Parametric Inverse Scattering for Lossless Dispersive Media with Enhanced Robustness

Hyoung-Jin Choi · Sang-Seol Lee

Abstract

The effects of high frequency noises on a perturbational inversion technique for a stratified dispersive medium are investigated in this paper. It is shown that the perturbational solution becomes unstable under high frequency noises. The physical origin of this instability is described. In order to enhance the robustness of the perturbational inverse scattering solution, a parametric inversion technique is introduced. The examples for the 2-pole and the 3-pole reflection coefficients are compared and contrasted, and improvement of the robustness of the solutions is shown.

Key words : Perturbational Inversion, Parametric Inversion, Robustness, Dispersive Medium, Riccati Equation.

Ⅰ. Introduction

The parametric inverse scattering theory is used to reconstruct electron density profiles. The one-dimensional potential profile V(z) is obtained based on the reflection coefficient $r(k)$ using a parametric inversion. This problem has been investigated by both the exact [1],[2] and the perturbational[3] inverse scattering theories. The perturbational solution may not be robust due to a high frequency noise since $r(k)$ is amplified for the high frequency during the inversion process by the dispersion relation. To enhance the robustness of the perturbational inversion, we apply the parametric inversion. The perturbational reconstruction solution with a proper bandlimitation is chosen as a trial function for the parametric inversion.

To illustrate this, the 2-pole and 3-pole reflection coefficients under noisy and bandlimited conditions are considered. The high frequency noise added to the reflection coefficient produces extremely large values in the perturbational inversion. The bandlimitation on the reflection coefficient affects both the maximum value of V(z) and the resolution of the reconstruction.

Ⅱ. Dispersion

We consider a time-harmonic wave that has the propagation factor of the form $e^{-i(\omega t - k\bar{n}\cdot\bar{r})}$ where ω and k are the angular frequency and the free-space wavenumber, respectively. Here, $\bar{n}$ is a complex refractive index vector and $\bar{r}$ is a position vector. If the phase factor is written as

$$ikn(\omega) = \alpha(\omega) + i\beta(\omega), \tag{1}$$

the wave propagates with an attenuation of α (nepers/m) and a phase shift of β (rad/m). A lossless nondispersive case can be stated as $\alpha(\omega)=0$ and $\beta(\omega)=\omega v_p$, where v_p is a phase velocity (m/s). If β is a function of ω, the various dispersions take places[4], and we consider the parametric one here.

At high frequencies, the dielectric constants become a frequency-dependent function. Thus we treat the molecules in the dielectrics as a dynamical system[4]. We consider the average dipole moment and polarization in a linear isotropic medium excited by an incident electric field. The forces on an electron due to the applied field $\bar{E}$ on an electron and the friction are

$$\bar{F}_E = -q\bar{E}_0 e^{-i\omega t} \tag{2}$$

$$\bar{F}_{Friction} = -R\,d\bar{r}/dt \tag{3}$$

where q and R are the charge of the electron (coulomb) and resistance (Ω), respectively, and $\bar{E}_0 e^{-i\omega t}$ is the applied field. If we treat the negative charge cloud as a single rigid body, the steady-state solution to the equation of motion is[5]

$$\bar{r} = iq\bar{E}_0 e^{-i\omega t} / [\omega m(g - i\omega)] \tag{4}$$

where m is the mass (Kg) of the electron and $g=R/m$. The microscopic definition of the current density with

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(4) becomes

$$\bar{J} = Nq^2\bar{E}_0 e^{-i\omega t}/[m(g-i\omega)] = \sigma'\bar{E}_0 e^{-i\omega t} \quad (5)$$

$$\sigma' = Nq^2/[m(g-i\omega)] \quad (6)$$

where N is the number of electrons per unit volume and σ' is the complex conductivity. With the complex permittivity ε' (F/m) and σ' of (6),

$$k^2 n(z) = \omega^2\mu_0\varepsilon\left[1 - \frac{Nq^2}{\varepsilon m(g^2+\omega^2)} + i\frac{Nq^2 g}{\omega\varepsilon m(g^2+\omega^2)}\right] \quad (7)$$

where μ_0 is the free-space permeability (H/m) and the real and imaginary parts of (7) can be derived from

$$\alpha^2 - \beta^2 = \omega^2\mu_0\varepsilon[1 - Nq^2/\{\varepsilon m(g^2+\omega^2)\}] \quad (8)$$

$$2\alpha\beta = Nq^2 g/[\omega\varepsilon m(g^2+\omega^2)]. \quad (9)$$

In a tenuous ionized gas, the damping factor g becomes negligible since the mean path of the electron is rather long. Thus (7) can be rewritten as

$$kn(\omega) = \omega[\mu_0\varepsilon_0]^{1/2}[1-\omega_c^2/\omega^2]^{1/2} \quad (10)$$

where $\omega_c^2 = Nq^2/(\varepsilon_0 m)$. When the electron density becomes $N(z)$ due to the inhomogeneous medium, the ω_c must be a function of z. Substituting (10) into (1), the refractive index becomes

$$n(\omega) = [1-\omega_c^2(z)/\omega^2]^{1/2} = [1-\mathrm{V}(z)/k^2]^{1/2} \quad (11)$$

where $\omega_c^2(z)/\omega^2 = V(z)/k^2$ and the profile potential V $(z)=qN(z)/(\varepsilon_0 mc^2)$ with the light velocity c. If $\omega >> \omega_c$, the ionized medium acts like the free space. To use the high frequency reflections, V(z) in (11) must be properly amplified during the inversion process, that causes the instability of the inversion.

Ⅲ. Perturbational Inversion

We consider the reflection by a normally incident time-harmonic planewave from an unknown plane-stratified inhomogeneous dispersive medium. The wave propagates along the z in the free space for $z<0$ and in the medium for $z\geq 0$. The local wave number $k(z)$ is described by

$$k(z) = \begin{cases} k & , \ z<0 \\ kn(z) & , \ z\geq 0 \end{cases} \quad (12)$$

where k is the free-space wavenumber. The refractive index is assumed to be lossless and nonmagnetic. A unit-amplitude incident wave illuminates the medium from the left-half space, and a reflected wave with the amplitude of the complex reflection coefficient $r(k)$ propagates in the negative z direction. The refractive index is to be determined from $r(k)$ which is related to the electron density profile.

The electrical characteristics of the profile to be reconstructed are defined by V(z) where V(z) is zero for $z<0$. The potential function is related to the relative dielectric permittivity.

$$\varepsilon_r(z,k) = 1 - \mathrm{V(z)/k^2}. \quad (13)$$

For the time-harmonic plane waves, the transverse component of $E(z,k)$ satisfies the wave equation for all z, and the wave equation becomes the Schrofinger equation under the dispersion relation given in (13). Using the one-dimensional free-space Green's function, the inhomogeneous solution of the equation becomes

$$E(z,k) = \int_0^\infty V(z')E(z',k)G(z,z')dz', \quad (14)$$

for $z<0$, where

$$G(z,\ z') = ie^{ik|z-z'|}/2k. \quad (15)$$

By using Born approximation inside the medium, only one unknown remains in (14) where $z\geq 0$. Then the total solution and the reflection coefficients become

$$E(z,k) = e^{ikz} + \left[\int_0^\infty \frac{i}{2k} e^{i2kz'} V_{pert}(z')dz'\right] e^{ikz} \quad (16)$$

and

$$r(k) = \int_0^\infty \frac{i}{2k} e^{i2kz'} V_{pert}(z')dz' \qquad \text{for } z<0 \quad (17)$$

where V_{pert} is the perturbation potential. Defining a new frequency variable, $p=-2ik$, (17) yields

$$r(p) \cong \frac{-1}{p}\int_0^\infty V_{pert}(z)e^{-pz}dz = \frac{-1}{p}L[V_{pert}(z)] \quad (18)$$

where L denotes the Laplace transform. Inversion of (18) produces a perturbational inversion as

$$V_{pert}(z) = L^{-1}[-pr(p)] \quad (19)$$

where L^{-1} denotes the inverse Laplace transform operator. Using Bromwitch integral, (19) can be expressed in k as

$$V_{pert}(z) = \frac{2i}{\pi}\int_{-\infty}^{\infty} kr(k)e^{-i2kz}dk. \quad (20)$$

This approximation is valid for small potential or in the high frequency regime[3]. We note that the $pr(p)$ term in (19) may produce instability in the reconstruction if there is a high frequency noise in $r(p)$.

To improve the robustness of the perturbational inversion under the bandlimited and noisy condition, the $V(z)$ is reconstructed using the perturbation inversion given by (20) where $r(k)$ has additive random noises as

$$r_R(k) = r_{ER}(k) + r_{NR}(k) \tag{21}$$

$$r_I(k) = r_{EI}(k) + r_{NI}(k) \tag{22}$$

where γ_{ER}, γ_{EI}, γ_{NR}, γ_{NI} and are the exact real, the exact imaginary, the noise real, and the noise imaginary reflection coefficients, respectively. The noise is generated by random process as

$$r_{NR}(k) = NA\cos[\phi(k)] \tag{23}$$

$$r_{NI}(k) = NA\sin[\phi(k)] \tag{24}$$

where NA is the noise amplitude which is a constant, and $\phi(k)$s are random numbers with uniform probability density ranging from zero to 2π. The exact reflection coefficients for the following two cases are given by [3]

$$r(p)_{2-pole} = (4k_1k_2)/[(p+2ik_1)(p+2ik_2)] \tag{25}$$

$$r(p)_{3-pole} = (8ik_1k_2k_3)/[(p+2ik_1)(p+2ik_2)(p+2ik_3)] \tag{26}$$

where $k_i(i = 1, 2, 3)$ designates the pole locations in the complex k plane. To investigate the effects of the bandlimited and noisy condition in the perturbational inversion, we use the reconstruction of the potentials for the 2-pole and 3-pole noiseless cases shown in Fig. 1, that provides a useful approximation. However, the reconstructed potential for the noisy 2-pole and 3-pole cases becomes instable when k_f is chosen relatively high (k_f=16). If k_f is decreased too much, the amplitude of the reconstructed potential will clearly become smaller. Hence, for the trade-off between the instability and the resolution, the k_f must be chosen carefully in the place where the slope of the magnitude of the $r(k)$ is approximately zero. However, at the place noise becomes dominant. The resolution has to be decreased due to the noise filtering.

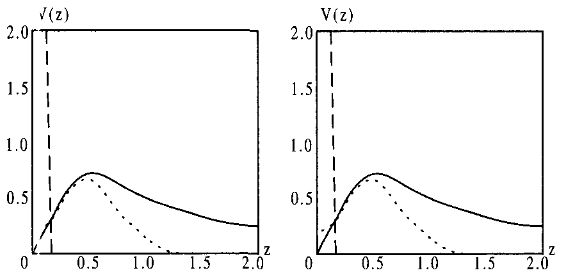


Fig. 1. Exact and perturbational reconstructions: Solid line (Exact solution), Dotted line(k_f = 4, NA=0), Dashed line(k_f = 16, NA=0.1).

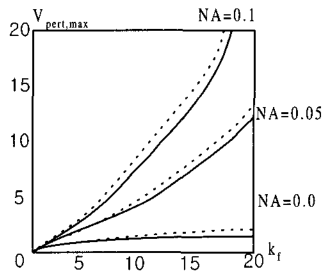


Fig. 2. Maximum values of perturbationally reconstructed potential for 2-pole and 3-pole causes under noisy and bandlimited conditions: Solid line (2-pole case), Dotted line(3-pole case).

The amount of noise is given using the signal-to-noise ratio (SNR) as

$$SNR=[\int_0^{kf} r_E(k)|^2dk] \ / \ [\int_0^{kf} |NA|^2dk] \tag{27}$$

where $r_E(k)$ is the exact value, and the denominator of the SNR becomes $(NA)^2k_f$ since NA is a constant. This SNR will increase if the NA and/or k_f is decreased. Therefore, applying low-pass filtering improves the SNR. As shown in Fig. 2, the maximum potential $V_{pert,max}$ increase rapidly as increasing of noises.

Ⅳ. Parametric Inversion

Since a parametric inversion requires a trial function, we use a properly bandlimited perturbational reconstruction solution as a trial function. Two parameters are introduced to the trial function. They will be determined by optimizing a root mean square error (RMSE). Mathematically, they are coordinate stretches in the x-y directions. Since reconstruction errors in both height and longitudinal placement are not uniform over the entire profile, we introduce more parameters such as

$$f(V) = \zeta_0 + \zeta_1 V + \zeta_2 V^2 + \cdots\cdots \tag{28}$$

$$f(z) = \eta_0 + \eta_1 z + \eta_2 z^2 + \cdots\cdots. \tag{29}$$

For simplicity, we use only 2 parameters ζ_1 and η_1 in (28) and (29) which are denoted by ζ and η.

A parametric inversion is based on the concept that the reflection data with undetermined parameters are compared with the exact or measured reflection. Then, the corresponding reflection coefficient is compared with measured data by RMSE. This process is repeated until the optimum value of ζ and η are obtained. The coordinates-stretched expression is given by

$$V_p(z)_{m+1} = \zeta V_p(\eta z)_m \tag{30}$$

where m is the number of iteration and V_p is a parametric solution. By using the perturbation solution, a trial function is

$$V_{pert}(z) = \frac{-4}{\pi}\int_0^\infty k[r_I(k)\cos(2kz) - r_R(k)\sin(2kz)]dk \tag{31}$$

which is obtained from (20) with the causality condition. For the simulation, we choose limit of integration with the k_f instead of infinity.

The trial function is used to get $r(k)$ in the Riccati equation:

$$\mathrm{d}r_R/\mathrm{d}z = -k[r_I(r_R-1) - \varepsilon_r(z) r_I(r_R-1)] \tag{32}$$

$$\mathrm{d}r_I/\mathrm{d}z = -0.5k[\{(1-r_R)^2 - r_I^2\} + \varepsilon_r(z)\{(1+r_R)^2 - r_I^2\}] \tag{33}$$

with $\varepsilon_r(z,k) = 1 - V(z)/k^2$. The reflection coefficient $r_O(k)$ is calculated by the Riccati equation, and we have the RMSE as

$$\mathrm{RMSE} = \sum_{k_i}^{k_f}\left[(r_{ER}-r_{OR})^2 + (r_{EI}-r_{OI})^2\right]^{1/2} \tag{34}$$

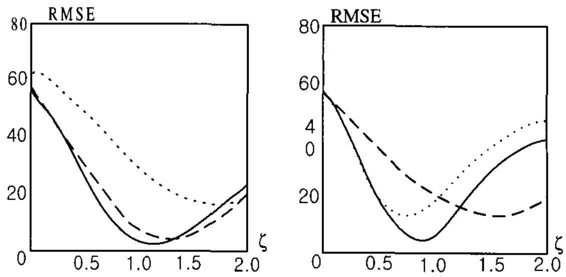
The RMSE is presented in Fig. 3 for the 2-pole and 3-pole cases.

Since the exact profiles are available from [1], [2] for the noiseless 2-pole and 3-pole cases, it allows us to calculate the following profile-to-error ratio (PER):

$$\mathrm{PER} = \left[\int_0^\infty V^2{}_{exact}\,dz\right] \Big/ \left[\int_0^\infty (V_{exact} - V_{recon})^2 dz\right] \tag{35}$$

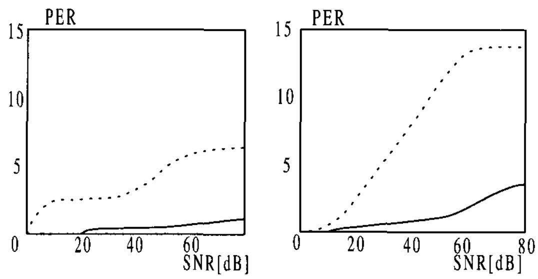
where V_{exact} is the exact potential and V_{recon} is the potential obtained by the perturbational or parametric inversion. The PER's for the 2-pole and 3-pole cases are shown in Fig. 4. It is evident that the large PER becomes the better the reconstruction process. Therefore we see that the parametric inversion improves the reconstruction obtained from the perturbational one.

V. Conclusion



(a) 2-pole case (b) 3-pole case

Fig. 3. RMSE.
(a) $NA = 0$: Solid line(η=2.225), Dashed line (η=1.025), Dotted line(η=3.5).
(b) $NA = 0$: Solid line(η=1.75), Dashed line (η=1.375), Dotted line(η=3.5).



(a) 2-pole case (b) 3-pole case

Fig. 4. Profile-to-error ratio : Solid line(perturbational inversion), Dotted line(parametric inversion).

It has been known that the perturbational inversion gives an excellent solution for the general shape of the potential function of the one-dimensional, inhomogeneous, lossless, dispersive, nonmagnetic media. However, it becomes unstable with high frequency noise added to the reflection coefficient. Physically, this instability comes from the fact that a wave propagates without much reflection if the wavenumber becomes much large in the dispersion relation.

To circumvent this instability, we have applied a parametric inversion technique and chosen a perturbational reconstruction solution as a trial function. Subsequently, the coordinate stretches are performed to optimize the reconstruction. We have shown that the proposed technique improves the robustness of a perturbational reconstruction. This improvement has been calculated quantitatively using the PER.

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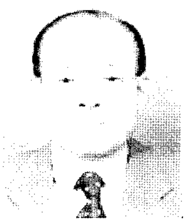

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