Control System Synthesis Using BMI: Control Synthesis Applications

Tae-Jin Chung, Hak-Joon Oh, and Chan-Soo Chung

Abstract: Biaffine Matrix Inequality (BMI) is known to provide the most general framework in control synthesis, but problems involving BMI's are very difficult to solve because nonconvex optimization should be solved. In the previous paper, we proposed a new solver for problems involving BMI's using Evolutionary Algorithms (EA). In this paper, we solve several control synthesis examples such as Reduced-order control, Simultaneous stabilization, Multi-objective control, H_∞ optimal control, Maxed H_2/H_∞ control design, and Robust H_∞ control. Each of these problems is formulated as the standard BMI form, and solved by the proposed algorithm. The performance in each case is compared with those of conventional methods.

Keywords: Biaffine matrix inequalities, nonconvex optimization, evolutionary algorithm.

1. INTRODUCTION

As mentioned in the previous paper [8] BMI provides more general frameworks in control synthesis than LMI. However, the complexity due to its nonconvex characteristic makes it hard to apply BMI's to control synthesis applications. Even though several algorithms have been proposed, their performances depend on how to relax given BMI problems. In addition, their computational time would increase significantly since they repeat relaxing BMI's to LMI's and solving Semidefinite Programming (SDP). In the previous paper [8], we proposed a new approach that utilizes the powerful evolution mechanism of Evolutionary Algorithm (EA) for BMI problems. In addition, a problem occured when applying EA to BMI was pointed out, and an efficient method was proposed to overcome this problem. This method is to evolve matrix variables instead of evolving elements of them. Evolution of matrices was performed by evolving their eigenvalues and eigenvectors separately and combining them into matrices again. This new method can solve several numerical examples and the performances could be better than those of Brand and Cut (B&C) algorithm [8].

In this paper, we will solve several control synthesis examples using the proposed algorithm. The examples are such as Reduced-order control, Simultaneous stabilization, Multi-objective control, H_∞ optimal control, Mixed H_2/H_∞ control design, and Robust H_∞ control. These control problems are actively studied in control society, and some of them are known as difficult to solve with conventional control algorithms

The reduced-order control has an additional constraint on the order of the controller in addition to the performance constraints. The conventional approaches to this problem are focused on the LMI's with additional rank conditions [11,17], or coupled two LMI's [12]. In [12], the $-XY$ centering algorithm for dual LMI problems was used. In this algorithm, X and Y matrix variables (referred R and S in other papers) related to the Lyapunov inequalities are coupled with each other by $X = Y^{-1}$ and they are iteratively updated until the condition $XY = I$ is satisfied. Ghaouli, *et. al.* [11] proposed the complementary linearization algorithm where the rank constraint is linearized as $Tr(XY)$. Tanaka *et. al.* [17] characterized fixed-order (or reduced-order) controllers based on the LMI approach, and provided necessary and sufficient conditions for the existence of reduced-order controllers.

The simultaneous stabilization problem is to design a single controller stabilizing multiple plants, which is known as NP-hard. In [7], an iterative LMI (ILMI) method is proposed and an example is solved. This example is also solved in [14] using the nonlinear programming approach, where SDP is solved iteratively. An interesting result is proposed in [1], where the desirable pole regions are formulated and a nonlinear constrained optimization technique is applied to place all the system poles into the specified regions.

The H_∞ optimal control has been the most famous design framework in control synthesis since the late of 1970's. One of the recent approaches to this problem

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is to use LMI's [6,9,16]. Since BMI is the general form of LMI, all of approaches in these papers could be an example of various applications of BMI.

The multi-objective control is to design a controller satisfying several design specifications. It is often desirable to design a controller stabilizing a plant as well as limiting control inputs or satisfying time transient response of output, etc. Several LMI's conditions for multi-objectives are presented in [13] and [10]. However, they have a conservative assumption which places all constraints on a common Lyapunov matrix.

The mixed H_2 / H_∞ control is a special example of multi-objective control. The desirable controller should satisfy the H_2 constraint in one channel and H_∞ constraint in the other channel. This problem is very useful in designing controllers with limited control input. A conventional LMI approach to solve this problem assumes a common Lyapunov matrix as same way in the multi-objective control case.

The robust H_∞ control arises when there exist uncertainties in plant models. When these uncertainties are expressed as a structured one, Linear Fractional Transformation (LFT) represents the uncertain system, and existing robust control algorithm could be applied. We will solve this problem in the BMI framework.

In this paper, the problems mentioned above are solved in the BMI framework and the performances are compared with those of the conventional algorithms. This paper is organized as follows. In Section 2, the newly proposed algorithm is reviewed briefly and each of control problems is solved in Section 3. In each cases the performances and the advantages are discussed. Finally, we conclude in Section 4.

2. A NEW APROACH TO BMI PROBLEM

The proposed approach [8] for BMI problems is to utilize the efficient evolutionary algorithm. An efficient method is also proposed to resolve the difficulty occurred in the matrix optimization using evolutionary algorithm. This is to evolve matrix variables by evolving its eigenvalues and evolving rotational angles for eigenvectors. A matrix could be evolved only in the positive definite conic space by restricting eigenvalues to positive real space. Given's rotation matrices, obtained from evolved rotation angles, rotate eigenvectors and produce another orthonormal eigenvectors. This new method can solve BMI problems more efficiently than conventional evolutionary algorithms. Algorithm 1 summarizes the proposed method.

In Step 16, we solve Alternating SDP with the current best individual as the initial point to accelerate searching mechanism, and the resulting local information is used for the next evolution. Since computational time of Alternating SDP algorithm is longer then that of evolution, it is applied every 1000 genreations. Normally, about 20 Alternating SDPs are solved for most BMI problems.

This algorithm is implemented in $C++$ using a mathematical library Meschanch and a random number generation library Ranlib, and its is compiled in Digital Alpha Station (CPU Alpha 21664 - 500MHz, 512MB Memory, DIGITAL UNIX 4.0).

Algorithm 1 Algorithm of the Proposed Approach

1: $t := 0$
2: Randomly select orthonormal eigenvectors
3: Initialize $P^{(0)} = \{(x_1, \sigma_1, \theta_1), \cdots, (x_n, \sigma_n, \theta_n)\}^{(1)}$
4: $FeasbleNO = 0$.
5: Evaluate $f(P^{(0)})$
6: **while** $(FeasbleNO < N)$ **do**
7: $t++$;
8: Generate $P^{(t)}$ by one step of EA. 2)
9: $P_{best}^{(t)} = max_i \left\{ f\left(P_i^{(t)}\right)\right\}, i = 1, \cdots, n$
10: **if** ($P_{best}^{(t)}$ is feasible) **then**
11: $FeasbleNO++$;
12: **else**
13: $Feasble = 0$;
14: **end if**
15: **if** $(t\%1000 == 0)$ **then**
16: Solve Alternating SDP. 3)
17: Insert the solution to $P^{(t)}$
18: **end if**
19: **end while**

1) x_i consist of variables for controller, rotational angles, and eigenvalues.
2) Mutation, Crossover, and Selection operators are applied. After evolving rotational angles, Given's rotation matrix is produced and N orthonormal eigenvectors are rotated.
3) This accelerates convergency rate of the algorithm.
Note) More details are in [8].

3. NUMERICAL EXAMPLE

In this section, the proposed approach is applied to several control synthesis problems to verify its performances.

3.1. Static output feedback controller design

The static output feedback design problem is to design a constant feedback gain stabilizing a system. It is known to be NP-hard [4]. The following system is cited from [3]. The goal is to design a stabilizing static output feedback controller with the maximum decay rate of closed loop system.

$$\left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 2 & 0 \end{array}\right] \quad (1)$$

This problem is equivalently expressed as the problem to find a scalar K, a symmetric positive definite matrix P, and the maximum decay rate such that the following BMI's are feasible.

$$\begin{aligned} A_{cl}^T(K)P + PA_{cl}(k) + 2\alpha P < 0 \\ P > 0 \end{aligned} \quad (2)$$

where $A_{cl}(K) = A + BKC$.

In [3], they used a BMI algorithm based on Benders decomposition to solve this problem. Their initial bounds for K and α were between -1.0 and -10.0, and between -0.1 and 1.0, respectively. They found a optimal solution $K = -9.4277$ with closed loop poles at -7.3196, -1.0541±0.5600j in 24 iterations (the maximum decay rate was not reported). According to their analysis of the algorithm, it required to solve 215 SDP problems.

The proposed algorithm found a controller $K = -9.9170$ with the maximized decay rate $\alpha = 0.6563$ in about 20 seconds. Moreover, this α can be obtained without bisection algorithm since it is just one variable in the BMI framework. The resulting closed loop poles were -7.8678, -1.0516±0.5600j. Comparing ITAE (Integral of time multiplied by the absolute value of error) performance indices of the two closed loop systems, we can notice that the proposed algorithm improved the performance by 4.7% than that of [3].

3.2 Reduced-order controller design

The reduced-order controller design is one of the most actively studied problems in control synthesis, since it is more challenging than full-order controller design and more preferable for practical purpose [12,15,17]. As we mentioned earlier, this problem is not formulated as LMI's only. In most cases applying LMI's for this problem, they use a couple of LMI's

Table 1. Controllers for mass-spring-damper system.

Order	Poles	Decay Rate
4	-0.0290±1.7475j -0.1316±0.9535j -0.2018±0.3176j -0.4106±0.4194j -0.4385±1.2891j	-0.0290
3	-1.1448 -0.0172±1.7418j -0.0269±0.9344j -0.0315±0.2969j -0.3419±0.8050j	-0.0172
2	-2.0871 -0.2024 -0.0101±0.0928j -0.0135±1.7362j -0.0173±0.1775j	-0.0101
1	-4.3884 -0.0019±0.3703j -0.0026±0.8989j -0.0155±1.7477j	-0.0019

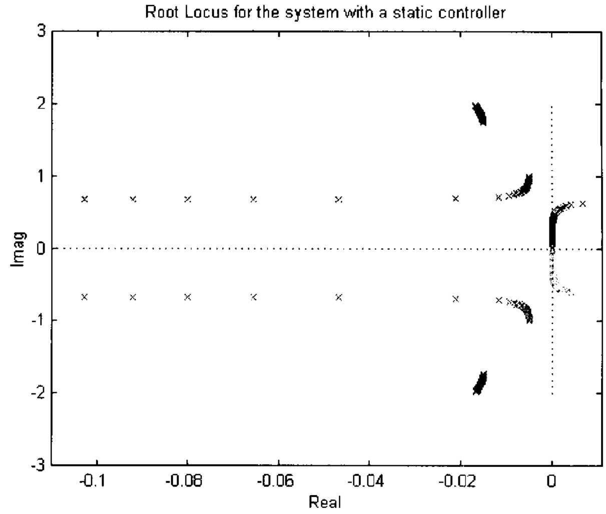

Fig. 1. Root Locus of the closed system ($N_c = 0$).

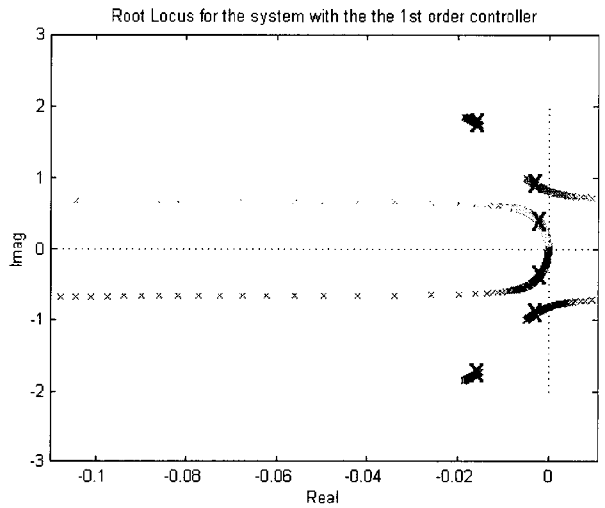

Fig. 2. Root Locus of the closed system ($N_c = 1$).

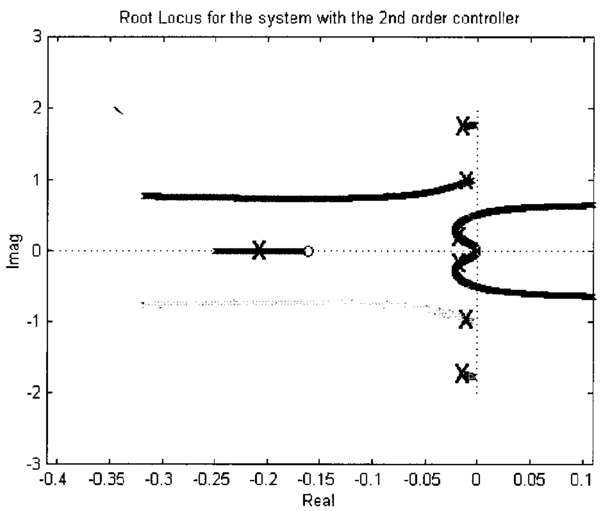

Fig. 3. Root Locus of the closed system($N_c = 2$).

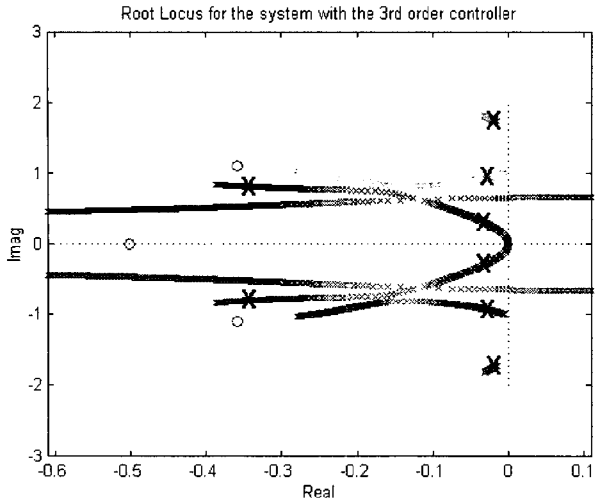

Fig. 4. Root Locus of the closed system($N_c = 3$).

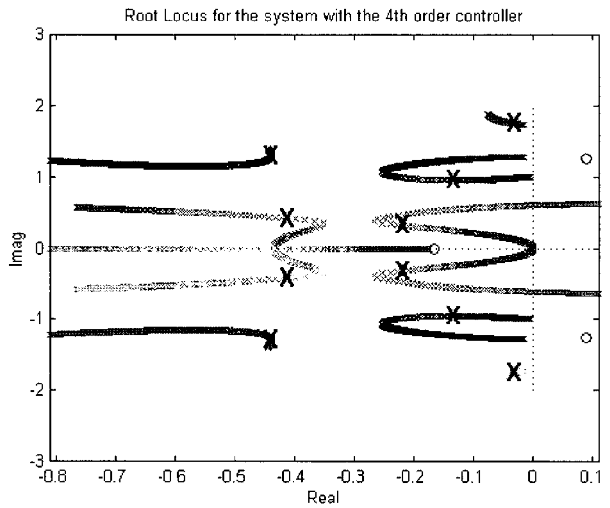

Fig. 5. Root Locus of the closed system ($N_c = 4$).

that are strongly coupled to each other. In [11], a cone complementary linearization algorithm was used to solve the coupled LMI's. In this subsection, we will design the several reduced-order controllers using the Algorithm 1. One of advantages in Algorithm 1 is that there is no conservatism due to linearization, because it solves BMI problems directly (without any modification). The plant to be stabilized is the 3-mass-spring-damper system.

The external disturbance ω is ignored to simplify the problem and the design constraint is set to stabilization of the closed loop system. The output feedback controller's orders are set to 4th(N_c =4), 3rd(N_c =3), 2nd(N_c =2), and 1st(N_c =1). Using the Algorithm 1, we can design reduced-order controllers for all cases. However, as the controller's order decreases, the poles of the closed loop system approach into the neighborhood of the imaginary axis. Table 1 shows the poles of the closed-loop systems with the designed controllers.

All poles are in the left-hand side in the complex plane, but the maximum decay are decreased as the controller's order decreases. Fig. 1, 2, 3, 4, and 5 show the root locus of the systems with each controllers including static output feedback case.

Table 2. Simultaneous controller design : Case 1.

a	Feedback Gain	Poles of the Closed loop System	
		For $P_1(s)$	For $P_2(s)$
-1.0	-2.5838	-3.6074 -0.3926	-0.5±0.5778j
-1.9	-2.0360	-3.9668 -0.0332	-0.9636 -0.9374

3.3. Simultaneous stabilization

The simultaneous stabilization is one of difficult control synthesis problems, and also known as NP-hard [1,5]. The problem is to design a single controller stabilizing multiple plants.

Consider the following arbitrary plants

$$\begin{aligned} \dot{x}_i(t) &= A_i x_i(t) + B_i u_i(t) \\ y_i(t) &= C_i x_i(t) \end{aligned} \tag{3}$$

for $i = 1, \cdots, m$.

The simultaneous stabilizing controller must stabilize all plants. In the conventional LMI method, all Lyapunov matrices P_i are set to a common P ,which produces conservatism. However, in the BMI framework, the problem is expressed as follows.

$$\begin{aligned} (A_i + B_i K C_i)^T P_i + P_i (A_i + B_i K C_i) + 2\alpha I < 0 \\ P_i > 0 \end{aligned} \tag{4}$$

In [14], a simultaneous controller was designed for following simple two plants.

$$P_1(s) = \frac{a}{(s+2)^2},\ P_2(s) = \frac{1}{(s+2)(s-1)} \tag{5}$$

It is known that there exists a static output feedback controller if $a > -2$. Using the nonlinear programming approach, Lee [14] designed controller for a = 5.0, 0.5, -1.0, -1.5, -1.9 and the designed feedback gains were -2.0718, -2.9735, -2.7413, -2.0027, -2.0064, respectively. We designed controller for the same plants with a = -1.0, -1.9 using the Algorithm 1, and the results are shown in Table 2.

It can be seen that the results are similar to those of [14]. Since there is no results on the CPU time in [14], we cannot compare the computational efficiency of two algorithms.

3.4. Simultaneous stabilization with pole placement

In this example, we will design another interesting control design problem, that is, simultaneous stabilization of four plants that places the closed loop poles in the described region. Consider the following four plants [1].

$$P_1(s)=\frac{s+4}{s+2}, P_2(s)=-\frac{2(s+2)}{s+3},$$
$$P_3(s)=\frac{s+2}{s^2-3s+4}, P_4(s)=\frac{s+3}{s^2-2s+1} \tag{6}$$

We would like to design a controller stabilizing these four plants simultaneously with the following first order controller.

$$K(s)=\frac{a_1 s+a_2}{s+a_3}=\left[\begin{array}{c|c} k_1 & k_2 \\ \hline k_3 & k_4 \end{array}\right] \tag{7}$$

We first design a stabilizing controller for those plants in similar ways in the previous subsection. The design controller is

$$K(s)=\frac{5.360s+381.194}{s+71.112} \tag{8}$$

and the poles of the closed-loop system are shown in Fig. 6. In this figure, it can be seen that all poles are in the left-half plane, but some poles are very close to the imaginary axis, which gives poorly damped mode to the system. Therefore, it is very desirable to place the poles in some desirable region. This can be achieved by adding additional constraints in the design step, which are pole placement constraints. Let's assume that the desirable region for the system is inside of the parabola or inside of the shaded triangular. These constraints can be expressed as nonlinear inequalities [1] or LMI's [16].

In [1], the authors used nonlinear optimization approaches to design a controller satisfying the nonlinear pole placement constraint in the characteristic polynomials. Their controller was

$$K_{s,D,1}(s)=\frac{13.52s+27.05}{s+2.39} \tag{9}$$

and the resulting poles of the system was placed in the parabola as in Fig. 7.

Even though all poles are inside of the desired parabola, some are outside of the triangular region (it does not mean that the region of triangular represents better characteristics of the system). We will design a controller producing more damping to the system, whose poles are inside of the triangular region.

The triangular region in the Fig. 6 can be expressed as intersection of a conic section with inner angle 0.3303(rad) and a vertical strip between -2.0 and -50.0. With these constraint, the closed-loop system will have at least -2.0 as the maximum decay rate and at least 0.9864 as the minimum damping ratio.

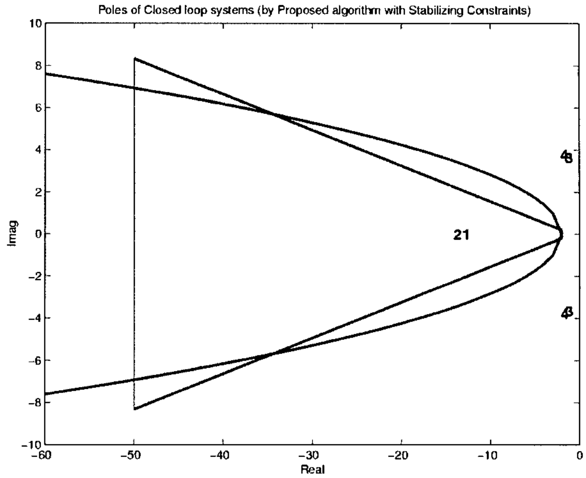

Fig. 6. Poles of the closed-loop systems (i: poles for i th plants).

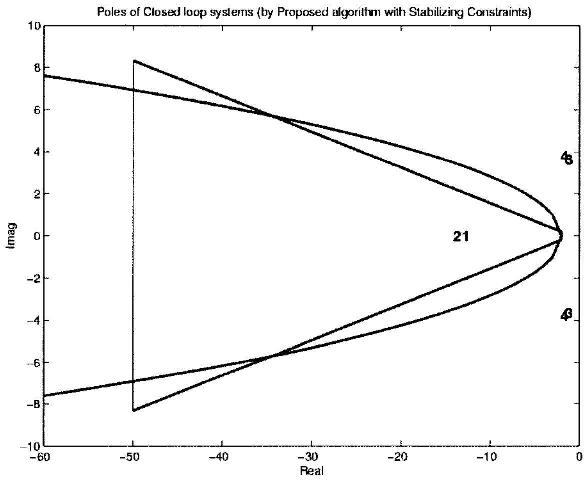

Fig. 7. Poles of the closed-loop systems by nonlinear optimization approach (i : poles for i th plants).

When formulating these constraints into BMI's form, we have eight Lyapunov matrices, In the conventional LMI approaches, all of these Lyapunov matrices are set to be common. But, we can use eight different Lyapunov matrices for different constraints, so that we can avoid conservatism of LMI approaches. We have four variables for the controller, 36 variables for eight Lyapunov matrices, and the dimension of the involving matrices are 70.

Using the proposed algorithm, we can design a controller satisfying all of the constraints. The designed controller is

$$K_{s,D,2}(s)=\frac{26.579s+1197.4}{s+43.799} \tag{10}$$

and the resulting poles are shown in Fig. 8.

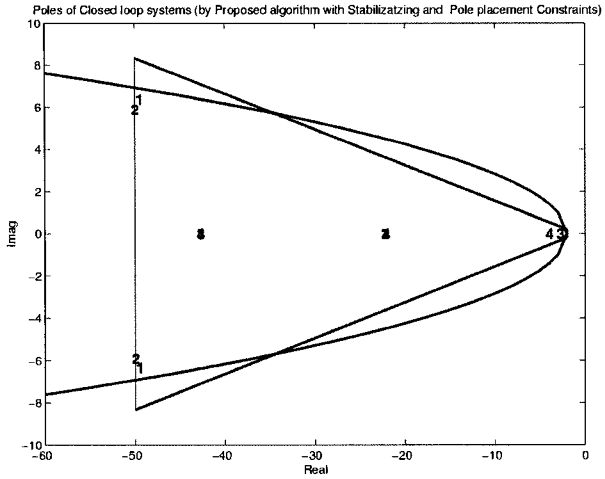

Fig. 8. Poles of the closed-loop systems by the proposed algorithm (i : poles for i th plants).

3.5. Structured controller design

The structured controller is to design a controller with a special structure. This problem has been known to be difficult to solve, since the special structure of a controller is difficult to be formulated in the existing controller design tools. When designing decentralized controller, the structure of a controller will be specified according to the structure of the network, or reduction of redundant sensors or actuators. In this example, we will design a couple of simple structured controllers

Let's consider a simplest PID controller. The design of a PID controller can be considered as a structured control design problem, since the structure of the controller is specified. Consider a plant

$$P(s) = \frac{0.8}{s(0.5s+1)} \tag{11}$$

and the designed controller should be of the form

$$K_{PID}(s) = K_P + K_I / s + K_D s \tag{12}$$

Using the Algorithm 1, we can find a PID gain such as 59.854, 3.5578, and 99.92, which stabilizes the plant.

As an another interesting example, we consider the 3-mass-spring-damper system again. However, at this time, as assume that the controller has a specified structure such as

$$K_{structured} = \left[\begin{array}{ccc|c} k_1 & 0 & 0 & k_2 \\ 0 & k_3 & 0 & k_4 \\ 0 & 0 & k_5 & k_6 \\ \hline k_7 & k_8 & k_9 & k_{10} \end{array}\right] \tag{13}$$

Table 3. H_∞ controller design #1.

Order	Poles	γ
2	-5.2692 -18.771 -46.779 -1.6172±3.4562j	9.7849
1	-5.2698 -18.700 -1.6199±3.4574j	10.4377

This specific controller is easily formulated as BMI form and the proposed algorithm yields the following stabilizing controller.

$$K_{structured} = \left[\begin{array}{ccc|c} -14.625 & 0 & 0 & -0.2868 \\ 0 & -0.2863 & 0 & 0.0499 \\ 0 & 0 & -0.5415 & -0.0963 \\ \hline -0.5415 & -0.0963 & 0.1411 & 0.6104 \end{array}\right] \tag{14}$$

3.6. H_∞ optimal controller design #1

The H_∞ optimal controller design has been the most popular design framework in controller design since it was proposed in the late of 1970's. One of the recent approaches to this problem is to use LMI framework [6,9,13,16]. Since BMI is a general form of LMI, all the controllers designed using LMI's can be also designed using BMI's. In this and the next examples, we will design a couple of H_∞ optimal controllers using the Algorithm 1.

The H_∞ optimal control problem is formulated as

$$\begin{bmatrix} A_c^T P + PA_c & PB_c & C_c^T \\ B_c^T P & -\gamma I & D_c^T \\ C_c & D_c & -\gamma I \end{bmatrix} < 0, \quad P > 0 \tag{15}$$

where γ is the H_∞ norm. The above matrix inequality is also a BMI form.

Consider the following unstable plant.

$$\dot{x} = \begin{bmatrix} 0.0 & 10.0 & 2.0 \\ -1.0 & 1.0 & 0.0 \\ 0.0 & 2.0 & -5.0 \end{bmatrix} x + \begin{bmatrix} 1.0 \\ 0.0 \\ 1.0 \end{bmatrix} \omega + \begin{bmatrix} 0.0 \\ 1.0 \\ 0.0 \end{bmatrix} u \tag{16}$$
$$y = x_2 + 2.0\omega$$

The H_∞ controller of order 2 can be designed using conventional method [2,19], or in the LMI framework. Using these conventional methods, we could get an optimal $\gamma = 9.5124$.

Using the Algorithm 1, we can obtain a similar controller, but γ is slightly increased. However, we can also design a reduced-order H_∞ controller with very similar γ. Recall that the reduced-order H_∞ control problem can not be formulated using conventional

H_∞ theories. The designed controllers are shown in (17) and (18) and Table 3 shows results of this example.

$$H_{2,H_\infty} = \left[\begin{array}{cc|c} -24.785 & 11.325 & 16.990 \\ -3.7160 & -48.640 & -24.473 \\ \hline -1.0615 & 1.9644 & 3.3708 \end{array}\right] \tag{17}$$

$$H_{1,H_\infty} = \left[\begin{array}{c|c} -26.602 & -7.6128 \\ \hline 24.511 & 3.3932 \end{array}\right] \tag{18}$$

3.7. H_∞ optimal controller design #2

In this example, another H_∞ optimal controller is designed to compare the performance of the proposed algorithm.

A system [18] is given as

$$A = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4560 \\ 0.0482 & -1.0100 & 0.0024 & -4.0200 \\ 0.1000 & 0.3680 & -0.7070 & 1.4200 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}, B_2 \begin{bmatrix} 0.4420 & 0.1760 \\ 3.5400 & -7.5900 \\ -5.5200 & 4.4900 \\ 0.0000 & 0.0000 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}, D_{12} = \begin{bmatrix} 0.0 & 0.0 \\ 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 0.0 & 1.0 & 0.0 & 0.0 \end{bmatrix}, D_{11} = D_{21} = D_{22} = 0.0$$

Here, we are looking for a static output feedback K so that the closed-loop H_∞ norm is less than a prescribed value. In [18], the min/max algorithm was used to design a controller $K = [1.06,\ \ 1.44]^T$, and the H_∞ norm was 12.4751.

We solve the same control problem using the Algorithm 1. The obtained controller is $K = [1.6198,\ 6.0181]^T$ after predefining H_∞ performance bound as 10.25. With the designed controller, the actual H_∞ performance was 10.0844, which is slightly improved. Fig. 9 shows the output response when impulse disturbance are induced. All of the responses converge to zero in about 25 seconds.

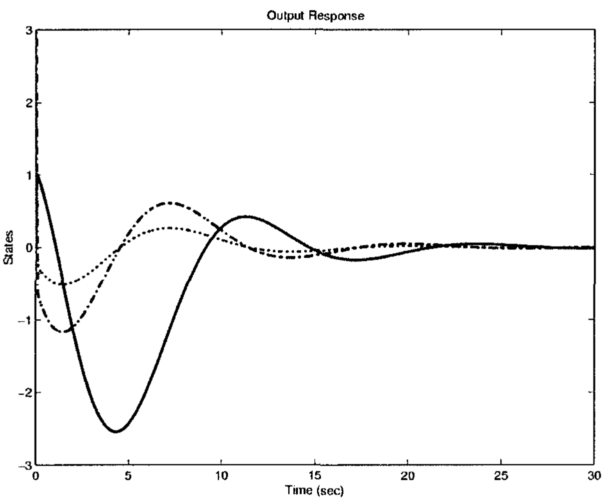

Fig. 9. Output response due to impulse disturbances (solid: y_1, Dotted: y_2, Dash: y_3).

3.8. Mixed H_2 / H_∞ controller design

The mixed H_2 / H_∞ control problem is to design a stabilizing controller satisfying H_2 performance in one channel and H_∞ performance in the other channel of the closed-loop system. Although it is difficult to formulate this control problem algebraically, LMI framework allows designers to design a mixed H_2 / H_∞ controller by sharing a common Lyapunov function [13,16]. In this example, we design a static output feedback H_2 / H_∞ controller for the following plant [18].

In [18], the Method of Centers algorithm was used to find a controller which is $K = [1.78,\ \ 5.26,\ \ -9.64]^T$, and the performance was $\|T_{z_2\omega}\| = 14.7106$ and $\|T_{z_\infty\omega}\| = 3.0$. Using the proposed algorithm, we find a similar controller, after setting H_∞ performance bound to 3.0 and H_2 performance bound to 15.0,

$$\dot{x} = \begin{bmatrix} -0.825 & 0.089 & 0.0 \\ 0.1870 & 0.430 & 0.0 \\ 0.0000 & 0.0 & 1.0 \end{bmatrix} x + \begin{bmatrix} 0.935 \\ 0.384 \\ 1.0 \end{bmatrix} \omega + \begin{bmatrix} 0.519 \\ 0.831 \\ 1.0 \end{bmatrix} u$$

$$z_2 = \begin{bmatrix} -0.946 & 0.324 & -1.0 \\ 0.019 & 0.331 & -0.465 \end{bmatrix} x + \begin{bmatrix} -1.0 \\ 1.0 \end{bmatrix} u$$

$$z_\infty = \begin{bmatrix} 0.167 & 0.487 & 1.0 \\ 0.179 & 0.435 & -0.116 \end{bmatrix} x + \begin{bmatrix} 1.0 \\ -1.0 \end{bmatrix} u$$

$K = [1.7920,\ \ 5.2929,\ \ -9.7600]^T$
and the performance is $\|T_{z_2\omega}\| = 14.7726$ and $\|T_{z_\infty\omega}\| = 2.982$ Fig. 10 and 11 show the output response of each channels when impulse disturbance is induced to the system.

3.9. Robust H_∞ controller design

As the last numerical example, we design a robust controller. The robust H_∞ control problem is to design a stabilizing controller for a uncertain system (19) and to satisfy H_∞ performance.

$$\begin{bmatrix}\dot{x}\\ z\\ y\end{bmatrix}=\left(\begin{bmatrix}A & B_\omega & B_u\\ C_z & D_{z\omega} & D_{zu}\\ C_y & D_{y\omega} & D_{yu}\end{bmatrix}+\begin{bmatrix}B_p\\ D_{zp}\\ D_{yp}\end{bmatrix}\Delta\left(I-D_{qp}\Delta\right)^{-1}\begin{bmatrix}C_q^T\\ D_{q\omega}^T\\ D_{qu}^T\end{bmatrix}\right)\begin{bmatrix}x\\ \omega\\ u\end{bmatrix} \tag{19}$$

with uncertainty Δ.

To illustrate the design example, we choose a two-mass-spring system. The nominal parameters are set to $m_{10}, m_{20}, k_0, b=0$ and the actual uncertain parameters are expressed as

$$k = k_0(1+\sigma_k\delta_k)$$
$$m_1 = m_{10}(1+\sigma_1\delta_1)$$
$$m_2 = m_{20}(1+\sigma_2\delta_2)$$

Consider

$$\Delta=\begin{bmatrix}\delta_k & 0 & 0\\ 0 & \delta_1 & 0\\ 0 & 0 & \delta_2\end{bmatrix}$$

where $|\delta_k|<\delta_k$, $|\delta_1|<\delta_1$, and $|\delta_2|<\delta_2$. Then, this uncertain system is described as the following LFT (Linear Fractional Transformation) form, where where , , and . Then, this uncertain system is described as the following LFT (Linear Fractional Transformation) form, where

$$\left[\begin{array}{c|ccc}A & B_p & B_\omega & B_u\\ \hline C_q & D_{qp} & D_{q\omega} & D_{qu}\\ C_z & D_{zp} & D_{z\omega} & D_{zu}\\ C_y & D_{yp} & D_{y\omega} & D_{yu}\end{array}\right] \tag{20}$$

stands for

$$\left[\begin{array}{cccc|ccccc}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
-\frac{k_0}{m_{10}} & \frac{k_0}{m_{10}} & 0 & 0 & \frac{\sigma_k k_0}{m_{10}} & -\sigma_2 & 0 & 0 & \frac{1}{m_{10}}\\
\frac{k_0}{m_{20}} & -\frac{k_0}{m_{20}} & 0 & 0 & -\frac{\sigma_k k_0}{m_{20}} & 0 & -\sigma_2 & \frac{1}{m_{20}} & 0\\
\hline
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-\frac{k_0}{m_{10}} & \frac{k_0}{m_{10}} & 0 & 0 & \frac{\sigma_k k_0}{m_{10}} & -\sigma_1 & 0 & 0 & \frac{1}{m_{10}}\\
\frac{k_0}{m_{20}} & -\frac{k_0}{m_{20}} & 0 & 0 & -\frac{\sigma_k k_0}{m_{10}} & 0 & -\sigma_2 & \frac{1}{m_{10}} & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

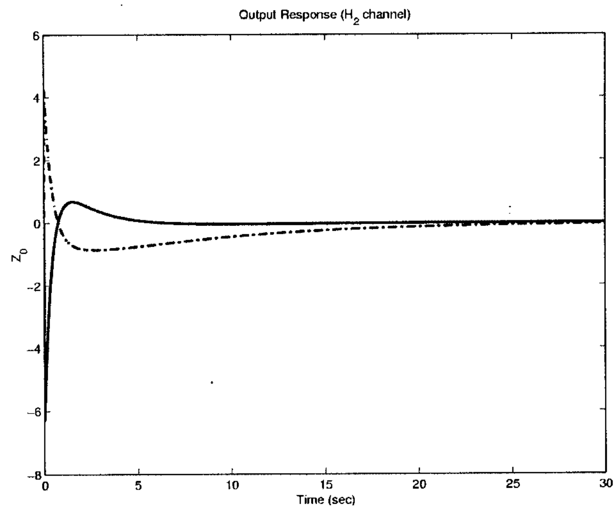

Fig. 10. Output response at H_2 channel due to impulse disturbance (solid: $z_{2,1}$, dotted: $z_{2,2}$).

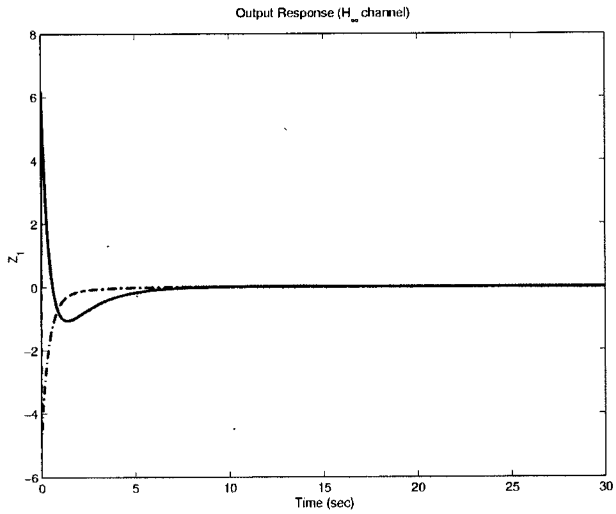

Fig. 11. Output response at channel due to impulse disturbance (solid:, dotted:).

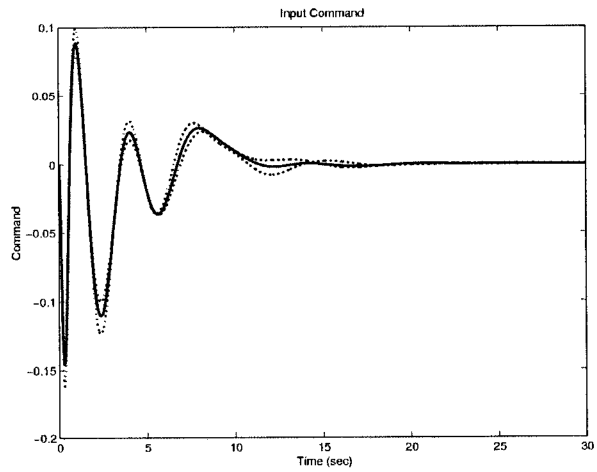

Fig. 12. Feed back input. (solid: nominal, dotted: perturbed).

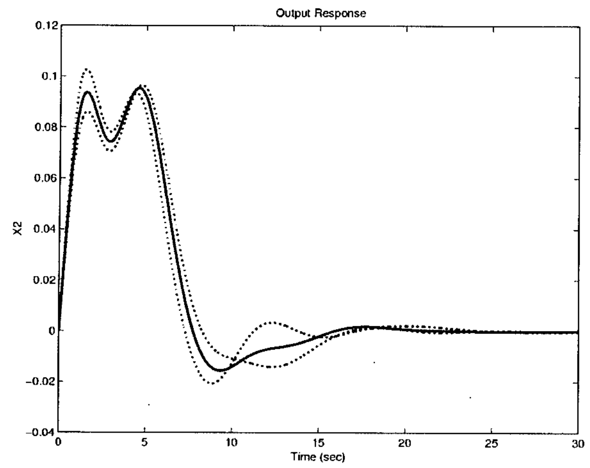

Fig. 13. Output responses (solid: nominal, dotted: perturbed).

We set parameters of the plant as $k_0 = 1.0$, $b = 0$ and assume that two masses and spring constant are varying to 10%, such that $\sigma_k = 1.0$, $\sigma_1 = 1.0$, and $\sigma_2 = 0.1$.

Using the Algorithm 1, we designed 4th order robust controller (21), and its H_∞ performance was 5.1586. Fig. 12 and 13 show the feedback inputs and the output responses when impulse disturbance is induced to the second mass. The dotted lines in each figures stand for feedback inputs and output responses, when the parameters are fixed to the extreme values of the perturbation bounds. The solid lines stand for the nominal case. In Fig. 13, we can see that the effects of the disturbances are regulated to zeros in about 25 seconds.

$$K = \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array}\right] = \left[\begin{array}{cccc|c} -5.1773 & 5.8608 & -8.6172 & 1.4154 & 0.5118 \\ 1.1940 & -3.7993 & 1.9284 & -9.3212 & 9.8969 \\ 8.0043 & 1.9746 & -1.4428 & -2.7642 & -8.7217 \\ -0.9366 & -2.4948 & 3.3555 & -0.8715 & 2.3398 \\ \hline 7.0940 & -7.1013 & 9.9004 & -2.3176 & 0.0000 \end{array}\right] \quad (21)$$

4. CONCLUSION

In this paper, we solved several control synthesis examples such as Reduced-order controller, Simultaneous stabilization, Multi-objective control, H_∞ optimal control, Mixed H_2 / H_∞ control design, and Robust H_∞ control in the BMI framework. Each of these problems are reformulated as the standard BMI forms, and solved by the proposed algorithm. The newly proposed algorithm is different from other methods by utilizing the evolutionary algorithm. Moreover, for matrix variables, an efficient evolution method was proposed.

The cited numerical problems are mostly attracting engineers interest recently and some of them are known as difficult to solve. Using the proposed algorithm, all of the control problems can be solved efficiently and their performances in each cases are not less than those of the conventional algorithms. From the computational point of view, this proposed algorithm can have more potentials since it provides an unified framework for various control problems to designers.

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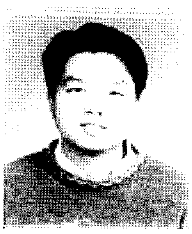

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