is bounded on $L^2_{a,q}$, then φ is in $\mathcal{B}_{q+1}$.*

PROOF. For $z \in B$, recall that $\mathcal{K}_z(w) = \frac{(1-\|z\|^2)^{\frac{n+1}{2}}}{(1-<w,z>)^{n+1}}$ is a unit vector in $L^2_a(B, d\nu)$.

$$
\begin{aligned}
&< \overline{\mathcal{K}_z}, h^q_{\overline{\varphi}} \mathcal{K}_z > \\
&= \int_B \overline{\mathcal{K}_z(w)}\ \overline{h^q_{\overline{\varphi}} \mathcal{K}_z(w)} d\mu_q(w) \\
&= \int_B \overline{\mathcal{K}_z(w)} \overline{\int_B \frac{\overline{\varphi}(u)\mathcal{K}_z(u)}{(1- < u, w >)^{n+q+1}} d\mu_q(u) d\mu_q(w)} \\
&= \int_B \overline{\mathcal{K}_z(u)} \varphi(u) \overline{\int_B \frac{\mathcal{K}_z(w)}{(1- < u, w >)^{n+q+1}} d\mu_q(w) d\mu_q(u)} \\
&= \int_B \overline{\mathcal{K}_z(u)} \varphi(u) \overline{\int_B \frac{(1- \parallel z \parallel^2)^{\frac{n+1}{2}}}{(1- < u, w >)^{n+q+1} (1- < w, z >)^{n+1}} d\mu_q(w) d\mu_q(u)} \\
&= \int_B \overline{\mathcal{K}_z(u)} \varphi(u) \overline{\frac{(1- \parallel z \parallel^2)^{\frac{n+1}{2}}}{(1- < u, z >)^{n+1}}} d\mu_q(u) \\
&= \int_B \varphi(u) \overline{\mathcal{K}^2_z(u)} d\mu_q(u) \\
&= \mathcal{V}_q \varphi(z).
\end{aligned}
$$

The function $\mathcal{V}_q\varphi$ is in L^∞ with $\| \mathcal{V}_q\varphi \|_\infty \leq \| h^q_{\overline{\varphi}} \|$. By Theorem 5, we have $\varphi \in \mathcal{B}_{q+1}$. □

THEOREM 10. *Suppose that $q > 0$ and φ is in $L^2_{a,q}$. If $h^q_{\overline{\varphi}}$ is compact on $L^2_{a,q}$, then φ is in $\mathcal{B}_{q+1,0}$.*

PROOF. Suppose that $h^q_{\overline{\varphi}}$ is compact on $L^2_{a,q}$. If f is in $H^\infty(B)$, then

$$\begin{aligned}
< f, \mathcal{K}_z > &= \int_B f(w)\overline{\mathcal{K}_z(w)}d\mu_q(w) \\
&= \int_B f(w)\frac{(1-\| z \|^2)^{\frac{n+1}{2}}}{(1-< z,w >)^{n+1}}d\mu_q(w) \\
&= (1-\| z \|^2)^{\frac{n+1}{2}} \int_B \frac{f(w)}{(1-< z,w >)^{n+1}}d\mu_q(w).
\end{aligned}$$

$$\begin{aligned}
| < f, \mathcal{K}_z > | &= (1-\| z \|^2)^{\frac{n+1}{2}} \left| \int_B \frac{f(w)}{(1-< z,w >)^{n+1}}d\mu_q(w)\right| \\
&\leq (1-\| z \|^2)^{\frac{n+1}{2}} \| f \|_\infty c_q \int_B \frac{(1-\| w \|^2)^q}{|(1-< z,w >)|^{n+1}}d\nu(w) \\
&\leq M(1-\| z \|^2)^{\frac{n+1}{2}}
\end{aligned}$$

for some constant M. The last inequality follows from Theorem 2. This implies that

$$| < f, \mathcal{K}_z > | \leq M(1-\| z \|^2)^{\frac{n+1}{2}} \to 0$$

as $\| z \| \to 1^-$. Since $H^\infty(B)$ is dense in $L^2_{a,q}(B)$, this shows that $\mathcal{K}_z \to 0$ weakly in $L^2_{a,q}$ as $\| z \| \to 1^-$.

Since $\mathcal{K}_z \to 0$ weakly in L^2_a as $\| z \| \to 1^-$,

$$\mathcal{V}_q\varphi(z) = < \overline{\mathcal{K}}_z, h^q_{\overline{\varphi}}\mathcal{K}_z > \to 0$$

as $\| z \| \to 1^-$. Since $\mathcal{V}_q\varphi \in C(\overline{B})$, $\mathcal{R}_q\mathcal{V}_q\varphi \in \mathcal{B}_{q+1,0}$ by Theorem 4. Since $\mathcal{R}_q\mathcal{V}_q\varphi = \frac{1}{c_{n+1}}\varphi$ by Theorem 5, we have $\varphi \in \mathcal{B}_{q+1,0}$. □

References

[1] J. Arazy, S. D. Fisher, and J. Peetre, *Hankel operators on weighted Bergman spaces*, Amer. J. Math. **110** (1988), 989–1054.

[2] S. Axler, *The Bergman spaces, the Bloch space and commutators of multiplication operators*, Duke Math. J. **53**, 315–332.

[3] C. A. Berger, L. A. Coburn, and K. H. Zhu, *Function theory on Cartan domain and the Berezin-Toeplitz symbol calculus*, Amer. J. Math. **110** (1988), 921–953.

[4] D. Bekolle, C. A. Berger, L. A. Coburn, and K. H. Zhu, *BMO in the Bergman metric on bounded symmetric domain*, J. Funct. Anal. **93** (1990), 310–350.

[5] K. T. Hahn and K. S. Choi, *Weighted Bloch spaces in* $\mathbb{C}^n$, J. Korean Math. Soc. **35** (1998), 171–189.

[6] S. Janson, J. peetre, and R. Rochberg, *Hankel Forms and the Fock Space*, Revista Math. Ibero-amer **3** (1987), 61–138.

[7] W. Rudin, *Function theory in the unit ball of* $\mathbb{C}^n$, Springer Verlag, New York (1980).

[8] R. M. Timoney, *Bloch functions of several variables*, J. Bull. London Math. Soc **12** (1980), 241–267.

[9] K. H. Zhu, *Duality and Hankel operators on the Bergman spaces of bounded symmetric domains*, J. Funct. Analysis **81** (1988), 262–278.

[10] ______, *Bloch type spaces of analytic functions*, Rocky Mountain J. Math. **23** (1993), 1143–1177.

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