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ON THE NILPOTENCY OF CERTAIN SUBALGEBRAS OF KAC-MOODY ALGEBRAS OF TYPE $A_N^{(r)}$

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Abstract. Let $\mathfrak{g} = \mathfrak{g}(A) = \mathfrak{N}_- \oplus \mathfrak{h} \oplus \mathfrak{N}_+$ be a symmetrizable Kac-Moody algebra with the indecomposable generalized Cartan matrix A and W be its Weyl group. Let θ be the highest root of the corresponding finite dimensional simple Lie algebra $\mathring{\mathfrak{g}}$ of $\mathfrak{g}$. For the type $A_N^{(r)}$, we give an element $w_0 \in W$ such that $w_0^{-1}(\mathring{\Delta}_+) = \mathring{\Delta}_-$. And then we prove that the degree of nilpotency of the subalgebra $S_w = \mathfrak{N}_+ \cap w(\mathfrak{N}_-)$ is greater than or equal to ht$\theta + 1$.

0. Introduction

Let $\mathfrak{g} = \mathfrak{g}(A)$ be a symmetrizable Kac-Moody algebra with the indecomposible generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$. It is well known that the generalized Cartan matrix $A = (a_{ij})_{i,j\in I}$ is either (i) finite (ii) affine or (iii) indefinite type (See Theorem 4.3 of [3]). Denote by Δ^{re}, Δ^{re}_+, Δ^{im}, and Δ^{im}_+ the set of all real, positive real, imaginary and positive imaginary roots, respectively. Let $\Pi = \{\alpha_0, \alpha_1, \dots, \alpha_n\}$ denote the set of simple roots and $Q = \sum_{i=0}^n Z\alpha_i$ denote the root lattice. For $\alpha, \beta \in Q$, we define $\alpha > \beta$ if $\alpha - \beta \in Q_+ = \sum Z_{\geq 0}\alpha_i$. For $\alpha = \sum_{i=0}^n k_i\alpha_i \in Q$ we define the height of α by $ht(\alpha) = \sum_{i=0}^n k_i$ and

$$S_w = \mathfrak{N}_+ \cap w(\mathfrak{N}_-) = \bigoplus_{\alpha \in \Delta^+(w)} \mathfrak{g}_\alpha$$

for $w \in W$ where $\Delta = \Delta_+ \cup \Delta_-$ is the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and $\Delta^+(w) = \{\alpha \in \Delta_+ \mid w^{-1}(\alpha) < 0\}$. The subalgebra S_w is finite-dimensional and nilpotent.

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In [1], Billig and Pianzola conjectured that the degree of nilpotency of S_w is bounded by a constant which depends on A but not on w.

In this paper, we give the partial proof for Billig and Pianzola's conjecture for the type $A_N^{(r)}$.

First, we recall the theory of Kac-Moody algebras. And then we survey the theories of the root system and Weyl group of affine Lie algebras of type $A_N^{(r)}$.

In the case of rank 2, we prove that the degree of nilpotency is 1 or 2.

For the type $A_l^{(1)}(l > 1)$ or $A_{2l-1}^{(2)}(l > 1)$, we give an element w_0 of the Weyl group W such that $w_0^{-1}(\mathring{\Delta}_+) = \mathring{\Delta}_-$. And then we prove that the degree of nilpotency is greater than or equal to the htθ.

In case of $A_{2l}^{(2)}(l > 2)$, we introduce an element $w_1 \in W$ such that $\{a\alpha_l + \delta, \alpha_l + 3\delta\} \cup \mathring{\Delta}_+ \subseteq \Delta^+(w)$ and we prove that the degree of nilpotency is greater than or equal to ht$\theta + 1$.

1. Preliminaries

Let $\alpha_{i_1}, \alpha_{i_2}, \cdots, \alpha_{i_t} \in \Pi$ (not necessarily distinct) and denote by $r_i = r_{\alpha_i}$ the simple reflection of W. When $w \in W$ is written as $w = r_{i_1} \cdots r_{i_t}$ ($\alpha_{i_j} \in \Pi$, t minimal), we call the expression reduced.

PROPOSITION 1.1 (See [6]). *Let $w = r_{i_1} \cdots r_{i_t} \in W$ be a reduced expression of w. Then we have*

$$\Delta^+(w) = \{\beta_1, \cdots, \beta_t\},$$

where $\beta_p = r_{i_1} \cdots r_{i_{p-1}}(\alpha_{i_p})$ $(1 \leq p \leq t)$ and the β_p's are all distinct.

We have the following lemma.

LEMMA 1.2. *Let A be a generalized Cartan matrix of affine type, and W the Weyl group of the associated Kav-Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$. Then*

$$\Delta^+(w) \subset \Delta_+^{re}$$

for all $w \in W$.

PROOF. Since Δ_+^{im} is W-invariant by Proposition 5.2(a) of [3], we have $\Delta^+(w) \cap \Delta_+^{im} = \emptyset$. This completes the proof. $\square$

2. Root systems of affine Lie algebras of type $A_N^{(r)}$

Let $A = (a_{ij})_{i,j=0}^n$ be an indecomposable generalized Cartan Matrix of affine type and let $\mathfrak{g} = \mathfrak{g}(A)$ be an associated Lie algebra. The Kac-Moody Lie algebra $\mathring{\mathfrak{g}} = \mathfrak{g}(\mathring{A})$ associated with the Cartan matrix $\mathring{A} = (a_{ij})_{i,j=1}^n$ is a finite dimensional simple Lie algebra. Let Δ and $\mathring{\Delta}$ denote the set of roots for $\mathfrak{g}$ and $\mathring{\mathfrak{g}}$, respectively. Note that $\Delta = \Delta_+ \cup \Delta_-$, $\mathring{\Delta} = \mathring{\Delta}_+ \cup \mathring{\Delta}_-$, $\Delta^{re} = \Delta_+^{re} \cup \Delta_-^{re}$, $\Delta^{im} = \Delta_+^{im} \cup \Delta_-^{im}$, where the subscript plus(minus) denotes the positive(negative) roots. Let $\mathring{\Pi} = \{\alpha_1, \cdots, \alpha_n\}$ and $\Pi = \{\alpha_0, \alpha_1, \cdots, \alpha_n\}$ denote the set of simple roots for $\mathring{\mathfrak{g}}$ and $\mathfrak{g}$, respectively. We denote by $\mathring{\Delta}_l$ ($\mathring{\Delta}_s$) the set of long(resp., short) roots for $\mathring{\mathfrak{g}}$.

Let $\mathfrak{g} = \mathfrak{g}(A)$ be the affine Lie algebra of type $A_N^{(r)}$ and let Δ be the root system of $\mathfrak{g} = \mathfrak{g}(A)$. We denote by Δ_s, Δ_m, and Δ_l the set of all short, medium, and long roots, respectively. We have the following Theorem.

THEOREM 2.1. *Let W be the Weyl group of Kac-Moody algebra. Then there exists $w_0 \in \mathring{W}$ such that $w_o^{-1}(\mathring{\Delta}_+) \subseteq \Delta_-$.*

PROOF. Since Π and $-\Pi$ are root bases, there exists $w_0 \in \mathring{W}$ such that $w_0^{-1}(\mathring{\Pi}) = -\mathring{\Pi}$ and hence $w_0^{-1}(\mathring{\Delta}_+) = \mathring{\Delta}_-$, we are done. □

We can take such an element w_0 in the above theorem as follows:

PROPOSITION 2.2. (a) *If A is of type $A_l^{(1)}$ ($l > 1$) and l is odd, then*

$$w_0 = (r_1 \cdots r_{l-1} r_l r_{l-1} \cdots r_1)(r_2 \cdots r_{l-1} r_l r_{l-1} \cdots r_2) \cdots (r_{\frac{l-1}{2}} r_{\frac{l+1}{2}} r_{\frac{l-1}{2}})(r_{\frac{l+1}{2}})$$

is an element of $\mathring{W}$ such that $w_0^{-1}(\mathring{\Delta}_+) = \mathring{\Delta}_-$. If l is even, then

$$w_0 = (r_1 \cdots r_{l-1} r_l r_{l-1} \cdots r_1)(r_2 \cdots r_{l-1} r_l r_{l-1} \cdots r_2) \cdots (r_{\frac{l}{2}} r_{\frac{l}{2}+1} r_{\frac{l}{2}})$$

is an element of $\mathring{W}$ such that $w_0^{-1}(\mathring{\Delta}_+) = \mathring{\Delta}_-$.

(b) *If A is of type $A_{2l}^{(2)}$ ($l > 1$), then*

$$w_0 = (r_1 \cdots r_{l-1} r_l r_{l-1} \cdots r_1)(r_2 \cdots r_{l-1} r_l r_{l-1} \cdots r_2) \cdots (r_{l-1} r_l r_{l-1}) r_l$$

is an element of $\mathring{W}$ such that $w_0^{-1}(\mathring{\Delta}_+) = \mathring{\Delta}_-$.

PROOF. (a) By simple calculation, we have the desired results.
(b) See Theorem 2.8 of [4].

The following proposition is well known.

PROPOSITION 2.3. (See [3]). *Let A be a generalized Cartan matrix of type $A_{2l}^{(2)}$ and let $\mathfrak{g} = \mathfrak{g}(A)$. Then we have the following:*
$\Delta_+^{re} = \{\frac{1}{2}(\alpha + (2n-1)\delta) \mid \alpha \in \mathring{\Delta}_l,\ n \in Z\} \cup \{\alpha + n\delta \mid \alpha \in \mathring{\Delta}_s,\ n \in Z\} \cup \{\alpha + 2n\delta \mid \alpha \in \mathring{\Delta}_l,\ n \in Z\}$.

(c) $\Delta_+^{re} = \{\frac{1}{2}(\alpha + (2n-1)\delta) \mid \alpha \in \mathring{\Delta}_l,\ n \in Z\} \cup \{\alpha + n\delta \mid \alpha \in \mathring{\Delta}_s,\ n \in Z\} \cup \{\alpha + 2n\delta \mid \alpha \in \mathring{\Delta}_l,\ n \in Z$ if A is of type $A_{2l}^{(2)}$. □

3. Degree of nilpotency of S_w

From now on, let $A = (a_{ij})_{i,j=0}^n$ be a generalized Cartan matrix of type $A_N^{(r)}$. For $w \in W$, define

$$
\begin{aligned}
S_w &= \mathfrak{N}_+ \cap w(\mathfrak{N}_-) = \bigoplus_{\alpha \in \Delta^+(w)} \mathfrak{g}_\alpha, \\
S_w^0 &= S_w, \\
S_w^k &= [S_w, S_w^{k-1}]
\end{aligned}
\tag{3.1}
$$

for $w \in W$, $(k = 1, 2, \cdots)$.

Billig and Pianzola (See [4]) conjectured that the least positive integer such that $S_w^k = \{0\}$ for all $w \in W$ is bounded by a constant which depends on the generalized Cartan matrix A but not on w.

We call this least positive integer the degree of nilpotency of S_w.

THEOREM 3.1. *Let A be of type $A_1^{(1)}$, $\mathfrak{g} = \mathfrak{g}(A)$, and W the Weyl group of $\mathfrak{g} = \mathfrak{g}(A)$. Then the degree of nilpotency of S_w is 1.*

PROOF. We know $\Delta^{re} = \{\pm\alpha_1 + n\delta \mid \alpha \in \mathring{\Delta},\ n \in Z\}$ where $\delta = \alpha_1 + \alpha_2$. Since sum of any two real roots in Δ^{re} is not in Δ^{re}, combining (3.1), we have

$$
\begin{aligned}
S_w^1 &= [S_w, S_w] \\
&= \Big[\bigoplus_{\alpha \in \Delta^+(w)} \mathfrak{g}_\alpha, \bigoplus_{\alpha \in \Delta^+(w)} \mathfrak{g}_\alpha \Big]
\end{aligned}
$$

$$\subset \bigoplus_{\alpha,\beta\in\Delta^+(w)} \mathfrak{g}_{\alpha+\beta}$$
$$\subset \bigoplus_{\alpha,\beta\in\Delta^{re}} \mathfrak{g}_{\alpha+\beta}$$
$$= \{0\}$$

for given $w \in W$. This completes the proof. □

THEOREM 3.2. *Let A be a generalized Cartan matrix of type $A_2^{(2)}$, $\mathfrak{g} = \mathfrak{g}(A)$ the associated Lie algebra and W the corresponding Weyl group. Then the degree of nilpotency of S_w is 2.*

PROOF. We know $\delta = 2\alpha_0 + \alpha_1$. By definition of root system of $A_2^{(2)}$, we have

$$\begin{aligned}\Delta_s^{re} &= \{x\alpha_0 + y\alpha_1 \mid 2x^2 - 8xy + 8y^2 = 2,\ x, y \in Z\} \\ &= \{x\alpha_0 + y\alpha_1 \mid x = 2y \pm 1,\ x, y \in Z\} \\ &= \left\{\frac{1}{2}(\pm\alpha_1 + (2n-1)\delta) \,|\, n \in Z\right\}\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\Delta_l^{re} &= \{x\alpha_0 + y\alpha_1 \mid 2x^2 - 8xy + 8y^2 = 4,\ x, y \in Z\} \\ &= \{x\alpha_0 + y\alpha_1 \mid x = 2y \pm 2,\ x, y \in Z\} \\ &= \{\pm\alpha_1 + 2n\delta \,|\, n \in Z\}.\end{aligned} \tag{3.2}$$

Combining (3.1), (3.2), and (3.3), we have

$$S_w = \Big(\bigoplus_{\alpha\in\Delta^+(w)\cap\in\Delta_s} \mathfrak{g}_\alpha\Big) \bigoplus \Big(\bigoplus_{\alpha\in\Delta^+(w)\cap\in\Delta_l} \mathfrak{g}_\alpha\Big) \tag{3.4}$$

$$\Big[\bigoplus_{\alpha\in\Delta^+(w)\cap\in\Delta_s} \mathfrak{g}_\alpha,\ \bigoplus_{\alpha\in\Delta^+(w)\cap\in\Delta_s} \mathfrak{g}_\alpha\Big] \subseteq \bigoplus_{\alpha\in\Delta^+(w)\cap\Delta_l} \mathfrak{g}_\alpha, \tag{3.5}$$

for given $w \in W$. Since

$$w^{-1}(\alpha_1 + m\delta) < 0 \ \text{ implies } \ w^{-1}(-\alpha_1 + m\delta) > 0$$

and

$$w^{-1}(-\alpha_1 + m\delta) < 0 \ \text{ implies } \ w^{-1}(\alpha_1 + m\delta) > 0,$$

combining (3.2), (3.3), we have

$$(3.6) \qquad \Big[\bigoplus_{\alpha\in\Delta^+(w)\cap\in\Delta_l} \mathfrak{g}_\alpha, \bigoplus_{\alpha\in\Delta^+(w)\cap\in\Delta_l} \mathfrak{g}_\alpha \Big] = \{0\}$$

and

$$(3.7) \qquad \Big[\bigoplus_{\alpha\in\Delta^+(w)\cap\in\Delta_s} \mathfrak{g}_\alpha, \bigoplus_{\alpha\in\Delta^+(w)\cap\in\Delta_l} \mathfrak{g}_\alpha \Big] = \{0\}$$

for $w \in W$. Thus we have

$$\begin{aligned} S_w^1 &= [S_w, S_w] \\ &= \Big[\bigoplus_{\alpha\in\Delta^+(w)} \mathfrak{g}_\alpha, \bigoplus_{\alpha\in\Delta^+(w)} \mathfrak{g}_\alpha \Big] \\ &\subset \bigoplus_{\alpha,\beta\in\Delta^+(w)} \mathfrak{g}_{\alpha+\beta} \\ &\subset \bigoplus_{\alpha\in\Delta_l^{re}} \mathfrak{g}_\alpha \end{aligned}$$

and

$$\begin{aligned} S_w^2 &= [S_w, S_w^1] \\ &= \Big[\bigoplus_{\alpha\in\Delta^+(w)} \mathfrak{g}_\alpha, \bigoplus_{\alpha\in\Delta_l^{re}\cap\Delta^+(w)} \mathfrak{g}_\alpha \Big] \\ &= \{0\}. \end{aligned}$$

On the other hand,

$$\Delta^+(r_1r_0r_1r_0) = \Big\{\alpha_1, \frac{1}{2}(\alpha_1+\delta), \alpha_1+2\delta, \frac{1}{2}(\alpha_1+3\delta)\Big\}$$

and hence

$$\{0\} \neq \mathfrak{g}_{\alpha_1+2\delta} \subset S^1_{r_1r_0r_1r_0}.$$

This completes the proof. □

We introduce the following important element:

$$\theta = \delta - a_0\alpha_0 = \sum_{i=1}^{l} a_i\alpha_i.$$

It is well known that θ is the highest root of $\mathring{\mathfrak{g}}$.

LEMMA 3.3 (See [4]). *Each $\beta \in \mathring{\Delta}_+$ can be written in the form $\alpha_{i_1} + \cdots + \alpha_{i_t}$ ($\alpha_{i_j} \in \Pi$ not necessarily distinct) in such a way that each partial sum is a root.*

THEOREM 3.4. *Let A be of Affine type $A_N^{(r)}$. Let θ be the highest root in $\mathring{\Delta}_+$ with $ht\ \theta = k$. Then there is a finite sequence $\{\theta_n\}$ in $\mathring{\Delta}_+$, with the following properties:*
(a) *$ht\ \theta_i = i$ for $1 \leq i \leq k$.*
(b) *Every root $\alpha \in \mathring{\Delta}_+$ can be represented as a connected subroot of θ_i for some i with $1 \leq i \leq ht\theta$.*

PROOF. By Lemma 3.3, θ can be written in the form $\alpha_{i_1} + \cdots + \alpha_{i_k}$ ($\alpha_{i_j} \in \Pi$ not necessarily distinct) in such a way each partial sum is a root. Construct a sequence $\{\theta_n\}$ by the following table:

$\mathfrak{g}(A)$	h	$\check{h}$	$\theta_n,\ \theta$
$A_l^{(1)}$	$l+1$	$l+1$	$\theta_n = \sum_{i=1}^n \alpha_i,\ \theta = \alpha_1 + \cdots + \alpha_l$
$A_{2l-1}^{(2)}$	$2l-1$	$2l$	$\theta_n = \begin{cases} \sum_{i=1}^n \alpha_i & \text{for } n \leq l, \\ \sum_{i=1}^l \alpha_i + \sum_{i=2l-n}^{l-1} \alpha_i & \text{for } l+1 \leq n \leq 2l-1, \end{cases}$ $\theta = 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{l-1} + \alpha_l$
$A_{2l}^{(2)}$	$2l+1$	$2l+1$	$\theta_n = \begin{cases} \sum_{i=1}^n \alpha_i & \text{for } n \leq l, \\ \sum_{i=1}^l \alpha_i + \sum_{i=2l-n}^{l-1} \alpha_i & \text{for } l+1 \leq n \leq 2l-1, \end{cases}$ $\theta = 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{l-1} + \alpha_l$

This completes the proof. □

THEOREM 3.5. *Let A be a generalized Cartan matrix of type $A_l^{(1)}$ or $A_{2l-1}^{(2)}$ and θ the highest root of $\mathfrak{g}(\mathring{A})$. Then the degree of nilpotency is greater than or equal to* htθ.

PROOF. Let w_0 be an element of W such that $w_0^{-1}(\mathring{\Delta}_+) = \mathring{\Delta}_-$ and let θ be the highest root. Let $\{\theta_n\}$ be the sequence in the above table. Then we have

$$\mathfrak{g}_{\theta_i} \subseteq S_{w_0}^{i-1} \quad \text{for} \quad i = 1, \cdots, \mathrm{ht}\theta.$$

In particular, $\{0\} \neq \mathfrak{g}_\theta \subseteq S_{w_0}^{ht\theta - 1}$. This complete the proof. □

LEMMA 3.6 (See [4]). *Let A be a generalized Cartan matrix of type ${A_{2l}}^{(2)}$. Then there exists an element $w_1 \in W$ such that*

$$\left\{\frac{1}{2}(\alpha_l + \delta), \frac{1}{2}(\alpha_l + 3\delta)\right\} \cup \mathring{\Delta}_+ \ \subset \mathring{\Delta}^+(w_1).$$

THEOREM 3.7. *Let A be a generalized Cartan matrix of type $A_{2l}^{(2)}$ and θ the highest root of $\mathfrak{g}(\mathring{A})$. Then the degree of nilpotency is greater than or equal to* ht$\theta + 1$.

PROOF. Let w_1 be an element of W such that

$$\left\{\frac{1}{2}(\alpha_l + \delta), \frac{1}{2}(\alpha_l + 3\delta)\right\} \cup \mathring{\Delta}_+ \subset \mathring{\Delta}^+(w_1).$$

Then we can construct a sequence $\{\gamma_n\}$ in $\mathring{\Delta}^+(w_1)$ such that $\gamma_1 = \frac{1}{2}(\alpha_l + \delta)$, $\gamma_2 = \frac{1}{2}(\alpha_l + 3\delta), \ldots, \gamma_{2k-1} = \gamma_{2k} = \alpha_{l-(k-1)}$ for $2 \leq k \leq l$. By direct calculation, we have

$$\mathfrak{g}_{\sum_{i=1}^{n} \gamma_i} \subseteq S_{w_1}^{n-1} \tag{3.8}$$

for $1 \leq n \leq 2l$. Since $\sum_{i=1}^{2l} \gamma_i = \theta + 2\delta$, combining (3.8), we have

$$\{0\} \neq \mathfrak{g}_{\theta+2\delta} \subseteq S_{w_1}^{ht\theta}.$$

This completes the proof. □

EXAMPLE 3.8. In $A_4^{(2)}$, let $w = r_1r_2r_1r_2r_0r_1r_0r_2r_1r_0r_2r_1r_0$. Then we have

$$\begin{aligned}\Delta^+(w) = \{&\alpha_1, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_2, \frac{1}{2}(2\alpha_1 + \alpha_2 + \delta), \alpha_1 + \alpha_2 + \delta, \\ &\frac{1}{2}(\alpha_2 + \delta), 2\alpha_1 + \alpha_2 + 2\delta, \alpha_1 + \alpha_2 + 2\delta, \frac{1}{2}(2\alpha_1 + \alpha_2 + 3\delta), \\ &\alpha_2 + 2\delta, \alpha_1 + \alpha_2 + 3\delta, \frac{1}{2}(\alpha_2 + 3\delta)\}.\end{aligned}$$

Put $\gamma_1 = \frac{1}{2}(\alpha_2 + \delta)$, $\gamma_2 = \frac{1}{2}(\alpha_2 + 3\delta)$, $\gamma_3 = \alpha_1$, $\gamma_4 = \alpha_1$. Then $\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 = 2\alpha_1 + \alpha_2 + 2\delta$ is a root, and hence

$$\{0\} \neq \mathfrak{g}_{2\alpha_1+\alpha_2+2\delta} \subset S_w^3.$$

References

[1] Yuly Billig and Arturo Pianzola, *Root Strings with Two Consecutive Real roots*, Tôhoku Math. J. **47** (1995), 391–403.

[2] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, 1972.

[3] V. G. Kac, *Infinite-Dimensional Lie Algebras*, Cambridge University Press, 1990.
[4] Yeonok Kim, *A Note on The Root System of an Affine Lie Algebra of Type* $A_{2l}^{(2)}$, Commun. Korean Math. Soc. **16** (2001), no. 1, 85–94.
[5] J. Morita, *Root strings with three or four real roots in Kac-Moody root systems*, Tôhoku Math. J. **40** (1988), no. 4, 645–650.
[6] Z. Wan, *Introduction to Kac-Moody Algebra*, World Scientific Publishing Co. Pte. Ltd., 1991.

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