Equivalent Image Method for the Analysis of Electrostatic Problems

Eui-Joong Kim[1] · Young-Soon Lee[1] · Kun-Woo Kim[2] · Young-Ki CHo[2]

Abstract

For the purpose of numerically efficient analysis for electrostatic problems, an improved equivalent image method, by which infinite exact image of a point charge in parallel-plate and rectangular conducting structures can be replaced by only a few equivalent images, is considered. Through this study, it is observed that the error is of the order of 0.001 % when the present method is used to evaluate electric potential for a point charge in parallel-plate and rectangular conducting planes.

Key words : PPW, Exact Image, Green's Function, Closed-Form.

Ⅰ. Introduction

Thus far, in order to accelerate the calculation of spatial Green's function, many research results on the closed-form Greens functions have been reported. Most of those were closed-form Green's functions method for microstrip structures[1]~[3]. The general expression of closed-form is of $\sum a_n e^{-jkr_n}/r_n$(where $r_n = \sqrt{\rho^2+(z+jb_n)^2}$), in which a_n and b_n are complex coefficients. Therefore the closed-form Green's functions methods have been called also complex image methods. Some have dealt with complex image method for the problem of the parallel-plate waveguide[4],[5]. Others have dealt with complex image method for the analysis of electrostatic problems[6],[7]. In [6], the complex image method was introduced to replace infinite images of a static point charge in parallel-plate conducting planes by a few complex images, in which case a_i and b_i were all real. For this reason, the complex images in [6] and [7] are called *equivalent images*. To avoid the confusion between images and equivalent images, infinite original images are called *exact images*, while the simulated images which can replace infinite images are called *equivalent images* hereafter.

Here, an improved equivalent image method, which can give more accurate results in comparison with those obtained by the previous method[6],[7] are proposed by employing the scheme of the generalized pencil of functions(GPOF)[3] as an approximation technique.

Ⅱ. Infinite Exact Image of Point Charge in Parallel-Plate Conducting Structure

Consider a parallel-plate structure with height h_z, in side which a point charge is located at (x_0, y_0, z_0), as shown in Fig. 1.

In order to calculate the electric potential the adoption of image method yields the exact images, which are discretely located on the infinite line ($x=x_0$, $y=y_0$), with positive and negative images in turn: the positive images

$$+q(x_0, y_0, z_m^+) \quad (1)$$

and the negative images

$$-q(x_0, y_0, z_m^-) \quad (2)$$

where $z_m^{\pm} = 2mh_z \pm z_0$, in which $m = 0, \pm1, \pm2, \cdots$.

Then the spatial electric potential of the point charge in the parallel-plate conducting structure is exactly

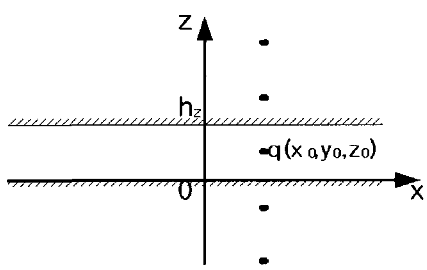

Fig. 1. Exact image of a point charge in a parallel-plate conducting structure.

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obtained by use of the Coulomb's law as

$$\Phi_{(x,y,z)} = \frac{q}{4\pi\varepsilon_0}\sum_{m=-\infty}^{+\infty}\left(\frac{1}{r_m^+}-\frac{1}{r_m^-}\right) \quad (3)$$

where $r_m^{\pm}=\sqrt{\rho^2+(z-z_m^{\pm})^2}$, $\rho=\sqrt{(x-x_0)^2+(y-y_0)^2}$

III. Equivalent Images of Point Charge in Parallel-Plate Conducting Structure

In order to derive the equivalent images which can replace the infinite exact images, three stages are performed as in [6]. First, through the Fourier transform for (3), the spectral potential of (3) is obtained as

$$\tilde{\Phi}(z,\gamma_z) = \frac{qe^{-jk_x x_0}e^{-jk_y y_0}}{2\varepsilon_0\gamma_z}\{e^{-\gamma_z|z-z_0|}-e^{-\gamma_z(z+z_0)}$$
$$+e^{-\gamma_z(2h_z-z+z_0)}-e^{-\gamma_z(2h_z-z-z_0)}+[e^{-\gamma_z(z-z_0)}-e^{-\gamma_z(z+z_0)}$$
$$+e^{-\gamma_z(2h_z-z+z_0)}-e^{-\gamma_z(2h_z-z-z_0)}]\frac{1}{e^{2\gamma_z h_z}-1}\Big\} \quad (4)$$

where $\gamma_z=\sqrt{k_x^2+k_y^2}$.

Next, the last expression of (4) is approximated in terms of a sum of decaying exponential functions as

$$\frac{1}{e^{2\gamma_z h_z}-1} \cong \sum_{i=1}^{M} a_i e^{b_i\gamma_z h_z} \quad (5)$$

where M is the number of exponential terms, and $a_i(b_i)$ is amplitude coefficients(exponents) which can be obtained for a given M by use of approximation technique, such as the Prony's method in [1] and the GPOF method in [3]. In the previous study[6],[7], a_i and b_i were calculated only when $M=3$ and $M=4$ by using the Prony's method as approximation technique. When the Prony's method were used, M was typically taken 2~4[6],[7], because the case of large M (e.g., 5~7) could give worse results rather than that of small M (e.g., 2~4).

On the other hand, the GPOF method has many advantages in comparison with the Prony's method from the viewpoint of the restriction of M and noise sensitivity of approximation method[3]. Some results for a_i and b_i obtained by use of the GPOF are given in Table 1, for the case that $M=5$, 6 and 7.

Finally, after a_i and b_i are determined, substituting (5) into (4) and taking the inverse Fourier transform yields

$$\Phi = \frac{q}{4\pi\varepsilon_0}\left[\left(\frac{1}{R_0^+}-\frac{1}{R_0^-}+\frac{1}{R_1^+}-\frac{1}{R_1^-}\right)+\sum_{i=1}^{M}a_i\left(\frac{1}{R_{0i}^+}-\frac{1}{R_{0i}^-}+\frac{1}{R_{1i}^+}-\frac{1}{R_{1i}^-}\right)\right] \quad (6)$$

where
$$R_0^{\pm}=\sqrt{\rho^2+(z\mp z_0)^2},$$
$$R_1^{\pm}=\sqrt{\rho^2+(z-2h_z\mp z_0)^2},$$
$$R_{0i}^{\pm}=\sqrt{\rho^2+(z-b_ih_z\mp z_0)^2},$$
$$R_{1i}^{\pm}=\sqrt{\rho^2+(z-2h_z+b_ih_z\mp z_0)^2}.$$

Thus the equivalent images are

$$\pm q(\pm z_0),\quad \pm a_i q(\pm z_0+b_ih_z),$$
$$\pm q(2h_z\pm z_0),\quad \pm a_i q(2h_z\pm z_0-b_ih_z) \quad (7)$$

where (·) means the z-coordinate of the images.

Fig. 2 presents a comparison of some evaluation results for the electric potentials obtained using the proposed method(dotted lines) with exact results obtained by infinite exact image method(solid lines) for M= 5, 6 and 7. A good agreements between the two sets of results are observed over the whole spatial range. Moreover, it is seen that Dirchlet's boundary condition(i.e. $\phi=0$ on $z=0$ and $z=h_z$ plane) is satisfied by a few equivalent images. To validate the present results in detail, Fig. 3 shows the relative error for the evaluation results presented in Fig. 2. It is seen that the case of large M(e.g., 7) give more accurate results in comparison with the case of small M(e.g., 5).

Consequently, the relative error is of the order of 0.001 % for the whole spatial range when $M=7$ in the present study, while the error is of the order of 0.1 % when $M=4$, in the previous studies [6], [7].

Table 1. a_i(Amplitude) and b_i(relative position) of equivalent images when M = 5, 6 and 7.

M	i	a_i	b_i
5	1	1.00457 + j0.0	− 2.00139 + j0.0
	2	12.23460 + j0.0	− 28.87936 + j0.0
	3	1.21285 + j0.0	− 4.13509 + j0.0
	4	2.23169 + j0.0	− 7.41100 + j0.0
	5	4.61085 + j0.0	− 13.92181 + j0.0
6	1	1.00030 + j0.0	− 2.00008 + j0.0
	2	12.38403 + j0.0	− 32.94351 + j0.0
	3	1.04183 + j0.0	− 4.02074 + j0.0
	4	2.58614 + j0.0	− 10.34143 + j0.0
	5	4.91036 + j0.0	− 17.52359 + j0.0
	6	1.46313 + j0.0	− 6.42160 + j0.0
7	1	1.00001 + j0.0	− 2.00000 + j0.0
	2	1.00487 + j0.0	− 4.00197 + j0.0
	3	12.47566 + j0.0	− 36.60324 + j0.0
	4	5.10214 + j0.0	− 20.89413 + j0.0
	5	1.13249 + j0.0	− 6.09201 + j0.0
	6	1.70090 + j0.0	− 8.83727 + j0.0
	7	2.83574 + j0.0	− 13.26257 + j0.0

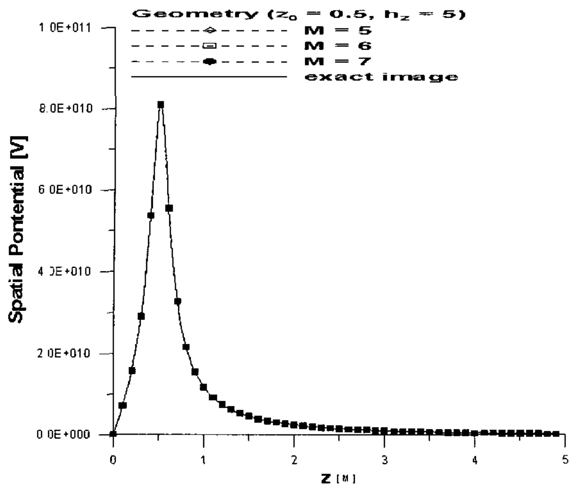

Fig. 2. Comparison between evaluation results of electric potentials obtained using the equivalent image method(dotted lines) and those(solid lines) obtain by exact image method.

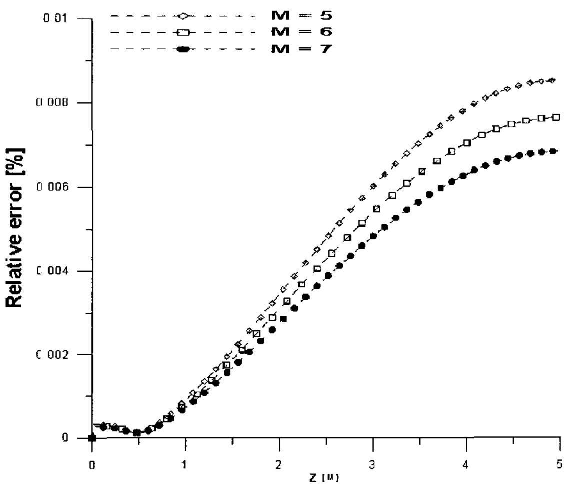

Fig. 3. The relative error for evaluation results of electric potentials.

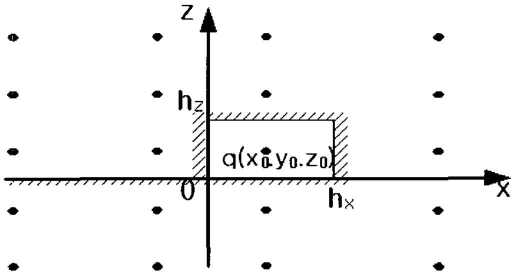

Fig. 4. Exact images of a point charge in a rectangular conducting planes.

Ⅳ. Application to Rectangular Conducting Planes

Consider a rectangular conducting planes with dimension $h_x \times h_z$, in which a point charge $q(x_0, y_0, z_0)$ is located, as shown in Fig. 4. As shown in Fig. 4, the exact images of a point charge in the rectangular conducting planes construct a plane array which consists of infinite line arrays.

To obtain the equivalent images, two steps are performed as in the exact image theory. First, on the line $x = x_0$(i.e., in the z direction), the corresponding infinite images are replaced to a few equivalent images of (7). Then the equivalent images in the x direction for each of those in the z direction can be obtained easily through the same procedures as in the parallel-plate conducting planes.

For instance, the equivalent images for the image $+qa_i(z_0 + b_i h_z)$ are of

$$\begin{aligned} &\pm q a_i (x_0, y_0, z_0 + b_i h_z), \\ &\pm q a_i (2h_x \pm x_0, y_0, z_0 + b_i h_z), \\ &\pm q a_i a_j (\pm x_0 - b_j h_x, y_0, z_0 + b_i h_z), \\ &\pm q a_i a_j (2h_x \pm x_0 - b_j h_x, y_0, z_0 + b_i h_z) \end{aligned} \tag{8}$$

in which a_i and b_i are the same as (5), which are listed in Table 1. On the analogy of above deduction, the equivalent images in the x direction of the other images can be also easily derived.

Ⅴ. Numerical Results

In order to check the validity of the present method,

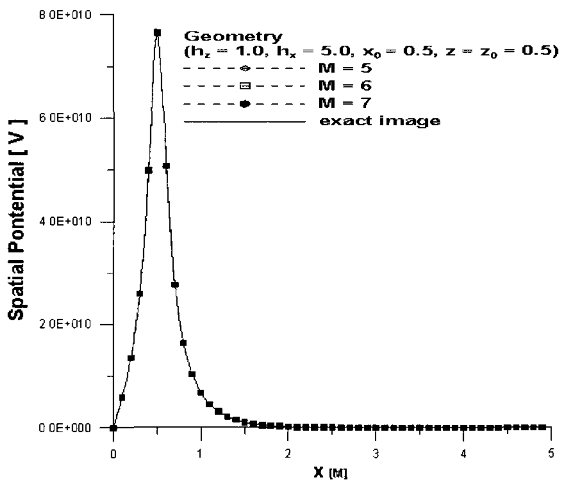

Fig. 5. Comparison between present results for electric potentials (dotted lines) and those(solid lines) obtained by exact image method.

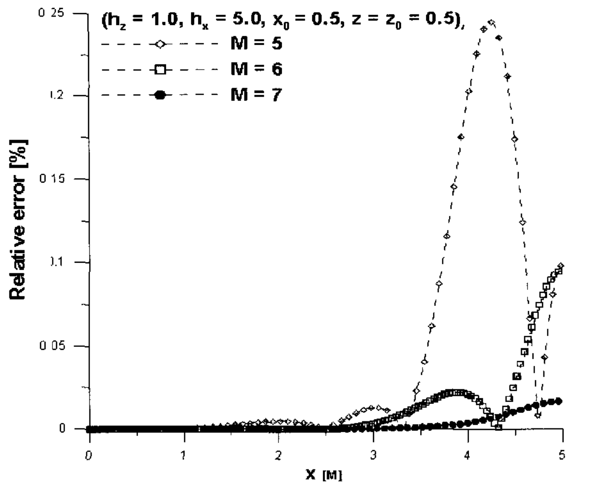

Fig. 6. The relative error for evaluation results of electric potentials.

electric potentials for a point charge in the rectangular conducting planes are evaluated by the present method and the exact image method. Fig. 5 shows comparison of results obtained by using the present method (dotted lines) with those obtained by the exact image method (solid lines). And also the relative errors are presented in Fig. 6. Once again, the relative error is of the order of 0.001 % when $M=7$.

Ⅵ. Conclusion

The present study has shown that only a few equivalent images are used to replace the infinite exact images, with the error kept within order of 0.001 % for the case of evaluation of the electric potential. Consequently the present method may help in analyzing the electrostatic problems relevant to parallel and rectangular conducting planes structures.

Besides the present method give very accurate evaluation results in comparison with the previous method. It is also observed that the Dirchlet's boundary condition is satisfied even with a few number of equivalent images. This study may also serve as an interesting example in a graduate course in the method of images.

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