An Estimation Approach to Robust Adaptive Control of Uncertain Nonlinear Systems with Dynamic Uncertainties

Choon-Ki Ahn, Beom-Soo Kim, and Myo-Taeg Lim

Abstract: In this paper, a novel estimation technique for a robust adaptive control scheme is presented for a class of uncertain nonlinear systems with a general set of uncertainty. For a class of introduced *more extended semi-strict feedback forms* which generalize the systems studied in recent years, a novel estimation technique is proposed to estimate the states of the fully nonlinear unmodeled dynamics without stringent conditions. With the introduction of powerful functions, the estimation error can be tuned to a desired small region around the origin via the estimator parameters. In addition, with some effective functions, a modified adaptive backstepping for dynamic uncertainties is presented to drive the output to an arbitrarily small region around the origin by an appropriate choice of the design parameters. With our proposed schemes, we can remove or relax the assumptions of the existing results.

Keywords: State estimation technique, dynamic uncertainties, adaptive backstepping, uncertain nonlinear systems, unmodeled dynamics.

1. INTRODUCTION

Adaptive control of a class of nonlinear systems has been given a lot of attention in control problems in recent years [1,2]. Most research results in adaptive nonlinear control were restricted to linearly parametrized and state feedback linearizable. A lot of earlier problems were all restricted to systems satisfying the matching condition and over-parametrization. One of the recent breakthroughs over-coming these difficulties in adaptive nonlinear control is the introduction of adaptive backstepping algorithms for feedback linearizable systems [3,4]. Marino and Tomei [5-7] combined it with their filtered transformations [8,9] to solve the adaptive output feedback problem for a class of nonlinear systems that has not since been enlarged.

Despite these advances in adaptive nonlinear control, most results deal with the case that all the nonlinearities of a system are known. In practice, almost every physical system is subject to various model uncertainties. Normally, the causes of model uncertainties are unknown physical parameters, external disturbances, and imprecise modeling. In [5, 6], a robust adaptive controller was constructed for systems in which parameter uncertainties are usually assumed to appear linearly with respect to (known) nonlinear functions.

In addition to these uncertainties, some systems are further subject to dynamic uncertainties that depend on the unmeasured states of unmodeled dynamics [10-14]. Practical examples are the dynamic friction models in [15] and the eccentric rotors in [16]. In [10] an adaptive controller was constructed for a class of extended strict feedback forms in which the unmeasured states enter the systems in a linear affine fashion. As pointed out in [10], it is unclear how to achieve robustness with respect to modeling errors such as uncertain nonlinearities. In [11], a robust adaptive control scheme was proposed using a dynamic dominating signal to determine the size of dynamic uncertainties. The result assumes that the unmodeled dynamics are ISpS (Input-to-State practically Stable). Also, the backstepping method based on small-gain arguments is presented assuming that the q-subsystem is ISpS[12]. More recently, Xu and Yao also constructed an adaptive robust controller via an observer for unmeasured states [13]. But the assumption of this approach is that observer error dynamics is exponentially ISS (Input-to-State Stable). In addition, the backstepping scheme is presented under the stringent assumption that the equilibrium point of an unmodeled dynamics is globally exponentially stable [14].

In this paper, we introduce *a more extended semi-strict feedback form* which is a more general form than the one studied in recent years. It has a generalized nonlinear unmodeled dynamics in contrast to [10, 13]. Note that the q-subsystem of this form has more

Manuscript received November 28, 2001; accepted August 13, 2002. This work has been supported by a Korea University Grant. The authors would like to thank Korea University and the anonymous reviewers whose constructive comments improved this paper.

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inputs than the form in [11,12,14]. Unfortunately, with the recent work, it is impossible to design a robust controller when the q-subsystem is only BIBS (Bounded-Input Bounded State) stable and the fully nonlinear system, and ς is unmeasured. However, the proposed novel estimation technique can estimate the states of an unmodeled dynamics under very mild conditions. In estimator design, the introduction of the ϖ function yields the complete elimination of the effect of parametric uncertainties. In addition, by the introduction of the g function, we can deal with the uncertain nonlinearities very effectively. With these functions, estimation error can be made as small as desired by an appropriate choice of the estimator design constants. Another novelty of this paper is the introduction of the γ function to handle dynamic uncertainties effectively. With a combination of powerful functions, a modification of the adaptive backstepping scheme is proposed. Our presented scheme can drive the output to an arbitrarily small region around the origin by an appropriate choice of controller design parameters. With these proposed schemes, we can deal with a more general form and guarantee a better performance, but we no longer require the stringent assumptions of the existing results.

The main contributions of this paper are summarized as follows: (1) The uncertain nonlinear systems under consideration are subject to a general set of uncertainty: parametric uncertainties, uncertain nonlinearities, and dynamic uncertainties. (2) In estimator design, we introduce the ϖ function to eliminate the effect of parametric uncertainties and the g function to handle uncertain nonlinearities successfully. (3) In controller design, we introduce the γ function to deal with the effect of dynamic uncertainties effectively. (4) With a novel estimation technique and a proposed modified adaptive backstepping scheme, we deal with a more general form and guarantee a better performance. (5) With these proposed tools, we can remove or smooth the assumptions of the existing studies.

The class of *more extended semi-strict feedback forms* is described in Section 2. In Section 3, a novel estimation technique is proposed to estimate the states of unmodeled dynamics under very mild conditions. The systematic controller design method for dynamic uncertainties is presented in Section 4. In Section 5, an overall design method is illustrated for the example system, and the simulation result is given. The conclusion is given in Section 6.

2. PROBLEM FORMULATION

The class of nonlinear systems to be controlled in this paper is *a more extended semi-strict feedback form* as follows:

$$\begin{aligned}
\dot{\varsigma} &= q(\varsigma, x_1, x_2, \ldots, x_l),\\
\dot{x}_i &= x_{i+1} + \theta^T \phi_i(x_1, \ldots, x_i) + \Delta_i\\
&\quad (1 \le i \le l-1),\\
\dot{x}_i &= x_{i+1} + \theta^T \phi_i(x_1, \ldots, x_i) + \varsigma^T \psi_i(x_1, \ldots, x_i) + \Delta_i \qquad (1)\\
&\quad (1 \le i \le n-1),\\
\dot{x}_n &= u + \theta^T \phi_n(x_1, \ldots, x_n) + \varsigma^T \psi_n(x_1, \ldots, x_n) + \Delta_n\\
y &= x_1,
\end{aligned}$$

where $u \in R$, $y \in R$ are the control input and the output, respectively, and $x = (x_1, \ldots, x_n)$ is part of the measured states. $\varsigma \in R^m$ represents the unmeasured state, and $\theta \in R^P$ is a vector of uncertain constant parameters. $\phi_i \in R^P$ and $\psi_i \in R^m$ are vectors of known smooth functions. Δ_i represent unknown nonlinear smooth funct-ions such as disturbances and modeling errors.

Throughout this paper, we need only the following assumptions.

Assumption 1: The unmodeled dynamics q-subsystem, with $\bar{x}_l = (x_1, \ldots, x_l)$ as the input and ς as the state, is bounded-input-bounded-state stable.

Assumption 2: For each $1 \le i \le n$, there exists an unknown positive constant p_i such that, for all $t \in R_+$, $\varsigma \in R^m$, and $x \in R^n$

$$|\Delta_i(t, \varsigma, x)| \le p_i \delta_i(x_1, x_2, \ldots, x_i), \qquad (2)$$

where δ_i is a known nonnegative smooth function.

The control objective in this paper is to construct a robust adaptive nonlinear control law so that the output y is driven to an arbitrarily small region around the origin with exponential convergence rate while maintaining a global uniform ultimate boundedness of all the signals and states in spite of various uncertainties.

Remark 1: (I) If there are no uncertain nonlinearities (*i.e.* $\Delta_i = 0, \forall_i$), system (1) with Assumption 1 and a special q-dynamic is the extended strict feedback form studied in [10]. Freeman and Kokotovic constructed an adaptive controller for this system in [10]. (II) If there are no dynamic uncertainties (i.e. there are no unmeasured states ς), system (1) with the assumption of knowing the bounds of parameter variations and uncertain nonlinearities reduces to the semi-strict feedback form studied in [17,18]. Yao and Tomizuka [17] also constructed an adaptive robust controller for this system. (III) If the q-dynamic has only one input, x_1, and satisfies the Lipschitz function, then system (1) with a small modification was considered in [11,12,14]. Jiang and his colleagues also constructed an adaptive controller in [11,12,14] under the assumption that the q-dynamic is ISpS. It should be noted that this assumption is more stringent than our Assumption 1. (IV) If the q-dynamic is a special form and the observer error dynamic is exponentially ISpS, system (1) with the assumption of knowing the bounds of parameter variations and uncertain nonlinearities is the extended semi-strict feedback form considered in [13]. X_u and Yao also constructed an adaptive robust controller for this system.

As seen in Remark 1, a class of *more extended semi-strict feedback forms* is more general than the systems considered in recent results. In spite of considering this general form, our assumptions are milder than the previous research. It should be noted that we can remove Assumption 2 and smooth Assumption 1 in [14], and also we can remove Assumption 3 and smooth Assumption 1 in [13]. In addition, Assumption 4.1 of [11] and Assumption 2 of [12] are relaxed by our proposed scheme.

Remark 2: From Assumption 2, since p_i is unknown, we do not need to have prior knowledge of the bounds of uncertain nonlinearities.

3. A STATE ESTIMATION OF UNMODELED DYNAMICS

In this section, we estimate the unmeasured states of an unmodeled dynamics via a proposed novel estimation technique. It will be shown that, in spite of various uncertain terms, we can construct a robust estimator. In estimator design, the g function is introduced to handle uncertain nonlinearities. In addition, we introduce the $\tilde{\omega}$ function to remove the effect of parametric uncertainties. The proposed estimation technique is based on the idea of introducing a state transformation [19]. We introduce a nonlinear transformation:

$$\chi_i = \varsigma_i - \omega_i(x_1,\ldots,x_l) + \theta^T \tilde{\omega}_i(x_1,\ldots,x_l), \tag{3}$$

where $i=1,\ldots m$, $\varsigma = \{\varsigma_1,\ldots,\varsigma_m\}^T \in R^m$, and $\omega_i(x_1,\ldots,x_l) \in R$, $\tilde{\omega}_i(x_1,\ldots,x_l) \in R^P$ are smooth design functions to be determined later. The time derivative of χ_i is represented as

$$\begin{aligned}\dot{\chi}_i &= \dot{\varsigma}_i - \dot{\omega}_i(\bar{x}_l) + \theta^T \dot{\tilde{\omega}}_i(\bar{x}_l)\\ &= q_i(\varsigma_i,\bar{x}_l) - \sum_{j=1}^{l}\frac{\partial\omega_i(\bar{x}_l)}{\partial x_j}(x_{j+1} + \theta^T\phi_j + \Delta_j)\\ &\quad - \frac{\partial\omega_i(\bar{x}_l)}{\partial x_l}\varsigma^T\psi_l + \theta^T\dot{\tilde{\omega}}_i(\bar{x}_l)\\ &= q_i(\varsigma_i,\bar{x}_l) - \sum_{j=1}^{l}\frac{\partial\omega_i}{\partial x_j}(x_{j+1} + \theta^T\phi_j)\\ &\quad - \sum_{j=1}^{l}\frac{\partial\omega_i}{\partial x_j}\Delta_j + \theta^T\dot{\tilde{\omega}}_i(\bar{x}_l) - \frac{\partial\omega_i}{\partial x_l}\varsigma_i\psi_{li}\\ &\quad - \sum_{j=1}^{i-1}\frac{\partial\omega_i}{\partial x_l}\varsigma_j\psi_{lj} - \sum_{j=i+1}^{l}\frac{\partial\omega_i}{\partial x_l}\varsigma_j\psi_{lj},\end{aligned} \tag{4}$$

where $\bar{x}_l = (x_1,\ldots,x_l) \in R^m$, $\dot{\varsigma}_i = q_i(\varsigma_i,\bar{x}_l)$ denotes the *i-th* component of the function $\dot{\varsigma} = q(\varsigma,\bar{x}_l)$, and ψ_{lj} represents the j-*th* element of function ψ_l.

In this case, (4) is parametrized as follows :

$$\begin{aligned}\dot{x}_i &= -(a_i + c_i)\varsigma_i - g_i(\bar{x}_l)\varsigma_i - \frac{\partial\omega_i}{\partial x_l}\varsigma_i\psi_{li}\\ &\quad - \sum_{j=1}^{l}\frac{\partial\omega_i}{\partial x_j}(x_{j+1} + \theta^T\phi_j) - \sum_{j=1}^{l}\frac{\partial\omega_i}{\partial x_j}\Delta_j\\ &\quad + \theta^T\dot{\tilde{\omega}}_i(\bar{x}_l) + v_i,\end{aligned} \tag{5}$$

where a_i and c_i are positive design constants, and $g_i(\bar{x}_l) \geq 0$ is a smooth design function to be determined later. The function v_i is the modeling error represented as

$$\begin{aligned}v_i &= q_i(\varsigma_i,\bar{x}_l) + \{a_i + c_i + g_i(\bar{x}_l)\}\varsigma_i\\ &\quad - \sum_{j=1}^{i-1}\frac{\partial\omega_i}{\partial x_l}\varsigma_j\psi_{lj} - \sum_{j=i+1}^{l}\frac{\partial\omega_i}{\partial x_l}\varsigma_j\psi_{lj}.\end{aligned} \tag{6}$$

Since $\varsigma_i = \chi_i - \theta^T\tilde{\omega}_i + \omega_i$, we have

$$\begin{aligned}\dot{x}_i &= -(a_i + c_i)\chi_i - (a_i + c_i)\omega_i - g_i(\bar{x}_l)(\chi_i + \omega_i)\\ &\quad - \frac{\partial\omega_i}{\partial x_l}\chi_i\psi_{li} - \frac{\partial\omega_i}{\partial x_l}\omega_i\psi_{li}\\ &\quad - \sum_{j=1}^{l}\frac{\partial\omega_i}{\partial x_j}x_{j+1} - \sum_{j=1}^{l}\frac{\partial\omega_i}{\partial x_j}\Delta_j\\ &\quad + \theta^T\left[\dot{\tilde{\omega}}_i + \left\{a_i + g_i(\bar{x}_l) + c_i + \frac{\partial\omega_i}{\partial x_l}\psi_{li}\right\}\tilde{\omega}_i\right.\\ &\quad \left. - \sum_{j=1}^{l}\frac{\partial\omega_i}{\partial x_j}\phi_j\right] + v_i.\end{aligned} \tag{7}$$

If we choose $\tilde{\omega}_i$ to satisfy the equality

$$\begin{aligned}&\dot{\tilde{\omega}}_i + \left\{a_i + g_i(\bar{x}_l) + c_i + \frac{\partial\omega_i}{\partial x_l}\psi_{li}\right\}\tilde{\omega}_i\\ &\quad - \sum_{j=1}^{l}\frac{\partial\omega_i}{\partial x_j}\phi_j = 0,\end{aligned} \tag{8}$$

then, (7) becomes

$$\begin{aligned}\dot{x}_i &= -\left\{a_i + c_i + g_i(\bar{x}_l) + \frac{\partial\omega_i}{\partial x_l}\psi_{li}\right\}(\chi_i + \omega_i)\\ &\quad - \sum_{j=1}^{l}\frac{\partial\omega_i}{\partial x_j}x_{j+1} - \sum_{j=1}^{l}\frac{\partial\omega_i}{\partial x_j}\Delta_j + v_i.\end{aligned} \tag{9}$$

It is interesting to note that, by the introduction of the ϖ function, the effect of the parametric uncertainties was eliminated from the χ_i-system (9)

Therefore, we introduce the following estimator:

$$\dot{\hat{x}}_i = -\left\{ a_i + c_i + g_i(\bar{x}_l) + \frac{\partial \omega_i}{\partial x_l}\psi_{li} \right\}(\hat{x}_i + \omega_i) - \sum_{j=1}^{l} \frac{\partial \omega_i}{\partial x_j} x_{j+1} \quad (10)$$

Denoting the state estimation error by $e_i = \hat{\chi}_i - \chi_i$ from (9) and (10), we have the following error equation:

$$\dot{e}_i = -b_i e_i - c_i e_i - g_i(\bar{x}_l)e_i + \sum_{j=1}^{l} \frac{\partial \omega_i}{\partial x_j}\Delta_j - v_i , \quad (11)$$

where $b_i = a_i + \frac{\partial \omega_i}{\partial x_l}\psi_{li}$ and the design constant b_i is chosen to be $b_i > 0$.

Since $\chi_i = \varsigma_i - \omega_i + \theta^T \tilde{\omega}_i$, we can obtain the equivalent representation of ς_i:

$$\varsigma_i = \hat{x}_i - \theta^T \tilde{\omega}_i + \omega_i - e_i . \quad (12)$$

Now, we are ready to state and prove the following theorem.

Theorem 1: If we use estimator (10), the solution of (11) satisfies the following properties as:

(1) e_i is globally uniformly ultimately bounded.

(2) $| e_i(t) | \leq \mu_i \exp(-b_i t) + Q_i \quad (i = 1, ..., m),$ (13)

where

$$\mu_i = \sqrt{2[W_i(0) - \rho_i]}, \quad Q_i = \sqrt{\frac{M_i}{b_i}},$$
$$\rho_i = \frac{M_i}{2b_i}, \quad M_i = \frac{\bar{v}_i^2}{4c_i} + \sum_{j=1}^{l} \frac{p_j^2}{4k_{ij}}. \quad (14)$$

$W_i(0)$ is the value of the i-th Lynapunov function at initial time and $\bar{v}_i = \sup_{\forall t \geq 0} v_i(t)$. In addition, k_{ij} is a positive design constant.

Proof: Refer to Appendix I. □

Under additional conditions, the exact regulation of the estimation error to the origin can be achieved without tuning the design constants appropriately.

Corollary 1: (Exact Regulation)

(1) *If* $\Delta_i = 0$ *for* $1 \leq i \leq l$ *, there exist constants* Π, Ξ *such that*

$$\int_0^t e_i(\tau)^2 d\tau \leq \Pi + \Xi \int_0^t v_i(\tau)^2 d\tau. \quad (15)$$

(2) *In addition, if* $v_i(t)$ *is square-integrable, then*

$$\lim_{t \to \infty} e_i(t) = 0. \quad (16)$$

Proof: Refer to Appendix II. □

In vector notations, (12) becomes

$$\varsigma = \hat{\chi} - \tilde{\omega}^T \theta + \omega - e , \quad (17)$$

where

$$\varsigma = [\varsigma_1, ..., \varsigma_m]^T \in R^m,$$
$$\hat{\chi} = [\hat{\chi}_1, ..., \hat{\chi}_m]^T \in R^m,$$
$$\tilde{\omega} = [\tilde{\omega}_1, ..., \tilde{\omega}_m] \in R^{p \times m},$$
$$\omega = [\omega_1, ..., \omega_m]^T \in R^m,$$
$$e = [e_1, ..., e_m]^T \in R^m.$$

Remark 3: Note that the choice of function $\omega_i(\bar{x}_l)$ is not unique. Since

$$b_i = a_i + \frac{\partial \omega_i}{\partial x_l}\psi_{li} > 0, \quad a_i > 0. \quad (18)$$

as long as ψ_{li} is non-zero, nonlinear design function w_i can be explicitly constructed as follows:

$$w_i(\bar{x}_l) = \int_0^{x_l} \frac{b_i - a_i}{\psi_{li}(\bar{x}_l)} dx_l \quad (19)$$

where $a_i \neq b_i$ .If ψ_{li} is zero, then we have $a_i = b_i > 0$ and $w_i(\bar{x}_l)$ is any smooth function.

Remark 4: As seen in property (2) of Theorem 1, if design constants a_i, b_i, c_j, k_{ij}, $(i = 1, \cdots, m$ and $j = 1, \cdots, l)$ are chosen appropriately, then it is possible to make the estimation error $e_i(t)$ as small as desired to any prescribed accuracy. Thus, we can achieve the robust regulation of $e_i(t)$ to a small region around the origin.

4. A MODIFICATION OF ADAPTIVE BACK-STEPPING FOR DYNAMIC UNCERTAINTIES

In this section, a modification of the adaptive backstepping methodology is proposed. We introduce η, l functions to handle uncertain nonlinearities and the γ function to deal with dynamic uncertainties effectively. Since the first l-1 steps do not include the unmeasured state ς, we will deal with the effect of unmeasured state ς from step l. Thus, we let $\gamma_i = 0, 1 \leq i \leq l-1$.

Step 1: $i = 1$

Let $z_1 = x_1, z_2 = x_2 - \alpha_1$ and $\bar{\Delta}_1 = \Delta_1$,where $\alpha_1 = \alpha_{1s} - \eta_1 - \gamma_1$. Since there is no effect of ς in Step 1, we let $\gamma_1 = 0$ and let $\tilde{\theta} = \hat{\theta} - \theta$, $\tilde{p}_1 = \hat{p}_1 - p_1$. From Assumption 2, $\bar{\Delta}_1$ satisfies $| \bar{\Delta}_1 | \leq \bar{p}_1 \bar{\delta}_1$, where $\bar{p}_1 = p_1$, $\bar{\delta}_1 = \delta_1$. Throughout this paper, $\hat{\bullet}$ represents the estimate of $\bullet$.

From the above definition, we can obtain the following:

$$\dot{z}_1 = z_2 + \alpha_1 + \theta^T \phi_1(z_1) + \Delta_1. \quad (20)$$

Consider the following Lyapunov function

$$V_1=\frac{1}{2}z_1^2+\frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta}+\frac{1}{2r_1}\tilde{\bar{p}}_1^2, \tag{21}$$

where Γ is the positive definite matrix for design, and r_1 is a positive constant for design.

The time derivative of V_1 is given by

$$\begin{aligned}\dot{V}_1&=z_1(z_2+\alpha_1+\hat{\theta}^T\phi_1(z_1)+\Delta_1)\\&+\tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}}+\frac{1}{r_1}\tilde{\bar{p}}_1\dot{\hat{\bar{p}}}_1-z_1\tilde{\theta}^T\phi_1(z_1).\end{aligned} \tag{22}$$

If the following are chosen as

$$\alpha_1=-k_1z_1-\hat{\theta}^T\phi_1(z_1)-\eta_1, \tag{23}$$

$$\tau_1=\Gamma[z_1\phi_1(z_1)-\sigma_\theta(\hat{\theta}-\theta^0)], \tag{24}$$

$$\dot{\hat{\bar{p}}}=r_1[l_1-\sigma_1(\hat{\bar{p}}-\bar{p}_1^0)], \tag{25}$$

where $k_1,\sigma_\theta,\sigma_1,\bar{p}_1^0$ are positive design constants, $\theta^0\in R^P$, $\Gamma\in R^P$ are positive definite matrices for design, and η_1, l_1 functions are yet to be designed. It can be shown that

$$\begin{aligned}\dot{V}_1&=z_1z_2-k_1z_1^2+z_1(\bar{\Delta}_1-\eta_1)+\tilde{\bar{p}}_1l_1+\tilde{\theta}^T\Gamma^{-1}(\dot{\hat{\theta}}-\tau_1)\\&-\sigma_\theta\tilde{\theta}^T(\hat{\theta}-\theta^0)-\sigma_1\tilde{\bar{p}}_1(\hat{\bar{p}}_1-\bar{p}_1^0).\end{aligned} \tag{26}$$

Noting that η_1, l_1 are chosen as any smooth functions satisfying the following two conditions

$$\begin{aligned}&(i)\quad z_1(\bar{\Delta}_1-\eta_1)+\tilde{\bar{p}}_1l_1\le\frac{\bar{p}_1}{\epsilon_1},\\&(ii)\quad -z_1\eta_1\le 0,\end{aligned} \tag{27}$$

where ε_1 is a positive design constant and condition (ii) is the passivity-like requirement not to interfere with the nominal structure, by completing the square as in [20]

$$\begin{aligned}-\sigma_\theta\tilde{\theta}^T(\hat{\theta}-\theta^0)&=-\frac{1}{2}\sigma_\theta|\tilde{\theta}|^2-\frac{1}{2}\sigma_\theta|\hat{\theta}-\theta^0|^2\\&+\frac{1}{2}\sigma_\theta|\theta-\theta^0|^2,\end{aligned} \tag{28}$$

$$\begin{aligned}-\sigma_1\tilde{\bar{p}}_1(\hat{\bar{p}}_1-\bar{p}_1^0)&=-\frac{1}{2}\sigma_1\tilde{\bar{p}}_1^2-\frac{1}{2}\sigma_1(\hat{\bar{p}}-\bar{p}_1^0)^2\\&+\frac{1}{2}\sigma_1(\bar{p}_1-\bar{p}_1^0)^2.\end{aligned} \tag{29}$$

It can be shown that

$$\begin{aligned}\dot{V}_1&\le z_1z_2-k_1z_1^2+\frac{\bar{p}_1}{\varepsilon_1}-\frac{1}{2}\sigma_\theta|\tilde{\theta}|^2-\frac{1}{2}\sigma_1\tilde{\bar{p}}_1^2\\&+\frac{1}{2}\sigma_\theta|\theta-\theta^0|^2+\frac{1}{2}\sigma_1(\bar{p}_1-\bar{p}_1^0)^2\\&+\tilde{\theta}^T\Gamma^{-1}(\dot{\hat{\theta}}-\tau_1)\\&\le -c_1V_1+z_1z_2+\lambda_1+\tilde{\theta}^T\Gamma^{-1}(\dot{\hat{\theta}}-\tau_1),\end{aligned} \tag{30}$$

where

$$\begin{aligned}\lambda_1&=\frac{\bar{p}_1}{\varepsilon_1}+\frac{1}{2}\sigma_\theta|\theta-\theta^0|^2+\frac{1}{2}\sigma_1(\bar{p}_1-\bar{p}_1^0)^2,\\c_1&=\min\left\{2k_1,\sigma_1r_1,\frac{\sigma_\theta}{\lambda_{\max}(\Gamma_1^{-1})}\right\}>0.\end{aligned} \tag{31}$$

Remark 5: For example, we can choose η and l functions as the following smooth functions

$$\eta_1=z_1\hat{\bar{p}}_1^2\bar{\delta}_1^2+\frac{\varepsilon_1^2}{4}z_1\bar{\delta}_1^2, \tag{32}$$

$$l_1=\varepsilon_1z_1^2\bar{\delta}_1^2, \tag{33}$$

where ε_1 is a positive design constant.

In this case, using $|z_1\bar{\delta}_1|\le\varepsilon_1z_1^2\bar{\delta}_1^2+\frac{1}{4\varepsilon_1}$,

$$\begin{aligned}z_1(\bar{\Delta}_1-\eta_1)+\tilde{\bar{p}}_1l_1&\le\left|z_1\bar{\delta}_1\right|\bar{p}_1-z_1\eta_1+\tilde{\bar{p}}_1l_1\\&\le-\bar{\delta}_1^2(z_1\hat{\bar{p}}_1-\frac{\varepsilon_1z_1}{2})^2+\frac{\bar{p}_1}{4\varepsilon_1}\\&\le\frac{\bar{p}_1}{4\varepsilon_1}.\end{aligned} \tag{34}$$

Therefore, condition (i) of (27) is satisfied. Also, since

$$\begin{aligned}-z_1\eta_1&\le-z_1^2\hat{\bar{p}}_1^2\bar{\delta}_1^2-\frac{\varepsilon_1^2}{4}z_1^2\bar{\delta}_1^2\\&\le 0,\end{aligned} \tag{35}$$

condition (ii) of (27) holds. Another selection is the scheme of the *tanh* function. In addition, the Gudermannian function scheme can satisfy (i) and (ii) of (27).

Step 2: $2\le i\le l-1$

Let $z_i=x_i\alpha_{i-1}$, where $\alpha_{i-1}=\alpha_{(i-1)s}-\eta_{i-1}-\gamma_{i-1}$. Since there is also no effect of ς in this step, we let $\gamma_{i-1}=0$. If we let $\bar{\Delta}_i=\Delta_i-\sum_{j=1}^{i-1}(\frac{\partial\alpha_{i-1}}{\partial x_j}\Delta_j)$, then we have $|\bar{\Delta}_i|\le\bar{p}_i\bar{\delta}_i(x_1,\ldots,x_i)$.

Lemma 1: At step i, $\forall 2\le i\le l-1$, we choose the desired control function α_i and the tuning as

$$\begin{aligned}\alpha_i(x_1,\ldots,x_i,\hat{\theta},\hat{\bar{p}}_1,\ldots,\hat{\bar{p}}_i)&=-z_{i-1}-k_iz_i\\&+\left[\sum_{j=1}^{i-2}z_{j+1}\frac{\partial\alpha_i}{\partial\hat{\theta}}\Gamma-\hat{\theta}^T\right]\bar{\phi}_i\\&+\sum_{j=1}^{i-1}\left(\frac{\partial\alpha_{j-1}}{\partial x_j}x_{j+1}+\frac{\partial\alpha_{j-1}}{\partial\hat{\bar{p}}_j}\dot{\hat{\bar{p}}}_j\right)\end{aligned}$$

$$+\frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}\tau_i - \eta_i, \tag{36}$$

$$\tau_i = \tau_{i-1} + \Gamma z_i \bar{\phi}_i, \tag{37}$$

where $\bar{\phi}_i = \phi_i - \sum_{j=1}^{i-1}\left(\frac{\partial \alpha_{i-1}}{\partial x_j}\phi_j\right)$, k_i is a positive design constant and η_i is the smooth design function.

If we choose the following adaptation law

$$\dot{\hat{p}}_i = r_i[l_i - \sigma_i(\hat{p}_i - \bar{p}_i^0)], \tag{38}$$

where $\tilde{\bar{p}}_i = \hat{\bar{p}}_i - \bar{p}_i$, r_i, α_i, $\tilde{\bar{p}}_i^0$ are positive design constants, and l_i is a smooth function for design, then it can be shown that the time derivative of the Lyapunov function candidate

$$V_i = V_{i-1} + \frac{1}{2}z_i^2 + \frac{1}{2r_i}\tilde{\bar{p}}_i^2, \tag{39}$$

satisfies

$$\begin{aligned}\dot{V}_i \le& -c_i V_i + z_i z_{i+1} + \lambda_i \\ &+\left[\sum_{j=1}^{i-1} z_{j+1}\frac{\partial \alpha_j}{\partial \hat{\theta}} - \tilde{\theta}^T\Gamma^{-1}\right](\tau_i - \dot{\hat{\theta}}),\end{aligned} \tag{40}$$

where

$$\begin{aligned}\lambda_i &= \sum_{j=1}^{i}\frac{\bar{p}_j}{\varepsilon_j} + \frac{1}{2}\sigma_\theta |\theta - \theta^0|^2 + \frac{1}{2}\sum_{j=1}^{i}\sigma_j(\bar{p}_j - \bar{p}_j^0)^2, \\ c_i &= \min\left\{2k_i, \sigma_i r_i, \frac{\sigma_\theta}{\lambda_{\max}(\Gamma_i^{-1})}\right\} > 0,\end{aligned} \tag{41}$$

and the i-th error subsystem is

$$\begin{aligned}\dot{z}_i =& z_{i+1} - k_i z_i - z_{i-1} + \bar{\Delta}_i - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}(\dot{\hat{\theta}} - \tau_i) \\ &- \eta_i + \left[\sum_{j=1}^{i-2} z_{j+1}\frac{\partial \alpha_j}{\partial \hat{\theta}}\Gamma - \tilde{\theta}^T\right]\bar{\phi}_i.\end{aligned} \tag{42}$$

Note that the η_i and l_i functions are any smooth robust control terms satisfying

$$\begin{aligned}&(i) \quad z_i(\bar{\Delta}_i - \eta_i) + \tilde{\bar{p}}_i l_i \le \frac{\bar{p}_i}{\varepsilon_i}, \\ &(ii) \quad -z_i\eta_i \le 0.\end{aligned} \tag{43}$$

Proof: Refer to Appendix III □

Remark 6: For example, we can choose the η_i and l_i functions as the following polynomial functions

$$\eta_i = z_i \hat{\bar{p}}_i^2 \bar{\delta}_i^2 + \frac{\varepsilon_i^2}{4} z_i \bar{\delta}_i^2, \tag{44}$$

$$l_i = \varepsilon_i z_i^2 \bar{\delta}_i^2, \tag{45}$$

where ε_i is a positive design constant. From the same method as in Remark 5, we can observe that conditions (i) and (ii) of (43) hold.

Step 3: $i = l$

From $i = l$, since there is the effect of unmeasured state ς, we introduce γ function to handle dynamic uncertainties. Using the equivalent expression of ς(17), we will deal with system (1) effectively. Let $z_l = x_l - \alpha_{l-1}$ where $\alpha_{l-1} = \alpha_{(l-1)s} - \eta_{l-1} - \gamma_{l-1}$. If we let $\bar{\Delta}_l = \Delta_l - \sum_{j=1}^{l-1}\left(\frac{\partial \alpha_{l-1}}{\partial x_j}\Delta_j\right)$, then we have $|\bar{\Delta}_l| \le \bar{p}_l\bar{\delta}_l(x_1,\ldots,x_l)$.

From the above definition and (17), we can obtain the following:

$$\begin{aligned}\dot{z}_l &= z_{l+1} + \alpha_l + \phi_l^T\theta + \psi_l^T\varsigma + \Delta t - \dot{\alpha}_{l-1} \\ &= z_{l+1} + \alpha_l + \phi_l^T\theta + \psi_l^T(\hat{\chi} - \varpi^T\theta + w - e) \\ &\quad + \Delta_l - \dot{\alpha}_{l-1} \\ &= z_{l+1} + \bar{\phi}_l^T\theta - \sum_{j=1}^{l-1}\left[\frac{\partial \alpha_{l-1}}{\partial x_j}x_{j+1} + \frac{\partial \alpha_{l-1}}{\partial \hat{\bar{p}}_j}\dot{\hat{\bar{p}}}_j\right] \\ &\quad + \bar{\Delta}_l - \frac{\partial \alpha_{l-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} + \psi_l^T(\hat{\chi} + w - e) + \alpha_l\end{aligned} \tag{46}$$

where $\bar{\phi}_l = \varpi\psi_l + \phi_l - \sum_{j=1}^{l-1}\left(\frac{\partial \alpha_{l-1}}{\partial x_j}\phi_j\right)$.

If we choose the following:

$$\begin{aligned}\alpha_l(x_1,\ldots,x_l,\hat{\theta},\hat{\bar{p}}_1,\ldots,\hat{\bar{p}}_l,\hat{\chi}) =& -z_{l-1} - k_l z_l \\ &+\left[\sum_{j=1}^{l-2} z_{j+1}\frac{\partial \alpha_j}{\partial \hat{\theta}}\Gamma - \hat{\theta}^T\right]\bar{\phi}_l - \eta_l - \gamma_l \\ &+\frac{\partial \alpha_{l-1}}{\partial \hat{\theta}}\tau_l + \sum_{j=1}^{l-1}\left(\frac{\partial \alpha_{l-1}}{\partial x_j}x_{j+1} + \frac{\partial \alpha_{l-1}}{\partial \hat{\bar{p}}_j}\dot{\hat{\bar{p}}}_j\right) \\ &-\psi_l^T\hat{\chi} - \psi_l^T w,\end{aligned} \tag{47}$$

$$\tau_l = \tau_{l-1} + \Gamma z_l \bar{\phi}, \tag{48}$$

$$\dot{\hat{\bar{P}}}_l = r_l[l_l - \sigma_l(\hat{\bar{p}}_l - \bar{p}_l^0)], \tag{49}$$

where $\tilde{\bar{p}}_l = \hat{\bar{p}}_l - \bar{p}_l$, k_l, r_l, σ_l, $\bar{p}_l^0$ are positive design constants, and η_l and γ_l are smooth design functions, then it can be shown that

$$\begin{aligned}\dot{z}_l =& z_{l+1} - k_l z_l - z_{l-1} + \bar{\Delta}_l - \frac{\partial \alpha_{l-1}}{\partial \hat{\theta}}(\dot{\hat{\theta}} - \tau_l) - \eta_l - \gamma_l \\ &+\left[\sum_{j=1}^{l-2} z_{j+1}\frac{\partial \alpha_j}{\partial \hat{\theta}}\Gamma - \tilde{\theta}^T\right]\bar{\phi}_l - \psi_l^T e.\end{aligned} \tag{50}$$

Consider the following Lyapunov function:

$$V_l = V_{l-1} + \frac{1}{2}z_l^2 + \frac{1}{2r_l}\tilde{\bar{p}}_l^2. \tag{51}$$

Using similar arguments developed in the previous subsections, it can be shown that

$$\begin{aligned}\dot{V}_l &= \dot{V}_{l-1} + z_l\dot{z}_l + \frac{1}{r_l}\tilde{\bar{p}}_l\dot{\hat{\bar{p}}}_l \\ &\le -\sum_{j=1}^{l} k_j z_j^2 + z_l z_{l+1} + \frac{1}{2}\sigma_\theta |\theta - \theta^0|^2 \\ &+ \frac{1}{2}\sum_{j=1}^{l}\sigma_j(\bar{p}_j - \bar{p}_j^0)^2 - \frac{1}{2}\sigma_\theta|\tilde{\theta}|^2 \\ &- \frac{1}{2}\sum_{j=2}^{l}\sigma_j\tilde{\bar{z}}_j^2 + z_l(\bar{\Delta}_l - \eta_l - \gamma_l - \psi_l^T e) \\ &+ \tilde{\bar{p}}_l l_l + \sum_{j=1}^{l-1}\frac{\bar{p}_j}{\varepsilon_j} + \left[\sum_{j=1}^{l-1} z_{j+1}\frac{\partial \alpha_j}{\partial \hat{\theta}} - \tilde{\theta}^T\Gamma^{-1}\right](\tau_l - \dot{\hat{\theta}}).\end{aligned} \tag{52}$$

Noting that η_l, γ_l, and l_l are chosen as any smooth functions satisfying the following two conditions

$$\begin{aligned}&(i) \quad z_l(\bar{\Delta}_l - \eta_l - \gamma_l - \psi_l^T e) + \tilde{\bar{p}}_l l_l \le \frac{\bar{p}_l}{\varepsilon_l} + \varepsilon_{l2}, \\ &(ii) \quad -z_l(\eta_l + \gamma_l) \le 0,\end{aligned} \tag{53}$$

where ε_l is a positive design constant and refers to Remark 7 for ε_{l2}, we can conclude that

$$\begin{aligned}\dot{V}_l &\le -c_l V_l + z_l z_{l+1} + \lambda_l \\ &+ \left[\sum_{j=1}^{l-1} z_{j+1}\frac{\partial \alpha_j}{\partial \hat{\theta}} - \tilde{\theta}^T\Gamma^{-1}\right](\dot{\hat{\theta}} - \tau_l),\end{aligned} \tag{54}$$

where

$$\begin{aligned}\lambda_l &= \sum_{j=1}^{l}\frac{\bar{p}_j}{\varepsilon_j} + \varepsilon_{l2} + \frac{1}{2}\sigma_\theta|\theta - \theta^0|^2 \\ &+ \frac{1}{2}\sum_{j=1}^{l}\sigma_j(\bar{p}_j - \bar{p}_j^0)^2 \\ c_l &= \min\left\{2k_l, \sigma_l r_l, \frac{\sigma_\theta}{\lambda_{\max}(\Gamma_l^{-1})}\right\} > 0.\end{aligned} \tag{55}$$

Remark 7: For example, we can select η_l, γ_l, and l_l as follows:

$$\eta_l = z_l\hat{\bar{p}}_l^2\bar{\delta}_l^2 + \frac{\epsilon_l^2}{4}z_l\bar{\delta}_l^2, \tag{56}$$

$$\gamma_l = \frac{z_l|\psi_l|^2}{4}, \tag{57}$$

$$l_l = \varepsilon_l z_l^2\bar{\delta}_l^2. \tag{58}$$

In this case,

$$\begin{aligned}&z_l(\bar{\Delta}_l - \eta_l - \gamma_l - \psi_l^T e) + \tilde{\bar{p}}_l l_l \le |z_l\bar{\delta}_l|\bar{p}_l - z_l\eta_l \\ &\quad - z_l\gamma_l + \tilde{\bar{p}}_l l_l - z_l\psi_l^T e \\ &\le -\bar{\delta}^2(z_l\tilde{\bar{p}}_l - \frac{\varepsilon_l z_l}{2})^2 - |\frac{z_l\psi_l}{2} + e|^2 + \frac{\bar{p}_l}{4\varepsilon_l} + |e|^2 \\ &\le \frac{\bar{p}_l}{4\varepsilon_l} + |e|^2 \\ &\le \frac{\bar{p}_l}{4\varepsilon_l} + \varepsilon_{l2}.\end{aligned} \tag{59}$$

From Theorem 1, there exists a small positive constant, ε_{l2}. Thus, condition (i) of (53) is satisfied. In addition, since

$$-z_l(\eta_l + \gamma_l) = -z_l^2\hat{\bar{p}}_l^2\bar{\delta}_l^2 - \frac{\varepsilon_l^2}{4}z_l^2\delta_l^2 - \frac{z_l^2|\psi|^2}{4} \le 0, \tag{60}$$

condition (ii) of (53) holds.

Step 4: $l+1 \le i \le n-1$

Let $z_i = x_i - \alpha_{i-1}$ where $\alpha_{i-1} = \alpha_{i-1,s} - \eta_{i-1} - \gamma_{i-1}$. If we let $\bar{\Delta}_i = \Delta_i - \sum_{j=1}^{i-1}\left(\frac{\partial \alpha_{i-1}}{\partial x_j}\Delta_j\right)$, then we have $|\bar{\Delta}_i| \le \bar{p}_i\bar{\delta}_i(x_1,\ldots,x_i)$. In this subsection, let $\bar{\psi}_i = \psi_i - \sum_{j=1}^{i-1}\left(\frac{\partial \alpha_{i-1}}{\partial x_j}\psi_j\right)$.

Lemma 2: At step i, $\forall l+1 \le i \le n-1$, we choose the desired control function α_i and the tuning function as

$$\begin{aligned}&\alpha_i(x_1,\ldots,x_i,\hat{\theta},\hat{\bar{p}}_1,\ldots,\hat{\bar{p}}_i,\hat{\chi}) = -z_{i-1} - k_i z_i \\ &+ \left[\sum_{j=1}^{i-2} z_{j+1}\frac{\partial \alpha_j}{\partial \hat{\theta}}\Gamma - \hat{\theta}^T\right]\bar{\phi}_i - \eta_i - \gamma_i \\ &+ \sum_{j=1}^{i-1}\left(\frac{\partial \alpha_{i-1}}{\partial x_j}x_{j+1} + \frac{\partial \alpha_{i-1}}{\partial \hat{\bar{p}}_j}\dot{\hat{\bar{p}}}_j\right) + \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}\tau_i \\ &- \bar{\psi}_i^T\hat{\chi} - \bar{\psi}_i^T\omega + \sum_{j=1}^{m}\frac{\partial \alpha_{i-1}}{\partial \hat{\chi}_j}\dot{\hat{\chi}}_j\end{aligned} \tag{61}$$

$$\tau_i = \tau_{i-1} + \Gamma z_i\bar{\phi}_i, \tag{62}$$

where $\bar{\phi}_i = \tilde{\omega}\psi_i + \phi_i - \sum_{j=1}^{i-1}\left(\frac{\partial \alpha_{i-1}}{\partial x_j}\phi_j\right)$, k_i is a positive design constant, and η_i, γ_i are smooth design functions for design. If we choose the following adaptation law

$$\dot{\hat{\bar{p}}}_i = r_i[l_i - \sigma_i(\hat{\bar{p}}_i - \bar{p}_i^0)] \tag{63}$$

where $\tilde{\bar{p}}_i = \hat{\bar{p}}_i - \bar{p}_i$, $r_i, \sigma_i, \bar{p}_i^0$ are positive design constants, and l_i function is a smooth design function, then it can be shown that the time derivative of the Lyapunov function candidate

$$V_i = V_{i-1} + \frac{1}{2}z_i^2 + \frac{1}{2r_i}\tilde{\bar{p}}_i^2, \tag{64}$$

satisfies

$$\begin{aligned}\dot{V}_i &\le -c_i V_i + z_i z_{i+1} + \lambda_i \\ &+ \left[\sum_{j=1}^{i-1} z_{j+1}\frac{\partial \alpha_j}{\partial \hat{\theta}} - \tilde{\theta}^T\Gamma^{-1}\right](\tau_i - \dot{\hat{\theta}}),\end{aligned} \tag{65}$$

where

$$\lambda_i = \sum_{j=1}^{i} \frac{\overline{p}_j}{\epsilon_j} + \sum_{j=l}^{i} \epsilon_{j2} + \frac{1}{2}\sigma_\theta |\theta - \theta^0|^2 + \frac{1}{2}\sum_{j=1}^{i}\sigma_j(\overline{p}_j - \overline{p}_j^0)^2, \tag{66}$$

$$c_i = \min\left\{2k_i, \sigma_i r_i, \frac{\sigma_\theta}{\lambda_{\max}(\Gamma^{-1})}\right\} > 0,$$

and the i-th error subsystem is

$$\dot{z}_i = z_{i+1} - k_i z_i - z_{i-1} + \overline{\Delta}_i - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}(\dot{\hat{\theta}} - \tau_i) - \eta_i - \gamma_i + \left[\sum_{j=2}^{i-2} z_{j+1}\frac{\partial \alpha_j}{\partial \hat{\theta}}\Gamma - \tilde{\theta}^T\right]\overline{\phi}_i - \overline{\psi}_i^T e. \tag{67}$$

In here, η_i, γ_i, and l_i functions are any smooth robust control terms satisfying

$$\begin{aligned} &(i) \quad z_i(\overline{\Delta}_i - \eta_i - \lambda_i - \overline{\psi}_i^T e) + \tilde{\overline{p}}_i l_i \le \frac{\overline{p}_i}{\varepsilon_i} + \varepsilon_{i2}, \\ &(ii) \quad -z_i(\eta_i + \gamma_i) \le 0, \end{aligned} \tag{68}$$

where ε_i is a positive design constant and we refer to the similar argument of Remark 7 for ε_{i2}

Proof: Refer to Appendix IV. □

Remark 8: For example, we can select η_i, γ_i, and l_i as follows:

$$\eta_i = z_i \hat{\overline{p}}_i^2 \overline{\delta}_i^2 + \frac{\varepsilon_i^2}{4} z_i \overline{\delta}_i^2, \tag{69}$$

$$\gamma_i = \frac{z_i |\overline{\psi}_i|^2}{4}, \tag{70}$$

$$l_i = \varepsilon_i z_i^2 \overline{\delta}_i^2. \tag{71}$$

By the same method as Remark 7, we can conclude that the selected η_i, γ_i, and l_i functions satisfy conditions (i) and
(ii) of (68)

Step 5: $i = n$

As the final step, by letting $u = x_{n+1}$, Lemma 2 applies to Step n, Since u is the actual input, we can choose it as

$$\begin{aligned} u &= \alpha_n(x_1, \ldots, x_n, \hat{\theta}, \hat{\overline{p}}_1, \ldots, \hat{\overline{p}}_n, \hat{\chi}) \\ &= -z_{n-1} - k_n z_n + \left[\sum_{j=1}^{n-2} z_{j+1}\frac{\partial \alpha_j}{\partial \hat{\theta}}\Gamma - \hat{\theta}^T\right]\overline{\phi}_n \\ &\quad -\eta_n - \gamma_n + \sum_{j=1}^{n-1}\left(\frac{\partial \alpha_{n-1}}{\partial x_j}x_{j+1} + \frac{\partial \alpha_{n-1}}{\partial \hat{\overline{p}}_j}\dot{\hat{\overline{p}}}_j\right) \\ &\quad + \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}}\tau_n - \overline{\psi}_n^T \hat{\chi} - \overline{\psi}_n^T \omega + \sum_{j=1}^{m}\frac{\partial \alpha_{n-1}}{\partial \hat{\chi}_j}\dot{\hat{\chi}}_j, \end{aligned} \tag{72}$$

where $\overline{\phi}_n = \tilde{\omega}\psi_n + \phi_n - \sum_{j=1}^{n-1}\left(\frac{\partial \alpha_{n-1}}{\partial x_j}\phi_j\right)$, k_n is a positive design function, and η_n and γ_n are smooth design functions.

If we choose the following as in Lemma 2:

$$\tau_n = \tau_{n-1} + \Gamma z_n \overline{\phi}, \tag{73}$$

$$\dot{\hat{\overline{p}}}_n = r_n[l_n - \sigma_n(\hat{\overline{p}}_n - \overline{p}_n^0)], \tag{74}$$

where $\tilde{\overline{p}}_n = \hat{\overline{p}}_n - \overline{p}_n$, r_n, σ_n, $\overline{p}_n^0$ are positive design constants, and the l_n function is a smooth design function, then it can be shown that

$$\dot{V}_n \le -c_n V_n + \lambda_n + \left[\sum_{j=1}^{n-1} z_{j+1}\frac{\partial \alpha_j}{\partial \hat{\theta}} - \tilde{\theta}^T\Gamma^{-1}\right](\tau_n - \dot{\hat{\theta}}), \tag{75}$$

where

$$\lambda_n = \sum_{j=1}^{n}\frac{\overline{p}_j}{\varepsilon_j} + \sum_{j=l}^{n}\varepsilon_{j2} + \frac{1}{2}\sigma_\theta|\theta - \theta^0|^2 + \frac{1}{2}\sum_{j=1}^{n}\sigma_j(\overline{p}_j - \overline{p}_j^0)^2, \tag{76}$$

$$c_n = \min\left\{2k_i, \sigma_i r_i, \frac{\sigma_\theta}{\lambda_{\max}(\Gamma_i^{-1})} : i = 1, \ldots, n\right\} > 0.$$

If we choose the parameter adaptation law as

$$\dot{\hat{\theta}} = \tau_n = \Gamma\left[\sum_{j=1}^{n} z_j \overline{\phi}_i - \sigma_\theta(\hat{\theta} - \theta^0)\right], \tag{77}$$

we can obtain

$$\dot{V}_n \le -c_n V_n + \lambda_n. \tag{78}$$

Note that as η_n, γ_n, and l_n functions are any robust control terms satisfying

$$\begin{aligned} &(i) \quad z_n(\overline{\Delta}_n - \eta_n - \gamma_n - \overline{\psi}_n^T e) + \tilde{\overline{p}}_n l_n \le \frac{\overline{p}_n}{\varepsilon_n} + \varepsilon_{n2}, \\ &(ii) - z_n(\eta_n + \gamma_n) \le 0, \end{aligned} \tag{79}$$

where ε_n is a positive design constant and refers to the similar argument of Remark 7 for ε_{n2}, we are ready to state and prove the following main theorem.

Theorem 2: Consider system (1), Under Assumptions 1 and 2, if we apply control input (72), parameter adaptation law (77), and the design procedure developed in the previous sections, then

(1) all signals are global uniform ultimate bounded,

$$(2) \quad |y(t)| \le \mu^* \exp\left(-\frac{c_n t}{2}\right) + k^* \tag{80}$$

where

$$\sigma_2 = \frac{\lambda_n}{c_n},\ k^* = \sqrt{\frac{2\lambda_n}{c_n}},\ \mu^* = \sqrt{2[V_n(0) - \sigma_2]} \quad (81)$$

and $V_n(0)$ is the value of the n th step Lyapunov function at initial time.

Proof: Refer to Appendix V. □

Remark 9: As in Remark 8, by substituting $i = n$ into (69)-(71), we can choose η_n, γ_n, and l_n as

$$\eta_n = z_n \hat{\bar{p}}_n^2 \bar{\delta}_n^2 + \frac{\varepsilon_n^2}{4} z_n \bar{\delta}_n^2, \quad (82)$$

$$\gamma_n = \frac{z_n |\bar{\psi}_n|^2}{4}, \quad (83)$$

$$l_n = \varepsilon_n z_n^2 \bar{\delta}_n^2. \quad (84)$$

By simple calculation, we can determine that the above functions satisfy conditions (i) and (ii) of (79). Under additional conditions, the exact regulation of the output to the origin can be achieved.

Corollary 2: (Exact Output Regulation)
In addition to Assumptions 1 and 2, if $\Delta_i = 0$ and $v_i \in L_2$ $(1 \le i \le n)$ are satisfied, then we can find an adaptive controller of form (72) with $\sigma_\theta = \sigma_j = 0$ $(1 \le j \le n)$ such that the output $y(t)$ satisfies

$$\lim_{t\to\infty} y(t) = 0. \quad (85)$$

Proof: Refer to Appendix VI. □

Remark 10: Theorem2 guarantees the boundedness of z_i $(i = 1,\dots,n)$. Thus, the boundedness of x_i $(i = 1, \dots, n)$. is also guaranteed. From Assumption 1 and (6), it is noted that the function v_i is also bounded.

Remark 11: As seen in (76), (80), and (81), since ε_i, σ_i, σ_θ, k_i, r_i, $\bar{p}_i^0$, θ^0, and Γ are constants for design, we can make y as small as desired and thus achieve regulation to a small region around the origin by an appropriate choice of design parameters.

Remark 12: Corollary 2 guarantees that, under additional conditions, it is possible to achieve the exact output regulation to the origin without tuning the design constants appropriately.

5. EXAMPLE

In this section, we illustrate the proposed design scheme of the robust adaptive controller for uncertain nonlinear systems. In order to demonstrate our scheme, the simulation result is obtained for the following system:

$$\dot{\varsigma} = -\varsigma + x_1^2,$$

$$\begin{aligned} \dot{x}_1 &= x_2 + \theta + \Delta_1, \\ \dot{x}_2 &= x_3 + \theta x_1^2 + 2\varsigma + \Delta_2, \\ \dot{x}_3 &= u, \\ y &= x_1, \end{aligned} \quad (86)$$

where θ is an unknown constant parameter. For the purpose of simulation, we let $\theta = 2$. In addition, ς is unmeasured. Let $\Delta_1 = 0.1$, $\Delta_2 = 0.6\sin(2t)$. Then, the bounding functions are $\bar{\delta}_1 = \delta_1 = 1$, $\delta_2 = 1 + \frac{\partial \alpha_1}{\partial x_1}$, $\bar{\delta}_3 = \frac{\partial \alpha_2}{\partial x_2} + \frac{\partial \alpha_2}{\partial x_1}$. From (19) and (115), the ω and g functions are

$$\begin{aligned} w(x_1, x_2) &= \int_0^{x_2} \frac{b_1 - a_1}{2} dx_2 \\ &= \frac{b_1 - a_1}{2} x_2, \end{aligned} \quad (87)$$

$$\begin{aligned} g(x_1, x_2) &= \sum_{j=1}^{2} \varepsilon_j \left(\frac{\partial \omega}{\partial x_j}\right)^2 \delta_j^2 \\ &= \varepsilon_2 \left(\frac{b_1 - a_1}{2}\right)^2, \end{aligned} \quad (88)$$

where a_1, b_1, ε_2 are positive design constants, and $a_1 \ne b_1$. A $\tilde{\omega}$ function is the output of the following stable filter:

$$\dot{\tilde{\omega}} = -\left\{ b_1 + c_1 + \varepsilon_2 \left(\frac{b_1 - a_1}{2}\right)^2 \right\} \tilde{\omega} + \left(\frac{b_1 - a_1}{2}\right) x_1^2, \quad (89)$$

where c_1 is a positive constant for design. By introducing the following estimator,

$$\begin{aligned} \dot{\hat{\chi}} = &-\left\{ b_1 + c_1 + \varepsilon_2 \left(\frac{b_1 - a_1}{2}\right)^2 \right\} \left(\hat{\chi} + \frac{b_1 - a_1}{2} x_2 \right) \\ &- \left(\frac{b_1 - a_1}{2}\right) x_3, \end{aligned} \quad (90)$$

we can obtain the equivalent expression of ς

$$\varsigma = \hat{\chi} - \tilde{\omega}\theta + \frac{b_1 - a_1}{2} x_2 - e. \quad (91)$$

This representation of ς will be used in the controller design. We are ready to design the robust adaptive controller. For the $i = 1$ case, the α_1 function, the tuning function, and the adaptation law of the bounding constant are

$$\alpha_1 = -k_1 x_1 - \hat{\theta} - \eta_1, \quad (92)$$

$$\tau_1 = \Gamma[x_1 - \sigma_\theta(\hat{\theta} - \theta^0)], \quad (93)$$

$$\dot{\hat{\bar{p}}}_1 = r_1[l_1 - \sigma_1(\hat{\bar{p}}_1 - \bar{p}_1^0)], \quad (94)$$

And, for the i = 2 case, we have the α_2 function, the tuning function, and the adaptation law of the bounding constant as

$$\alpha_2 = -x_1 - k_2(x_2-\alpha_1) - \eta_2 - \gamma_2 + \frac{\partial \alpha_1}{\partial \hat{\theta}}\tau_2 + \frac{\partial \alpha_1}{\partial x_1}x_2 + \frac{\partial \alpha_1}{\partial \hat{p}_1}\dot{\hat{p}}_1 - 2\hat{\chi} - (b_1-a_1)x_2, \quad (95)$$

$$\tau_2 = \Gamma\left[x_1 + (x_2-\alpha_1)\left(2\hat{\varpi} + x_1^2 - \frac{\partial \alpha_1}{\partial x_1}\right) - \sigma_\theta(\hat{\theta}-\theta^0)\right], \quad (96)$$

$$\dot{\hat{p}}_2 = r_2[l_2 - \sigma_2(\hat{p}_2 - \bar{p}_2^0)]. \quad (97)$$

For i = 3 case, the parameter adaptation law and the adaptation law of the bounding function are

$$\dot{\hat{\theta}} = \Gamma\left[x_1 + (x_2-\alpha_1)\left(2\hat{\varpi} + x_1^2 - \frac{\partial \alpha_1}{\partial x_1}\right) - (x_3-\alpha_2)\left(\frac{\partial \alpha_2}{\partial x_2}x_1^2 + \frac{\partial \alpha_2}{\partial x_1}\right) - \sigma_\theta(\hat{\theta}-\theta^0)\right], \quad (98)$$

$$\dot{\hat{p}}_3 = r_3[l_3 - \sigma_3(\hat{p}_3 - \bar{p}_3^0)]. \quad (99)$$

In this case, the η, γ, and l functions are selected for the case i = 1,

$$\eta_1 = x_1\hat{p}_1^2 + \frac{\varepsilon_1^2}{4}x_1, \quad (100)$$

$$\gamma_1 = 0, \quad (101)$$

$$l_1 = \varepsilon_1 x_1^2. \quad (102)$$

For the case i = 2,

$$\eta_2 = (x_2-\alpha_1)\left(1 + \left|\frac{\partial \alpha_1}{\partial x_1}\right|\right)^2\left[\hat{p}_2^2 + \frac{\varepsilon_2^2}{4}\right] \quad (103)$$

$$\gamma_2 = x_2 - \alpha_1, \quad (104)$$

$$l_2 = \varepsilon_2(x_2-\alpha_1)^2\left(1 + \left|\frac{\partial \alpha_1}{\partial x_1}\right|\right)^2. \quad (105)$$

For the case i = 3,

$$\eta_3 = (x_3-\alpha_2)\left(\left|\frac{\partial \alpha_2}{\partial x_2}\right| + \left|\frac{\partial \alpha_2}{\partial x_1}\right|\right)^2\left[\hat{p}_3^2 + \frac{\varepsilon_3^2}{4}\right], \quad (106)$$

$$\gamma_3 = (x_3-\alpha_2)\left(\frac{\partial \alpha_2}{\partial x_3}\right)^2, \quad (107)$$

$$l_3 = \varepsilon_3(x_3-\alpha_2)^2\left(\left|\frac{\partial \alpha_2}{\partial x_2}\right| + \left|\frac{\partial \alpha_2}{\partial x_1}\right|\right)^2. \quad (108)$$

Finally, we can design the control input as follows:

$$u = -(x_2-\alpha_1) - k_3(x_3-\alpha_2) + \left[(x_2-\alpha_1)\frac{\partial \alpha_1}{\partial \hat{\theta}}\Gamma - \hat{\theta}\right] \times\left(-\frac{\partial \alpha_2}{\partial x_2}x_1^2 - \frac{\partial \alpha_2}{\partial x_1}\right) - \eta_3 - \gamma_3 + \left(\frac{\partial \alpha_2}{\partial x_1}x_2 + \frac{\partial \alpha_2}{\partial \hat{p}_1}\dot{\hat{p}}_1\right) + \left(\frac{\partial \alpha_2}{\partial x_2}x_3 + \frac{\partial \alpha_2}{\partial \hat{p}_2}\dot{\hat{P}}_2\right) + \frac{\partial \alpha_2}{\partial \hat{\theta}}\tau_2 - 2\hat{\chi} - (b_1-a_1)x_2 + \frac{\partial \alpha_2}{\partial \hat{\chi}}\dot{\hat{\chi}} \quad (109)$$

For the simulation result, the design parameters and initial conditions for our design scheme are given as follows:

$$\begin{aligned} &\varsigma(0) = x_1(0) = x_2(0) = x_3(0) = 1, \\ &\hat{\varpi}(0) = \hat{\chi}(0) = 0, \hat{\theta} = (0) = 1, \\ &\hat{p}_1(0) = \hat{p}_2(0) = \hat{p}_3(0) = 1, \\ &a = 6, b = 10, c = 2, \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 2, \\ &r_1 = r_2 = r_3 = 1, k_1 = k_2 = k_3 = 1, \\ &\sigma_1 = \sigma_2 = \sigma_3 = \sigma_\theta = 1, \\ &\Gamma = 2, \theta^0 = 0, \bar{p}_1^0 = \bar{p}_2^0 = \bar{p}_3^0 = 1. \end{aligned} \quad (110)$$

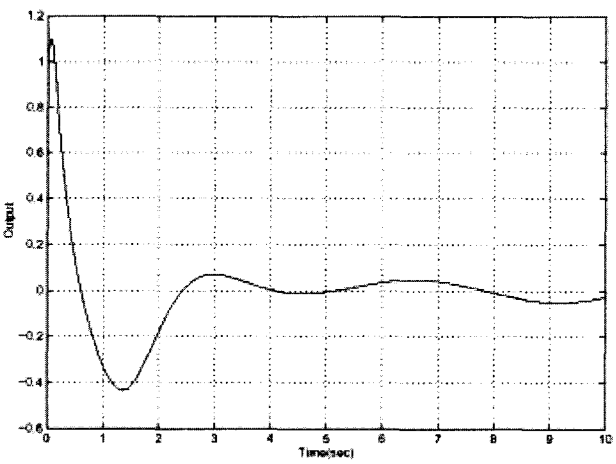

Fig. 1. Robust output regulation result.

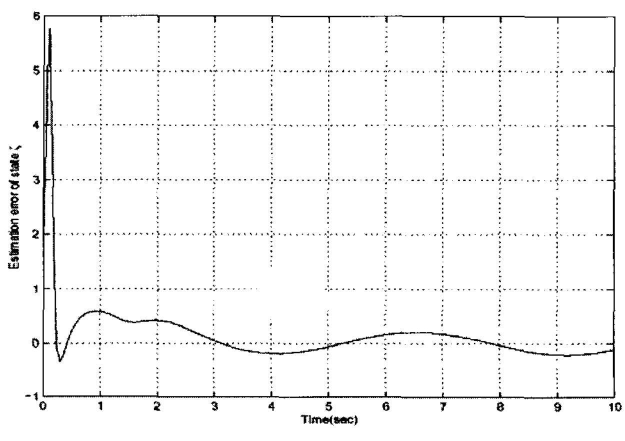

Fig. 2. The estimation error of the unmeasured state ς.

The plot in Fig. 1 shows that our control design scheme can achieve a robust regulation result of output y. As can be seen in Fig. 2, the proposed novel estimation technique can drive the estimation error of the unmeasured state ς to an arbitrary small region around the origin by an appropriate choice of the design parameters.

6. CONCLUSION

In this paper, a novel estimation technique-based robust adaptive control scheme is presented for a class of uncertain nonlinear systems with a general set of uncertainty. In contrast to other existing results, the proposed schemes deal with a more general form, but require less assumptions. With the novel estimation technique and the modified adaptive backstepping for dynamic uncertainties, the overall scheme achieves robust regulation of the output by appropriate choice of design constants. In addition, with these tools, we can remove or relax the assumptions of existing research.

APPENDIX I
PROOF OF THEOREM 1

We consider the following Lyapunov function candidate:

$$W_i = \frac{1}{2} e_i^2 \qquad (i = 1, \ldots, m). \tag{111}$$

The time derivative of W_i is computed as

$$\begin{aligned} \dot{W}_i &= e_i \dot{e}_i \\ &= -b_i e_i^2 - c_i e_i^2 - g_i e_i^2 - e_i v_i + \sum_{j=1}^{l} \frac{\partial \omega_i}{\partial x_j} \Delta_j e_i. \end{aligned} \tag{112}$$

Using $|zp| \le k_{ij} z^2 + \frac{p^2}{4k_{ij}}. \forall k_{ij} > 0$, then

$$\begin{aligned} \sum_{j=1}^{l} \frac{\partial \omega_i}{\partial x_j} \Delta_j e_i &\le \left| \sum_{j=1}^{l} \frac{\partial \omega_i}{\partial x_j} \Delta_j e_i \right| \\ &\le \sum_{j=1}^{l} \left| \frac{\partial \omega_i}{\partial x_j} e_i \right| p_j \delta_j \\ &= \sum_{j=1}^{l} \left| \frac{\partial \omega_i}{\partial x_j} \delta_j e_i p_j \right| \\ &\le \sum_{j=1}^{l} \left[k_{ij} \left(\frac{\partial \omega_i}{\partial x_j} \right)^2 \delta_j^2 e_i^2 + \frac{p_j^2}{4k_{ij}} \right]. \end{aligned} \tag{113}$$

By (113), it can be shown that

$$\begin{aligned} \dot{W}_i &\le -b_i e_i^2 - c_i e_i^2 - g_i e_i^2 - e_i v_i \\ &\quad + \sum_{j=1}^{l} \left[k_{ij} \left(\frac{\partial \omega_i}{\partial x_j} \right)^2 \delta_j^2 e_i^2 + \frac{p_j^2}{4k_{ij}} \right], \end{aligned} \tag{114}$$

where k_{ij} $(i = 1, \ldots, m,$ and $j = 1, \ldots, l)$ is a positive constant for design. If the following function is chosen as

$$g_i = \sum_{j=1}^{l} \left[k_{ij} \left(\frac{\partial \omega_i}{\partial x_j} \right)^2 \delta_j^2 \right] \ge 0, \tag{115}$$

it is shown that

$$\dot{W}_i \le -b_i e_i^2 - c_i e_i^2 - e_i v_i + \sum_{j=1}^{l} \frac{p_j^2}{4k_{ij}}. \tag{116}$$

By completing the square,

$$\dot{W}_i \le -b_i e_i^2 + \frac{v_i^2}{4c_i} + \sum_{j=1}^{l} \frac{p_j^2}{4k_{ij}}. \tag{117}$$

Noting that the modeling error v_i is bounded (refer to Remark10.), from (14), we have

$$\dot{W}_i \le -2b_i W_i + M_i. \tag{118}$$

If we let

$$\rho_i = \frac{M_i}{2b_i}, \tag{119}$$

then

$$0 \le W_i \le \rho_i + [W_i(0) - \rho_i] e^{-2b_i t}. \tag{120}$$

Since W_i is radially unbounded with regard to e_i, property (1) directly follows from (120). By the definition of W_i in (111), (120) satisfies

$$|e_i(t)| \le \sqrt{\frac{M_i}{b_i} + 2[W_i(0) - \rho_i] e^{-2b_i t}}, \quad (i = 1, \ldots, m). \tag{121}$$

If we choose μ_i and Q_i as (14), then we can obtain property (2) of Theorem 1.

APPENDIX II
PROOF OF COROLLARY 1

If $\Delta_i = 0$ for $1 \le i \le l$, (117) becomes

$$\dot{W}_i \le -b_i e_i^2 + \frac{v_i^2}{4c_i}. \tag{122}$$

Integration of both sides yields

$$W_i(t) - W_i(0) \le -b_i \int_0^t e_i^2(\tau) d\tau + \frac{1}{4c_i} \int_0^t v_i^2(\tau) d\tau. \tag{123}$$

Therefore,

$$\begin{aligned} \int_0^t e_i(\tau)^2 d\tau &\le \frac{1}{b_i} [W_i(0) - W_i(t)] \\ &\quad + \frac{1}{4b_i c_i} \int_0^t v_i^2(\tau) d\tau. \end{aligned} \tag{124}$$

If the following constants are chosen as

$$\Pi = \frac{1}{b_i}\sup_{\forall t\geq 0}[W_i(0) - W_i(t)] \qquad \frac{1}{4b_i c_i} = \Xi, \tag{125}$$

then we have property (1) of Corollary 1. Since e_i is bounded, Π is finite and $e_i \in L_\infty$ holds. From property (1), if $v_i(t)$ is square-integrable, then $e_i \in L_2$. Therefore, we can obtain $e_i \in L_2 \cap L_\infty$. By the boundedness of v_i (refer to Remark 10.), the righthand side of (11) is bounded. Therefore, $\dot{e}_i \in L_\infty$. According to Barbalat's lemma [1],[2], we can get property (2).

APPENDIX III PROOF OF LEMMA 1

From the induction process, let us assume that step i is valid, and then show that it is also true for step $i+1$. Choose $\alpha_{i+1}, \tau_{i+1}, \hat{\bar{p}}_{i+1}$ as forms of step i. By easy calculation, it is possible to demonstrate that

$$\begin{aligned}\dot{z}_{i+1} &= z_{i+2} - k_{i+1}z_{i+1} - z_i + \bar{\Delta}_{i+1} - \frac{\partial \alpha_i}{\partial \hat{\theta}}(\dot{\hat{\theta}} - \tau_{i+1}) \\ &- \eta_{i+1} + \left[\sum_{j=1}^{i-1} z_{j+1}\frac{\partial \alpha_j}{\partial \hat{\theta}}\Gamma - \tilde{\theta}^T\right]\bar{\phi}_{i+1}.\end{aligned} \tag{126}$$

Therefore, it can be shown that

$$\begin{aligned}\dot{V}_{i+1} &= \dot{V}_i + z_{i+1}\dot{z}_{i+1} + \frac{1}{r_{i+1}}\tilde{\bar{p}}_{i+1}\dot{\tilde{\bar{p}}}_{i+1} \\ &\leq -\sum_{j=1}^{i+1} k_j z_j^2 + z_{i+1}z_{i+2} + \frac{1}{2}\sigma_\theta\left|\theta - \theta^0\right|^2 \\ &+ \frac{1}{2}\sum_{j=1}^{i}\sigma_j(\bar{p}_j - \bar{p}_j^0)^2 - \frac{1}{2}\sigma_\theta\left|\tilde{\theta}\right|^2 \\ &+ z_{i+1}(\bar{\Delta}_{i+1} - \eta_{i+1}) - \frac{1}{2}\sum_{j=1}^{i}\sigma_j\tilde{z}_j^2 \\ &+ \tilde{\bar{p}}_{i+1}[l_{i+1} - \sigma_{i+1}(\hat{\bar{p}}_{i+1} - \bar{p}_{i+1}^0)] \\ &+ \left[\sum_{j=1}^{i-1} z_{i+1}\frac{\partial \alpha_j}{\partial \hat{\theta}} - \tilde{\theta}^T\Gamma^{-1}\right]\left[\tau_{i+1} - \Gamma z_{i+1}\bar{\phi}_{i+1} - \dot{\hat{\theta}}\right] \\ &+ z_{i+1}\left(\sum_{j=1}^{i-1} z_{j+1}\frac{\partial \alpha_j}{\partial \hat{\theta}}\Gamma - \tilde{\theta}\right)^T\bar{\phi}_{i+1} \\ &= -\sum_{j=1}^{i+1} k_j z_j^2 + z_{i+1}z_{i+2} + \frac{1}{2}\sigma_\theta\left|\theta - \theta^0\right|^2 \\ &+ \frac{1}{2}\sum_{j=1}^{i+1}\sigma_j(\bar{p}_j - \bar{p}_j^0)^2 - \frac{1}{2}\sigma_\theta\left|\tilde{\theta}\right|^2 - \frac{1}{2}\sum_{j=1}^{i+1}\sigma_j\tilde{z}_j^2 \\ &+ z_{i+1}(\bar{\Delta}_{i+1} - \eta_{i+1}) + \tilde{\bar{p}}_{i+1}l_{i+1} + \sum_{j=1}^{i}\frac{\bar{p}_j}{\varepsilon_j} \\ &+ \left[\sum_{j=1}^{i} z_{j+1}\frac{\partial \alpha_j}{\partial \hat{\theta}} - \tilde{\theta}^T\Gamma^{-1}\right](\tau_{j+1} - \dot{\hat{\theta}}).\end{aligned} \tag{127}$$

At step $i+1$, if η_{i+1}, and l_{i+1} are the same forms as (43), then we can obtain

$$\begin{aligned}\dot{V}_{i+1} &\leq -c_{i+1}V_{i+1} + z_{i+1}z_{i+2} + \lambda_{i+1} \\ &+ \left[\sum_{j=1}^{i} z_{j+1}\frac{\partial \alpha_j}{\partial \hat{\theta}}\tilde{\theta}^T\Gamma^{-1}\right](\dot{\hat{\theta}} - \tau_{i+1}),\end{aligned} \tag{128}$$

where

$$\begin{aligned}\lambda_{i+1} &= \sum_{j=1}^{i+1}\frac{\bar{p}_j}{\varepsilon_j} + \frac{1}{2}\sigma_\theta\left|\theta - \theta^0\right|^2 + \frac{1}{2}\sum_{j=1}^{i+1}\sigma_j(\bar{P}_j - \bar{P}_j^0)^2, \\ c_{i+1} &= \min\left\{2k_{i+1}, \sigma_{i+1}r_{i+1}, \frac{\sigma_\theta}{\lambda_{\max}(\Gamma_{i+1}^{-1})}\right\} > 0.\end{aligned} \tag{129}$$

This completes the induction process.

APPENDIX IV PROOF OF LEMMA 2

Using the induction process as Lemma 1, we choose $\alpha_{i+1}, \tau_{i+1}, \hat{\bar{p}}_{i+1}$ as the *i th* forms. By easy calculation, it can be shown that

$$\begin{aligned}\dot{z}_{i+1} &= z_{i+2} + \theta^T\bar{\phi}_{i+1} + \bar{\Delta}_{i+1} - \frac{\partial \alpha_i}{\partial \hat{\theta}}\dot{\hat{\theta}} + \alpha_{i+1} - \sum_{j=1}^{m}\frac{\partial \alpha_i}{\partial \hat{\chi}_j}\dot{\hat{\chi}}_j \\ &- \sum_{j=1}^{i}\left[\frac{\partial \alpha_i}{\partial x_j}x_{j+1} + \frac{\partial \alpha_i}{\partial \hat{\bar{p}}_j}\dot{\hat{\bar{p}}}_j\right] + \bar{\psi}_{i+1}^T(\hat{\chi} + w - e) \\ &= z_{i+2} - k_{i+1}z_{i+1} - z_i + \bar{\Delta}_{i+1} - \frac{\partial \alpha_i}{\partial \hat{\theta}}(\dot{\hat{\theta}} - \tau_{i+1}) - \eta_{i+1} \\ &- \gamma_{i+1} - \bar{\psi}_{i+1}^T e + \left[\sum_{j=1}^{i-1} z_{j+1}\frac{\partial \alpha_j}{\partial \hat{\theta}}\Gamma - \tilde{\theta}^T\right]\bar{\phi}_{i+1}.\end{aligned} \tag{130}$$

Therefore, by the method done in the proof of Lemma 1, we can obtain the following:

$$\begin{aligned}\dot{V}_{i+1} &\leq -\sum_{j=1}^{i+1} k_j z_j^2 + z_{i+1}z_{i+2} + \frac{1}{2}\sigma_\theta\left|\theta - \theta^0\right|^2 + \sum_{j=1}^{i}\frac{\bar{p}_j}{\varepsilon_j} \\ &+ \frac{1}{2}\sum_{j=1}^{i+1}\sigma_j(\bar{p}_j - \bar{p}_j^0)^2 - \frac{1}{2}\sigma_\theta\left|\tilde{\theta}\right|^2 - \frac{1}{2}\sum_{j=1}^{i+1}\sigma_j\tilde{z}_j^2 \\ &+ z_{i+1}(\bar{\Delta}_{i+1} - \eta_{i+1} - \gamma_{i+1} - \bar{\psi}_i^T e) + \tilde{\bar{p}}_{i+1}l_{i+1} \\ &+ \left[\sum_{j=1}^{i} z_{j+1}\frac{\partial \alpha_j}{\partial \hat{\theta}} - \tilde{\theta}^T\Gamma^{-1}\right](\tau_{i+1} - \dot{\hat{\theta}}).\end{aligned} \tag{131}$$

At step $i+1$, if η_{i+1}, γ_{i+1} and l_{i+1} are the same forms as (68), then we can obtain

$$\begin{aligned}\dot{V}_{i+1} &\leq -c_{i+1}V_{i+1} + z_{i+1}z_{i+2} + \lambda_{i+1} \\ &+ \left[\sum_{j=1}^{i} z_{j+1}\frac{\partial \alpha_j}{\partial \hat{\theta}} - \tilde{\theta}^T\Gamma^{-1}\right](\tau_{i+1} - \dot{\hat{\theta}}),\end{aligned} \tag{132}$$

where

$$\lambda_{i+1} = \sum_{j=1}^{i+1} \frac{\bar{P}_j}{\varepsilon_j} + \sum_{j=l}^{i+1} \varepsilon_{j2} + \frac{1}{2}\sigma_\theta \left|\theta - \theta^0\right|^2 + \frac{1}{2}\sum_{j=1}^{i+1} \sigma_j (\bar{p}_j - \bar{p}_j^0)^2 \quad (133)$$

$$c_{i+1} = \min\left\{2k_{i+1}, \sigma_{i+1} r_{i+1}, \frac{\sigma_\theta}{\lambda_{\max}(\Gamma_{i+1}^{-1})}\right\} > 0.$$

This completes the induction process.

APPENDIX V
PROOF OF THEOREM 2

Let $\rho_2 = \frac{\lambda_n}{c_n}$. (78) satisfies

$$0 \le V_n \le \rho_2 + [V_n(0) - \rho_2] e^{-c_n t}. \quad (134)$$

Since V_n is radially unbounded with regard to $(z_i, \tilde{\theta}, \tilde{\bar{p}}_i)$, property (1) directly follows from (134).

By the definition of V_n, (134) satisfies

$$|y(t)| \le \sqrt{\frac{2\lambda_n}{c_n} + 2[V_n(0) - \rho_2] e^{-c_n t}}. \quad (135)$$

Thus, we can obtain the property (2) of Theorem 2.

APPENDIX VI
PROOF OF COROLLARY 2

From Corollary 1, if $\Delta_i = 0$ and $v_i \in L_2 (1 \le i \le n)$ are satisfied, then we have $\lim_{t\to\infty} e_i(t) = 0$. In addition, by (59), $\varepsilon_{l2} = 0$ is obtained. The application of the same method for $l \le j \le n$ yields $\varepsilon_{j2} = 0$. Since $\Delta_i = 0$, we also have $\bar{p}_i = 0\ (1 \le i \le n)$. Therefore, (76) becomes $\lambda_n = 0$. By (80) and (81), we can obtain (85).

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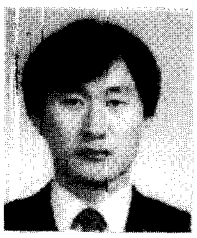

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