# Mixed $H_2/H_\infty$- Controller Realization with Entropy Integral

**Sang-Hyuk Lee and Ju-Sik Kim**

**Abstract**: An $H_2/H_\infty$-controller realization is carried out by considering an entropy integral. Using *J*-spectral factorization, the parametrizations of all $H_\infty$ stabilizing controllers are derived. By the relation of a mixed $H_2/H_\infty$ control problem and a minimum entropy/$H_\infty$ control problem, the mixed $H_2/H_\infty$-controller state-space realization is presented.

**Keywords**: Mixed $H_2/H_\infty$ control, *J*-spectral factorization, minimum entropy.

## 1. INTRODUCTION

The fundamental differences between $H_2$ control design and $H_\infty$ control theory can be discriminated by the modeling and exogenous disturbances treatment. Mustafa [1] showed the entropy of a system which satisfies an $H_\infty$ norm bound, and derived some important properties, including that entropy is an upper bound on $H_2$ cost. Entropy has been established as an important notation, with a wide applicability in a number of diverse subjects, used in spectral analysis.

Recently, mixed $H_2/H_\infty$ optimal control problems have received a great deal of attention and several authors have investigated these problems for continuous-time linear systems, by means of different approaches. The problem of maximizing the entropy of a stabilized closed-loop system has been solved by Mustafa and Glover [2]. Their solution exploits the parameterization of all closed-loop systems that satisfy an $H_\infty$ norm bound. Bernstein and Haddad [3] provided a solution to mixed $H_2/H_\infty$ control problems by designing a LQG control subject to a constraint on $H_\infty$ disturbance attenuation. Motivated by this work, Doyle *et al.* [4] and Zhou *et al.* [5] derived sufficient conditions for another kind of mixed $H_2/H_\infty$ problem to be solvable. It was shown later by Yeh *et al.* [6] that the solution of Bernstein and Haddad and the solution presented by Doyle *et al.* [4] are actually dual to each other.

We propose a parametrization of the class of all controllers in terms of the Youla parametrization. The $H_\infty$-controller design procedure is built upon the *J*-spectral factorization approach to $H_\infty$ control. However, Mustafa and Glover [7] proposed that the minimum entropy controller is the *central solution* in the parameterization of all stabilizing controllers satisfying $\| H \|_\infty < \gamma$. Therefore, the *central solution* satisfies the minimum entropy controller. Using the results of Mustafa [1], we present a $H_2/H_\infty$-controller state-space realization. In this paper, we introduce a minimum entropy/$H_\infty$ control problem and a mixed $H_2/H_\infty$ control problem. Also for a linear time invariant system, a mixed $H_2/H_\infty$-controller realization is derived. As is customary, let $\Re H_\infty$ denote the principal ring of proper stable rational functions of a complex variable with real coefficients. Thus, $m(\Re H_\infty)$ is the set of matrices with elements in $\Re H_\infty$.

## 2. STATEMENT OF THE PROBLEM

The control problem addressed in this paper concerns the finite-dimensional linear time-invariant feedback system depicted in Fig. 1.

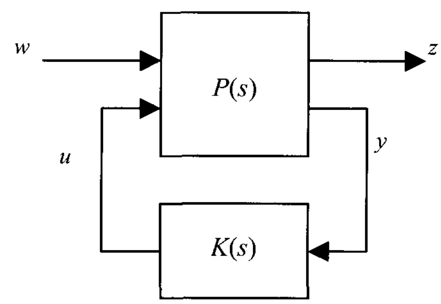


Fig. 1. Block diagram of the closed loop system.

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Consider an $n$-state plant $P(s)$ with a state space description

$$\dot{x}(t) = Ax(t) + B_w w(t) + Bu(t), \quad (1)$$

$$z(t) = C_z x(t) + D_{12} u(t), \quad (2)$$

$$y(t) = Cx(t) + D_{21} w(t), \quad (3)$$

where $x(t) \in R^n$ is the state variable and $w(t) \in R^{n_w}$ represents all external inputs, including disturbances and sensor noise. $z(t) \in R^{n_z}$ is the error output and $u(t) \in R^m$ and $y(t) \in R^r$ are the control input and the measured output, respectively.

In the following, we assume that

A1) $(A, B)$ can be stabilized and $(C, A)$ is detectable.

A2) $D_{12}^T D_{12} = I,\ D_{21} D_{21}^T = I,\ B_w D_{21}^T = 0,\ C_z^T D_{12} = 0.$

A3) $\begin{bmatrix} A - \lambda I & B \\ C_z & D_{12} \end{bmatrix}$ and $\begin{bmatrix} A - \lambda I & B_w \\ C & D_{21} \end{bmatrix}$ are, respectively, the full column and row rank for all $\lambda$, and $\lambda + \bar{\lambda} = 0$, where $\bar{\lambda}$ is the complex conjugate of $\lambda$.

Connecting an $n$-state feedback controller $K(s)$ with state-space description gives

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t), \quad (4)$$

$$u(t) = C_c x_c(t). \quad (5)$$

The closed-loop transfer function from $w$ to $z$ is as follows

$$H(s) := \left[\begin{array}{c|c} \bar{A} & \bar{D} \\ \hline \bar{E} & 0 \end{array}\right] = \bar{E}(sI - \bar{A})^{-1} \bar{D}, \quad (6)$$

with the state-space description

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{D}w(t), \quad (7)$$

$$z(t) = \bar{E}\bar{x}(t), \quad (8)$$

where $\bar{A} = \begin{bmatrix} A & BC_c \\ B_c C & A_c \end{bmatrix}$, $\bar{D} = \begin{bmatrix} B_w \\ B_c D_{21} \end{bmatrix}$,

$$\bar{E} = \begin{bmatrix} C_z & D_{12} C_c \end{bmatrix}.$$

$H(s)$ is a strictly proper real rational matrix which has no poles on the imaginary axis such that

$$\| H \|_\infty = \sup_{\omega \in R} \sigma_{\max} \left[ H(j\omega) \right] < \gamma,$$

where $\sigma_{\max}(\cdot)$ denotes the maximum singular value. In general, there is a class of $H(s)$ satisfying bound $\gamma$. For such a transfer matrix $H(s)$, a formal definition of entropy integral is stated as follows.

**Definition 1:** The entropy integral at infinity of the closed-loop transfer function $H(s)$, for a tolerance $\gamma$ such that, $\| H \|_\infty < \gamma$ is defined by

$$\mathrm{I}(H, \gamma) = -\frac{\gamma^2}{2\pi} \int_{-\infty}^{\infty} \ln \left| \det \left[ I - \gamma^{-2} H^*(j\omega) H(j\omega) \right] \right| d\omega \quad (9)$$

where $H^*(\cdot)$ is the complex conjugate transpose of $H(\cdot)$.

This definition clearly shows that the entropy is well-defined since $\| H \|_\infty < \gamma$ implies $0 < I - \gamma^{-2} H^*(j\omega)\ H(j\omega) \le 1$ and nonnegative. The closed loop entropy integral (9) is a useful measure of how close $H$ is to the upper bound $\gamma$ on the maximum singular value of $H(j\omega)$. The entropy gives us a guaranteed upper bound on the actual quadratic cost [3]. If $H(s)$ is strictly proper, then $\mathrm{I}(H, \gamma) \ge \| H \|_\infty$ is derived.

This entropy integral at infinity is the most interesting and important case because of the strong connections with other control problems [9]. For our problem, $H(s)$ corresponds to no direct feed through terms from exogenous inputs $w(t)$ to controlled outputs $z(t)$.

Next, we introduce definitions of the two problems considered here.

**The minimum entropy/ $H_\infty$ control problem:** Find a feedback controller $K(s)$ that stabilizes the plant $P(s)$ such that

1) The closed-loop transfer function $H[K(s)] = H$ satisfies the $H_\infty$ norm bound $\| H \|_\infty < \gamma$, where $\gamma \in R$ is given.

2) The closed-loop entropy integral $\mathrm{I}(H, \gamma)$ is minimized.

**Definition 2:** The auxiliary cost associated with $H(s)$, where $\| H \|_\infty < \gamma$, is defined by

$$\mathrm{J}(H, \gamma) = \mathrm{Trace}(Q_s \bar{E}^T \bar{E})$$

where $Q_s > 0$ is the stabilizing solution of the algebraic *Riccati* equation

$$\bar{A} Q_s + Q_s \bar{A}^T + \gamma^{-2} Q_s \bar{E}^T \bar{E} Q_s + \bar{D}\bar{D}^T = 0.$$

**The mixed $H_2 / H_\infty$ control problem:** Find a feedback controller $K(s)$ that stabilizes the plant $P(s)$ such that

1) The closed-loop transfer function $H[K(s)] = H$ satisfies the $H_\infty$ norm bound $\| H \|_\infty < \gamma$, where $\gamma \in R$ is given.

2) The auxiliary cost $J(H, \gamma)$ is minimized.

The mixed $H_2/H_\infty$-controller realization is obtained as follows. By the results of Mustafa [1], the mixed $H_2/H_\infty$ control problem and the minimum entropy/$H_\infty$ control problem are equivalent. We thus present the controller state-space realization that solves the aforementioned two problems.

## 3. MIXED $H_2/H_\infty$-CONTROLLER REALIZATION

We seek to determine the controllers $K(s)$ yielding the closed loop transfer function $H(s)$ such that $\| H \|_\infty < \gamma$. Then the class of controllers satisfying $H_\infty$ norm bound is given by

$$K = \left\{ (W_{11} - QW_{21})^{-1}(W_{12} - QW_{22}) : Q \le m(\Re H_\infty) \right\}, \tag{10}$$

where $Q \in m(\Re H_\infty)$. These results are obtained from determining matrices $V_1$, $W_1$, and $G_1$ satisfying

$$V_1 \, J_{rn_z}(\gamma)\tilde{V}_1 = \begin{bmatrix} P_{21} & 0 \\ P_{11} & I \end{bmatrix} J_{n_w n_z}(\gamma) \begin{bmatrix} \tilde{P}_{21} & \tilde{P}_{11} \\ 0 & I \end{bmatrix},$$

$$G_1 = \hat{J} \, V_1 \, \hat{J}^T \begin{bmatrix} P_{12} & 0 \\ -P_{22} & I \end{bmatrix},$$

$$\tilde{G}_1 \, J_{n_z r}(\gamma) G_1 = \tilde{W}_1 \, J_{mr}(\gamma) W_1,$$

where

$$\tilde{V}_1 = V_1(-\bar{s})^*, \quad \tilde{W}_1 = W_1(-\bar{s})^*, \quad J_{pq}(\gamma) = \begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix},$$

$$\hat{J} = \begin{bmatrix} 0 & -I_{n_z} \\ I_r & 0 \end{bmatrix}, \quad P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \left[ \begin{array}{c|cc} A & B_w & B \\ \hline C_z & 0 & D_{12} \\ C & D_{21} & 0 \end{array} \right],$$

$$V_1, \; W_1, \; V_1^{-1}, \; W_1^{-1} \in \Re H_\infty,$$

and $\bar{s}$ is the complex conjugate of $s \in \mathrm{C}$.

We define, $V_1 := W_1^{-1}$; that is

$$W_1 \, V_1 = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} = I.$$

The minimum entropy solution of (9) is obtained by setting the arbitrary stable contraction $Q$ in (10) to zero, i.e. by choosing the *central solution* out of the set of all admissible controllers. In the following proposition, minimum entropy controller realization is derived in terms of the state-space. This result is obtained from the relation of the minimum entropy/$H_\infty$ control problem and the mixed $H_2/H_\infty$ control problem.

**Proposition:** The controller solving the minimum entropy/ $H_\infty$ control problem takes the following state-space realization

$$K = L_1(sI - A + M_1 C + M_2 C_z + BL_1 - M_2 D_{12} L_1)^{-1} M_1$$

$$:= \left[ \begin{array}{c|c} A - M_1 C - M_2 C_z - BL_1 + M_2 D_{12} L_1 & M_1 \\ \hline L_1 & 0 \end{array} \right]$$

$$= \left[ \begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array} \right]$$

where

$$M_1 = YC^T + B_w D_{21}^T, \quad M_2 = -\gamma^{-2} Y C_z^T,$$

$$\begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = \begin{bmatrix} D_{12}^T C_z + B^T X \\ -(C + \gamma^{-2} D_{21} B_w^T X) \end{bmatrix} (I - \gamma^{-2} YX)^{-1},$$

and $X$ and $Y$ are, respectively, *Riccati* solutions of $H_X$ and $H_Y$ such that

$$H_X = \begin{bmatrix} A - BD_{12}^T C_z & \gamma^{-2} B_w B_w^T - BB^T \\ -C_z^T C_z + C_z^T D_{12} D_{12}^T C_z & -A^T + C_z D_{12} B^T \end{bmatrix},$$

$$H_Y = \begin{bmatrix} A^T - C^T D_{21} B_w^T & \gamma^{-2} C_z^T C_z - C^T C \\ -B_w B_w^T + B_w D_{21}^T D_{21} B_w^T & -A + B_w D_{21}^T C \end{bmatrix}.$$

**Proof:** The minimum entropy controller is the central solution in the parameterization of all stabilizing controllers and satisfies $\|H(s)\|_\infty < \gamma$ [7]. By the results of Seo *et al.*[8], the matrix $W_1$ is given by

$$W_1 = \begin{bmatrix} A - M_1 C - M_2 C & B - M_2 D_{12} & M_1 \\ L_1 & I & 0 \\ L_2 & 0 & I \end{bmatrix}.$$

Hence the central solution of all stabilizing controllers takes the form $K(s) = W_{11}^{-1} W_{12}$. It is also accepted that the minimum entropy controller satisfy the mixed $H_2/H_\infty$- controller [1,7], therefore we can propose the state-space realization as above. □

## 4. CONCLUSION

We illustrated a minimum entropy/$H_\infty$-controller and mixed $H_2/H_\infty$-controller design problem. Both of the controllers guarantee closed-loop stability, $H_\infty$ norm bound and $H_2$ performance. Using the equivalence of the mixed $H_2/H_\infty$ control problem and the minimum entropy/ $H_\infty$ control problem, the controller state-space realization was derived.

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