

Unified Estimations for Parameter Changes in the Uniform Distribution

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Abstract

We shall propose several estimators for the scale parameter in the uniform distribution when the parameter is functions of a known exposure level, and obtain expectations and variances for their proposed estimators. And we shall compare numerically relative efficiencies for proposed estimators of the scale parameter in the small sample sizes.

Keywords : Parameter change, relatively efficiency, Closer estimator, Symmetric Estimator

1. Introduction

The uniform distribution is used to represent the distribution of roundoff errors and arise as a result of the probability integral transformation. And the uniform distribution often provide useful tools for constructing goodness-of fit statistics, simulation of complex statistical procedures, and testing the quality of pseudorandom number generators. Here we shall consider the parametric estimations in the uniform distribution when its scale parameter is a function of a known exposure level t , which often occurs in the engineering and physical phenomena. Woo & Yoon(1990) considered unified estimations for parameter changes in a Pareto distribution. Woo & Ali(1994) considered the jackknife parametric estimations in the exponential distribution when its scale and location parameters change a function of environment dosage. And Woo & Lee(2000) considered an application of the Weibull distribution to the strength of materials when its shape and scale parameters are functions of a known exposure level.

The purpose of this work is to estimate the effects on the scale parameter in the uniform

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distribution when the scale parameter changes a function of an environment dosage, say t . First, we assume an uniform model and estimate parameters by the maximum likelihood, minimum risk and other methods. The derived estimators will be shown to be (asymptotically) unbiased and MSE-consistent under nice conditions.

Throughout the numerical values of relative efficiencies of the proposed estimators for scale parameter in the small sample sizes when the scale parameter changes a function of an environment dosage, we shall compare its efficiencies.

2. Estimations for Parameter Changes

We shall consider the uniform distribution with the p.d.f.

$$f(x; \theta(t)) = \begin{cases} \frac{1}{\theta(t)}, & 0 < x < \theta(t), \\ 0, & \text{elsewhere,} \end{cases}$$

where $\theta(t)$ are positive for $t > 0$, denoted by $X \sim UNIF(0, \theta(t))$. Gibbons(1974) investigated parametric estimators of the scale parameter in a population uniformly distributed over $(0, \theta)$. Fan(1991) considered a property of the interior and exterior trimmed means in the uniform distribution over $(0, \theta)$.

Here, we shall consider unified estimation for the parameter change of exposure levels in the uniform distribution even when the scale parameter $\theta(t)$ is a polynomial of t ;

$$\theta(t) = b_0 + b_1 t + \cdots + b_r t^r, \quad t > 0 \text{ and } b_i > 0, \text{ for all } i = 0, 1, \dots, r.$$

Assume X_{1j}, \dots, X_{nj} be a simple random samples(SRS) taken from $X_j \sim UNIF(0, \theta(t_j))$, $j = 1, \dots, r+1$, and X_1, \dots, X_{r+1} be independent, $t_i \neq t_k$ for $i \neq k$. And Let $X_{(1)j}, \dots, X_{(n)j}$ be corresponding the order statistics for X_{1j}, \dots, X_{nj} . Note that the complete and sufficient statistics for the scale parameter in the uniform distribution over $(0, \theta(t_j))$ are $X_{(n)j}$.

Define the following notation :

$$\det [t_i^0, \dots, t_i^r] = \begin{vmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^r \\ 1 & t_2 & t_2^2 & \cdots & t_2^r \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & t_{r+1} & t_{r+1}^2 & \cdots & t_{r+1}^r \end{vmatrix}.$$

By the maximum likelihood(ML) method, we can obtain the ML estimator for b_j , $j = 0, 1, \dots, r$;

$$\hat{b}_j^{(1)} = \frac{\det [t_i^0, \dots, t_i^{j-1}, X_{(n_i)i}, t_i^{j+1}, \dots, t_i^r]}{\det [t_i^0, \dots, t_i^r]}.$$

Note that

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{11} & a_{12} & \cdots & a_{11} \end{vmatrix} = a_{k1}A_{k1} + a_{k2}A_{k2} + \cdots + a_{kn}A_{kn}, \quad (2.1)$$

where $A_{kj} = (-1)^{k+j} D_{kj}$ and D_{kj} is minor determinant for a_{kj} eliminated k -row and j -column.

From (2.1), the expectations and variances of these MLE's $\hat{b}_j^{(1)}$ for b_j can be obtained by

$$E(\hat{b}_j^{(1)}) = \sum_{k=0}^r b_k \cdot \frac{\det [t_i^0, \dots, t_i^{j-1}, \frac{n_i}{n_i+1} \cdot t_i^k, t_i^{j+1}, \dots, t_i^r]}{\det [t_i^0, \dots, t_i^r]}$$

and

$$\begin{aligned} VAR(\hat{b}_j^{(1)}) &= \sum_{k=1}^{r+1} \frac{\det^2 [t_i^0, \dots, t_i^{j-1}, t_i^{j+1}, \dots, t_i^r]_{i \neq k}}{\det^2 [t_i^0, \dots, t_i^r]} \\ &\quad \cdot \frac{n_k}{(n_k+1)^2(n_k+2)} \cdot \theta^2(t_k), \end{aligned} \quad (2.2)$$

where $\det [t_i^0, \dots, t_i^{j-1}, t_i^{j+1}, \dots, t_i^r]_{i \neq k}$ is a minor determinant eliminated k -row and $(j+1)$ -column in the determinant, $\det [t_i^0, \dots, t_i^r]$.

Since MLE's $\hat{b}_j^{(1)}$ for b_j are biased estimators, we propose unbiased estimators $\hat{b}_j^{(2)}$ for b_j , $j=0, 1, \dots, r$, in an assumed uniform distribution as follows :

$$\hat{b}_j^{(2)} = \frac{\det [t_i^0, \dots, t_i^{j-1}, \frac{n_i+1}{n_i} \cdot X_{(n_i)i}, t_i^{j+1}, \dots, t_i^r]}{\det [t_i^0, \dots, t_i^r]}.$$

From (2.1), variances of these unbiased estimators $\hat{b}_j^{(2)}$ for b_j , $j=0, 1, \dots, r$, can be obtained by

$$\begin{aligned} VAR(\hat{b}_j^{(2)}) &= \sum_{k=1}^{r+1} \frac{\det^2 [t_i^0, \dots, t_i^{j-1}, t_i^{j+1}, \dots, t_i^r]_{i \neq k}}{\det^2 [t_i^0, \dots, t_i^r]} \\ &\quad \cdot \frac{1}{n_k(n_k+2)} \cdot \theta^2(t_k). \end{aligned} \quad (2.3)$$

Since the minimum risk estimator(MRE) for the scale parameter $\theta(t_j)$ in the assumed

uniform distribution over $(0, \theta(t_j))$ is $\hat{\theta}(t_j) = (n_j+2)/(n_j+1)X_{(n_j)i}$, we can propose following estimator $\hat{b}_j^{(3)}$ for b_j , $j=0, 1, \dots, r$:

$$\hat{b}_j^{(3)} = \frac{\det [t_i^0, \dots, t_i^{j-1}, \frac{n_i+2}{n_i+1} \cdot X_{(n_j)i}, t_i^{j+1}, \dots, t_i^r]}{\det [t_i^0, \dots, t_i^r]}.$$

From (2.1), the expectations and variances of these estimators $\hat{b}_j^{(3)}$ for b_j are given by

$$E(\hat{b}_j^{(3)}) = \sum_{k=0}^r b_k \cdot \frac{\det [t_i^0, \dots, t_i^{j-1}, \frac{n_i(n_i+2)}{(n_i+1)^2} \cdot t_i^k, t_i^{j+1}, \dots, t_i^r]}{\det [t_i^0, \dots, t_i^r]}$$

and

$$\begin{aligned} VAR(\hat{b}_j^{(3)}) &= \sum_{k=1}^{r+1} \frac{\det^2 [t_i^0, \dots, t_i^{j-1}, t_i^{j+1}, \dots, t_i^r]_{i \neq k}}{\det^2 [t_i^0, \dots, t_i^r]} \\ &\quad \cdot \frac{n_k(n_k+2)}{(n_k+1)^4} \cdot \theta^2(t_k). \end{aligned} \tag{2.4}$$

As Johnson(1950) proposed closer estimator for the scale parameter θ than MRE in the uniform distribution over $(0, \theta)$, the estimator's $\hat{b}_j^{(4)}$ for b_j are defined as follows ;

$$\hat{b}_j^{(4)} = \frac{\det [t_i^0, \dots, t_i^{j-1}, 2^{1/n_i} \cdot X_{(n_j)i}, t_i^{j+1}, \dots, t_i^r]}{\det [t_i^0, \dots, t_i^r]}.$$

The expectations and variances of these closer estimator's $\hat{b}_j^{(4)}$ for b_j can be obtained by

$$E(\hat{b}_j^{(4)}) = \sum_{k=0}^r b_k \cdot \frac{\det [t_i^0, \dots, t_i^{j-1}, 2^{\frac{1}{n_i}} \cdot \frac{n_i}{(n_i+1)} \cdot t_i^k, t_i^{j+1}, \dots, t_i^r]}{\det [t_i^0, \dots, t_i^r]}$$

and

$$\begin{aligned} VAR(\hat{b}_j^{(4)}) &= \sum_{k=1}^{r+1} \frac{\det^2 [t_i^0, \dots, t_i^{j-1}, t_i^{j+1}, \dots, t_i^r]_{i \neq k}}{\det^2 [t_i^0, \dots, t_i^r]} \\ &\quad \cdot 2^{2/n_k} \cdot \frac{n_k}{(n_k+1)^2(n_k+2)} \cdot \theta^2(t_k). \end{aligned} \tag{2.5}$$

As Gibbons(1974) proposed symmetric estimator for the scale parameter θ in the uniform distribution over $(0, \theta)$, we propose the following estimator's $\hat{b}_j^{(5)}$ for b_j ;

$$\hat{b}_j^{(5)} = \frac{\det [t_i^0, \dots, t_i^{j-1}, X_{(1)i} + X_{(n_j)i}, t_i^{j+1}, \dots, t_i^r]}{\det [t_i^0, \dots, t_i^r]}.$$

The expectations and variances of these estimator's $\hat{b}_j^{(5)}$ for b_j can be obtained by

$$E(\hat{b}_j^{(5)}) = b_j$$

and

$$\begin{aligned} VAR(\hat{b}_j^{(5)}) &= \sum_{k=1}^{r-1} \frac{\det^2 [t_i^0, \dots, t_i^{j-1}, t_i^{j+1}, \dots, t_i^r]_{i=k}}{\det^2 [t_i^0, \dots, t_i^r]} \\ &\quad \cdot \frac{2}{(n_k+1)(n_k+2)} \cdot \theta^2(t_k). \end{aligned} \quad (2.6)$$

From the results (2.2) through (2.6), the proposed estimators $\hat{b}_j^{(i)}$, $i=2,5$, are unbiased and MSE-consistent for b_j , and the proposed other three estimators $\hat{b}_j^{(i)}$, $i=1,3,5$, are asymptotically unbiased and MSE-consistent for b_j , respectively.

3. Concluding Remarks

In this paper, we proposed several estimators for the scale parameter in the uniform distribution when its scale parameter is a function of a known exposure level t and derived expectations and variances for their estimators. From the results (2.2) through (2.6), Table 1 shows the numerical values of relative efficiencies of $\hat{b}_j^{(i)}$, $i=2,3,4,5$, in relation to $\hat{b}_j^{(1)}$ in an assumed uniform distribution for sample sizes $n_1=5(5)25$, $n_2=5(5)25$, $b_0=0$, $b_1=1$, and $t_1=1$, $t_2=2$ when $r=1$. From Table 1, the MLE $\hat{b}_0^{(1)}$ is more efficient than other proposed estimators for b_0 when values for sample sizes n_1 and n_2 are similar but the closer estimator $\hat{b}_0^{(4)}$ is more efficient than other proposed estimators for b_0 as differences between n_1 and n_2 are increasing. And the closer estimator $\hat{b}_1^{(4)}$ is more efficient than other proposed estimators for b_1 in an assumed uniform distribution in the small sample sizes n_1 and n_2 .

References

- [1] Fan, D.Y.(1991). On a Property of the Order Statistics of the Uniform Distribution, *Communications in Statistics, Theory and Method*, Vol. 20, 1903-1909.
- [2] Gibbons, J.D.(1974). Estimation of the Unknown Upper Limit of Uniform Distribution,

Sankya, Series B, Vol. 36, 29–40.

- [3] Johnson, N.L.(1950). On the Comparison of Estimators, *Biometrika*, Vol. 37, 281–287.
- [4] Woo, J.S. and Ali, M.M.(1994). Unified Jackknife Estimation for Parameter Changes in the Exponential Distribution, *Journal of Statistical Studies*, Vol. 14, 20–23
- [5] Woo, J.S. abd Lee, C.S.(2000). Jackknife Estimates for Parameter Changes in the Weibull Distribution, *The Korean Communications in Statistics*, Vol 7(1), 199–209.
- [6] Woo, J.S. and Yoon. G.E.(1990). Unified Estimation for Parameter Changes in a Two Pareto Distribution, *Journal of Nature Sciences*, Vol 9, 41–48.

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Table 1. Relative efficiencies for other estimators in relation to ML estimator parameter changes in the uniform distribution ($b_0 = 0$, $b_1 = 1$, $t_1 = 1$, $t_2 = 2$).

size		parameter	Relative efficiencies			
			$\hat{b}_j^{(2)}$	$\hat{b}_j^{(3)}$	$\hat{b}_j^{(4)}$	$\hat{b}_j^{(5)}$
n_1	n_2					
5	5	b_o	1.44000	1.36111	1.31951	2.40000
		b_1	1.12500	1.06944	1.04526	1.87500
	10	b_o	1.13667	1.09596	1.05903	1.93333
		b_1	1.3000	1.25826	1.21587	2.27273
	15	b_o	0.95179	0.92174	0.89534	1.61026
		b_1	1.25149	1.21504	1.16909	2.17822
	20	b_o	0.85500	0.82937	0.80983	1.44000
		b_1	1.13478	1.10266	1.06161	1.95652
	25	b_o	0.79827	0.77495	0.75993	1.34054
		b_1	1.03366	1.00366	0.96985	1.76829
10	5	b_o	1.13667	1.09596	1.05903	1.93333
		b_1	0.84552	0.81682	0.80136	1.41791
	10	b_o	1.21000	1.19008	1.14870	2.20000
		b_1	0.97581	0.96134	0.94179	1.77419
	15	b_o	1.10000	1.08665	1.05448	2.02000
		b_1	1.10000	1.08813	1.06002	2.04082
	20	b_o	0.98000	0.96970	0.94763	1.80000
		b_1	1.15000	1.13915	1.10580	2.14286
	25	b_o	0.89165	0.88297	0.86863	1.63529
		b_1	1.13515	1.12508	1.09145	2.11340

size		parameter	Relative efficiencies			
n_1	n_2		$\hat{b}_j^{(2)}$	$\hat{b}_j^{(3)}$	$\hat{b}_j^{(4)}$	$\hat{b}_j^{(5)}$
15	5	b_o	0.95179	0.92174	0.89534	1.61026
		b_1	0.75695	0.73374	0.72385	1.26682
	10	b_o	1.1000	1.08665	1.05448	2.02000
		b_1	0.82365	0.81436	0.80750	1.50246
	15	b_o	1.13778	1.12891	1.09682	2.13333
		b_1	0.92754	0.92103	0.90975	1.73913
	20	b_o	1.08111	1.07445	1.04840	2.03889
		b_1	1.01945	1.01386	0.99623	1.93267
	25	b_o	1.00163	0.99624	0.97763	1.89126
		b_1	1.07817	1.07300	1.05038	2.05275
20	5	b_o	0.85500	0.82937	0.80983	1.44000
		b_1	0.71482	0.69378	0.68711	1.19469
	10	b_o	0.98000	0.96970	0.94763	1.80000
		b_1	0.74773	0.74026	0.74013	1.36364
	15	b_o	1.08111	1.07445	1.04840	2.03889
		b_1	0.82245	0.81778	0.81645	1.54519
	20	b_o	1.10250	1.09751	1.07177	2.10000
		b_1	0.90369	0.90000	0.89416	1.72131
	25	b_o	1.06805	1.06405	1.04238	2.04211
		b_1	0.97615	0.97289	0.96184	1.87225
25	5	b_o	0.79827	0.77495	0.75993	1.34054
		b_1	0.69039	0.67046	0.66588	1.15291
	10	b_o	0.89165	0.88297	0.86863	1.63529
		b_1	0.70377	0.69717	0.70112	1.28272
	15	b_o	1.00163	0.99624	0.97763	1.89126
		b_1	0.75811	0.75429	0.75906	1.42461
	20	b_o	1.06805	1.06405	1.04238	2.05411
		b_1	0.82346	0.82064	0.82266	1.57061
	25	b_o	1.08160	1.07840	1.05702	2.08000
		b_1	0.88947	0.88711	0.88494	1.71053