Parametric Approaches to Sliding Mode Design for Linear Multivariable Systems

Kyung-Soo Kim and Young-jin Park

Abstract: The parametric approaches to sliding mode design are newly proposed for the class of multivariable systems. Our approach is based on an explicit formula for representing all the sliding modes using the Lyapunov matrices of full order. By manipulating Lyapunov matrices, the sliding modes which satisfy the design criteria such as the quadratic performance optimization and robust stability to parametric uncertainty, etc., can be easily obtained. The proposed approach enables us to adopt a variety of Lyapunov- (or Riccati-) based approaches to the sliding mode design. Applications to the quadratic performance optimization problem, uncertain systems, systems with uncertain state delay, and the pole-clustering problem are discussed.

Keywords: Sliding mode control, Lyapunov inequality, linear matrix inequalities.

1. INTRODUCTION

For several decades, much effort has been made to obtain the desired performances in sliding mode. Sliding mode behavior is known to be insensitive to the matched uncertainties or disturbances through the reduced order dynamics. Note that sliding mode design has typically been accomplished by handling the reduced order system obtained by nonsingular canonical transformation (e.g., see [1,2]). In the literature, many of the standard approaches to sliding mode control have been proposed based on the reduced order system (e.g., see [3,4] or, [2] and the references therein).

Among the design methodologies in sliding mode control, the parametric approaches based on the application of the Riccati equation (or, Lyapunov-type constraints) have drawn much attention since the original effort made by Utkin and Yang [1] in which the standard linear quadratic regulator (LQR) method is applied to a certain reduced order system. Recently, there have been several types of Lyapunov approaches for dealing with parameter uncertainties (e.g., [5-7]) or the pole placement problem [8]. In particular, we point out that the explicit methods in [5] and [7], which use the Lyapunov matrix of full order, significantly simplify the system description by handling the full order system instead of the reduced order one.

The concern of the paper is to develop a systematic method that enables, for sliding manifold design, the adoption of a variety of multiplier theories which have been developed for the full state feedback with Lyapunov-type constraints. In the linear control theory, many of the design objectives have been represented by the parametric constraints containing the Lyapunov matrix, and rewritten by the linear matrix inequalities (LMIs) using the Schur complement and the change of variables (see [9]). Once the constraints are stated in LMIs, the design problem can be easily solved thanks to the convexity. For example, the uncertain delayed systems (e.g., [10,11] and references therein), parametric uncertain systems (e.g., [12,13] among many), pole-clustering problems ([14]) and so on have been effectively dealt with in Lyapunov approaches in the parameter space.

The basic idea of the paper comes from the results in [6] and [15]. In the presence of uncertainties, it was shown that the quadratic stabilizability condition for full state feedback can be used for selecting the robust sliding mode according to the parameterization technique that manipulates the Lyapunov matrix of full order [6]. Then, the method is further extended to respond to several issues raised in [15]. Based on the previous results, it will be shown that the linear sliding mode can be parameterized by partitioning and augmenting the Lyapunov matrix without loss of generality in Section 2. Then, using the explicit formula, several topics such as quadratic performance optimization, time-delayed systems, uncertain systems, and the pole placement constraint will be discussed in detail. Also, we address further application to the multiobjective approach in which multiple design criteria should be met. Finally, the conclusion follows in Section 3.

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The notations used in the paper are fairly standard. Among them, $\|\bullet\|$ and $\lambda(\bullet)$ represents the Euclidean norm and the set of eigenvalues of the argument matrix, respectively. The inequality signs for matrices denote the sign-definiteness for the real symmetric matrices.

2. MAIN RESULTS

2.1. Problem descriptions and preliminary

Consider the system

$$\dot{x} = Ax + B(u + w), \qquad (1)$$

where $x \in \Re^n$ and $u \in \Re^m$ are the state and the control input, respectively, and $w \in \Re^l$ is the disturbance whose element is bounded as $|w_j(t)| \le \bar{w}_j$, $\forall j \in [1,l]$, for the known $\bar{w}_j$. The stabilizability of the pair (A,B) is assumed. Also, for the simplicity of description, suppose the system is in the regular form [1,2]

$$\begin{cases} \dot{x}_1 = A_{11}x_1 + A_{12}x_2, \\ \dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_2(u + w), \end{cases} \qquad (2)$$

where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \begin{bmatrix} \Re^{n-m} \\ \Re^m \end{bmatrix}$ and B_2 is nonsingular. Without loss of generality, consider the sliding function

$$s(t) = Sx_1 + x_2, \qquad (3)$$

for some $S \in \Re^{m\times(n-m)}$. Suppose that an appropriate control is given to admit the reachability condition such that $\dot{s}(t)^T s(t) < 0$ for all $t \ge 0$. For example, using the control

$$\begin{cases} 0, & (\|s(t)\| = 0) \\ -B_2^{-1}\{GAx + \beta s + Z(t)sign(s)\}, & (\|s(t)\| > 0) \end{cases} \qquad (4)$$

where $sign(s) = [sign(s_1), \cdots, sign(s_m)]^T$, $\beta > 0$, $G = [S, I_m]$ and $Z(t) = diag[z_1, \cdots, z_m]$ for z_i defined as

$$z_i = \sum_{k=1}^{l} |(GB)_{ik}| \bar{w}_k, \qquad (5)$$

one may show $s(t) = 0$ for all $t \ge t_s$ for some t_s. Then, the system can be rewritten, when $s(t) = 0$, by

$$\begin{cases} \dot{x}_1 = (A_{11} - A_{12}S)x_1, \\ \dot{x}_2 = -S\dot{x}_1. \end{cases} \qquad (6)$$

Note that the system state is stabilized as long as the reduced order dynamics is stable. For convenience, the system is considered to be in the sliding mode when the sliding function vanishes with the stability of (6).

In general, it has long been standard to manipulate the reduced order system to obtain the stabilizing sliding function coefficient because the dynamics of reduced order is only related to sliding mode behavior. In the paper, however, we propose a unified approach to sliding mode by handling the full order system (1) instead of (6) without loss of generality. The proposed approach is advantageous in that the problem description is significantly simplified by manipulating the full order system, and the synthesis can be done by the convex approach using LMIs [9].

2.2. All stabilizing sliding function coefficients

One of the major concerns in sliding mode design is to find a stabilizing sliding function coefficient for the reduced order system (6). In the following, we present an explicit formula for representing all the possible sliding function coefficients.

Theorem 1: There exist some sliding modes if and only if there exist some $P > 0$ and $K \in \Re^{m\times n}$ satisfying

$$(A - BK)^T P + P(A - BK) + Q < 0, \qquad (7)$$

for a $Q \ge 0$. Moreover, for any feasible P, the matrix

$$S \triangleq P_{22}^{-1} P_{12}^T \qquad (8)$$

is the stabilizing sliding function coefficient, where P_{ij}'s are defined as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \in \begin{bmatrix} \Re^{(n-m)\times(n-m)} & \Re^{(n-m)\times m} \\ \Re^{m\times(n-m)} & \Re^{m\times n} \end{bmatrix}.$$

Proof: (Necessity) Let S be the stabilizing sliding function coefficient, which guarantees the stability of the matrix $A_s := A_{11} - A_{12}S$. Then, there should exist some $P_r > 0$ satisfying, for any $Q_r \ge 0$,

$$A_s^T P_r + P_r A_s + Q_r < 0. \qquad (9)$$

Let us define as $R := A_s^T P_r + P_r A_s + Q_r$, which is negative definite, for future reference.

Now, for an arbitrary $P_{22} > 0$, define the matrices

$$P_{12} = S^T P_{22}, P_{11} = P_r + P_{12}^T P_{22}^{-1} P_{12}, \qquad (10)$$

and

$$K = [K_1, K_2], \qquad (11)$$

where, for an $\varepsilon > 0$,

$$K_1 = B_2^{-1}\{A_{21} + P_{22}^{-1}P_{12}^T A_{11} + P_{22}^{-1}A_{12}^T P_r + \tfrac{\varepsilon}{2} P_{22}^{-2} P_{12}^T\},$$
$$K_2 = B_2^{-1}\{A_{22} + P_{22}^{-1}P_{12}^T A_{12} + \tfrac{\varepsilon}{2} P_{22}^{-1}\}.$$

Using the matrices $P := \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}$ and K defined above, it may be shown that, through some manipulations,

$$T\left\{A_K^T P + PA_K + Q\right\}T^T = \begin{bmatrix} R & 0 \\ 0 & -\varepsilon I_m \end{bmatrix} < 0 \qquad (12)$$

where $A_K = A - BK$ and $T = \begin{bmatrix} I_{n-m} & -P_{12}P_{22}^{-1} \\ 0 & I_m \end{bmatrix}$, which implies (7) owing to the nonsingularity of T.

(Sufficiency) Define $T_r := [I_{n-m}, -P_{12}P_{22}^{-1}]$. Then, pre- and post multiplying (7) by T_r and T_r^T, respectively, yields

$$\left(A_{11} - A_{12}P_{22}^{-1}P_{12}^T\right)^T P_r + P_r\left(A_{11} - A_{12}P_{22}^{-1}P_{12}^T\right) + T_r Q T_r^T < 0 \qquad (13)$$

where $P_r = P_{11} - P_{12}^T P_{22}^{-1} P_{12}$, which is positive definite since $P > 0$. Hence, choosing as $S = P_{22}^{-1}P_{12}^T$, the stability of the matrix $A_{11} - A_{12}S$ is guaranteed by the Lyapunov stability theorem. This completes the proof. □

Theorem 1 shows that all admissible sliding function coefficients can be obtained from the Lyapunov matrix of full order. Note that, in the literature, similar results utilizing the Lyapunov inequality (or Riccati-type equations) have been proposed to generate the sling function coefficients (e.g., see [8,7,5]). From this point of view, Theorem 1 is consistent with them. However, we point out that Theorem 1 also suggests the necessary condition which generalizes the usage of the Lyapunov inequality of full order in the siding mode design.

In addition, Theorem 1 provides a useful way to adopt a variety of Lyapunov (or Riccati) approaches developed in linear control theory to the design of sliding modes. This will be exten1sively dealt with in the following sections.

Remark 1: The feasibility of Lyapunov inequality (7) is the stabilizability of the pair (A, B). Hence, the stabilizability of the nominal system is the existence of sliding modes, which extends the controllability condition in [1].

2.3. Quadratic performance optimization

In this section, we solve the quadratic performance optimization problem in the sliding mode based on the parameterization technique in Theorem 1. The problem of concern is as follows: Find the optimal sliding function coefficient S to minimize the quadratic performance index

$$J(S) = \int_{t_s}^{\infty} x^T Q x \, dt \qquad (14)$$

where t_s is the starting time of the sliding mode, subject to the system (6). In the following, let us present the result:

Theorem 2: Given a stabilizing sliding function coefficient S, the cost function (14) is bounded as

$$J(S) < x_1(t_s)^T P_r x_1(t_s) \qquad (15)$$

for P satisfying $S = P_{22}^{-1}P_{12}^T$ and the inequality (7), where $P_r = P_{11} - P_{12}P_{22}^{-1}P_{12}^T$.

Proof: First, recall, from the proof for the necessity of Theorem 1, that there should exist some $P > 0$ and K satisfying (7) with the relation $S = P_{22}^{-1}P_{12}^T$. Then, for the matrices P and K, observe that

$$\begin{aligned} s(t) &= x_2 + Sx_1 = P_{22}^{-1}\left[P_{12}^T, P_{22}\right]x \\ &= (P_{22}^{-1}B_2^{-T})B^T P x, \end{aligned} \qquad (16)$$

which implies that $B^T P x = 0$ on $s(t) = 0$ for all $t \ge t_s$. Now, let us consider the derivative of a quadratic function $V = x^T P x$ for $t \ge t_s$ as follows:

$$\begin{aligned} \dot{V} &= x^T\left\{(A - BK)^T P + P(A - BK)\right\}x \\ &\quad + 2x^T PB(u + w + Kx) < -x^T Q x \end{aligned} \qquad (17)$$

since $B^T P x = 0$. Integrating both sides in (17) with respect to the time, we have

$$\int_{t_s}^{\infty} x^T Q x dt < x_1(t_s)^T (P_{11} - P_{12}P_{22}^{-1}P_{12}^T) x_1(t_s) \qquad (18)$$

using $x_2(t_s) = -P_{22}^{-1}P_{12}^T x_1(t_s)$. This completes the proof. □

Remark 2: Through some technical procedures, it may be shown that the quadratic term $x_1(t_s)^T P_r x_1(t_s)$ in (15) is the least upper bound. To show this, one may manipulate the Lyapunov equation instead of the Lyapunov inequality in (7). However, we do not discuss it here in detail.

Using the result of Theorem 2, the performance optimization can be redefined based on the LMIs method [9] that utilizes the change of variables such as $Y \,\square\, P^{-1}$ and $L \,\square\, KP^{-1}$. Note the matrix inversion property

$$Y = P^{-1} = \begin{bmatrix} (P_{11} - P_{12}P_{22}^{-1}P_{12}^T)^{-1} & \star \\ \star & \star \end{bmatrix} \qquad (19)$$

to deal with the performance bound (15), where $\star$ positions are of no concern. Thus, we can summarize the quadratic performance optimization in the sliding mode as follows:

Quadratic performance optimization: *Given $Q \ge 0$, minimize γ with respect to $Y > 0$ and L satisfying*

$$\begin{bmatrix} AY + YA^T - BL - L^T B^T & YC_q \\ C_q^T Y & -I \end{bmatrix} < 0 \qquad (20)$$

$$\begin{bmatrix} \gamma & x_1(t_s)^T \\ x_1(t_s) & U_1 Y U_1^T \end{bmatrix} > 0 \qquad (21)$$

where $Q = C_q C_q^T$ *and* $U_1 = [I_{n-m}, 0_{(n-m)\times m}]$.

As a matter of fact, the quadratic performance optimization was first introduced and solved in [1] by adopting the LQR method in the reduced order space. In the literature, it is noted that (i) the invertibility of the matrix Q, (ii) the controllability of the pair (A, B), and (iii) the detectability of the reduced order system depending on the matrix Q are assumed to describe the optimization problem. On the other hand, our approach does not require the conditions (i) and (iii) and, moreover, the stabilizability of the pair (A, B) is necessary instead of (ii). Note that such advantages result from the usage of the Lyapunov inequality of full order.

2.4. Pole placement problem

To improve the transient response in the sliding mode, the pole placement methods or the eigenstructure assignments have been the major subject in sliding mode control theory. To deal with the issue, let us define a set

$$Z(c,\rho) \triangleq \{ z \in C \mid |z + c| < \rho, \mathrm{Re}(z) < 0 \} \qquad (22)$$

for positive scalars c and ρ. Now, we introduce the result:

Theorem 3: There exist some S so that the set of the eigenvalues of the matrix $A_{11} - A_{12}S$ belong to the set $Z(c,\rho)$, that is,

$$\lambda(A_{11} - A_{12}S) \subset Z(c,\rho) \qquad (23)$$

if and only if there exist some $K \in \Re^{m\times n}$ and $P > 0$ satisfying

$$\begin{bmatrix} \rho P & (A - BK)^T P + cP \\ P(A - BK) + cP & \rho P \end{bmatrix} > 0. \qquad (24)$$

Also, the matrix $S = P_{22}^{-1} P_{12}^T$ for any feasible P guarantees the pole-clustering property (23).

Proof: To improve the readability, we start with a rough sketch of the proof. Using the results in [14] and [16], it is noted that the constraint (23) holds if and only if there exist some $P_r > 0$ and S satisfying

$$\begin{bmatrix} \rho P_r & * \\ P_r(A_{11} - A_{12}S) + cP_r & \rho P_r \end{bmatrix} > 0 \qquad (25)$$

where $*$ denotes the transpose of the off-diagonal term for saving the space. Hence, we should show the equivalence between the feasibility of (24) and (25).

(Sufficiency:(25) $\leftarrow$ (24)) Define as $T := \left[\begin{array}{c|c} T_r & \\ \hline & T_r \end{array}\right]$, where $T_r = [I_{n-m}, -P_{12}P_{22}^{-1}]$. Then, pre- and post multiplying (24) by T and T^T, respectively, leads to (25) by the choice of $S = P_{22}^{-1}P_{12}^T$.

(Necessity: (25) $\rightarrow$ (24)) Given the sliding function coefficient S and any matrix $P_{22} > 0$, define the matrices

$$P_{12} := S^T P_{22}, P_{11} := P_r + P_{12}P_{22}^{-1}P_{12}^T. \qquad (26)$$

Also, consider the matrices

$$H := \left[\begin{array}{cc|cc} 0 & I_m & 0 & 0 \\ \hline 0 & 0 & 0 & I_m \end{array}\right],$$

$$T := \left[\begin{array}{c|c} I_{n-m} - P_{12}P_{22}^{-1} & 0 \\ \hline 0 & I_{n-m} - P_{12}P_{22}^{-1} \end{array}\right],$$

which would make the matrix $[H^T, T^T]^T$ nonsingular. For simplicity, let $L_{(24)}$ and $L_{(25)}$ denote the left hand sides of (24) and (25), respectively. In the following, with the above matrices (also, $P := \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}$), it will be shown that $L_{(24)} > 0$ can be satisfied for some matrix K. To this end, through some manipulations, note that

$$\begin{bmatrix} H \\ T \end{bmatrix} L_{(24)} \begin{bmatrix} H \\ T \end{bmatrix}^T = \left[\begin{array}{c|c} HL_f H^T & HL_f T^T \\ \hline TL_f H^T & TL_f T^T \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} \rho P_{22} & N & 0 & A_{12}^T P_r \\ N^T & \rho P_{22} & M & 0 \\ \hline 0 & M^T & \multicolumn{2}{c}{\multirow{2}{*}{$L_{(25)}$}} \\ P_r A_{12} & 0 & & \end{array}\right] \qquad (27)$$

where

$$\begin{aligned} N &= A_{12}^T P_{12} + (A_{22} - B_2K_2)^T P_{22} + cP_{22}, \\ M &= P_{12}^T(A_{11} - A_{12}S) + P_{22}(A_{21} - B_2K_1) \\ &\quad - P_{22}(A_{22} - B_2K_2). \end{aligned} \qquad (28)$$

Note that N and M can be made for arbitrarily values thanks to K_1 and K_2. Since $L_{(25)} > 0$, the positive definiteness of the transformed quantity in (27) is equivalent to that of the matrix

$$\begin{bmatrix} \rho P_{22} & N \\ N^T & \rho P_{22} \end{bmatrix} - \begin{bmatrix} 0 & A_{12}^T P_r \\ M & 0 \end{bmatrix} L_{(25)}^{-1} \begin{bmatrix} 0 & A_{12}^T P_r \\ M & 0 \end{bmatrix} \qquad (29)$$

which can be always positive definite by selecting proper P_{22} and N. For example, choose $M = N = 0$ and a sufficiently large $P_{22} > 0$. Then, it may be shown that $L_{(24)} > 0$. This completes the proof. □

Note that the inequality (24) is also the necessary and sufficient condition for the existence of the full state feedback which places the closed loop poles in the specified region such that $\lambda(A-BK) \subset Z(c,\rho)$.

2.5. Application to time delay systems

The time delayed systems has been of concern in some literature (e.g., see [2] and the references therein). While the complete solution remains unsolved yet, we here propose a new method based on the parametric approach.

Consider the system

$$\dot{x}(t) = Ax(t) + A_d x(t-\tau) + B(u+w), \tag{30}$$

where $0 \le \tau \le \tau_{\max}$ for the known $\tau_{\max}$. Also, without loss of generality, we assume that the system exists in the canonical form similar to (6).

We start with the control law (4), instead of (5), having

$$z_i = \sum_{j=1}^{n} |(GA_d)_{ij}| \bar{x}_j + \sum_{k=1}^{l} |(GB)_{ik}| \bar{w}_k, \tag{31}$$

where $\bar{x}_j(t) \triangleq \sup_{\xi \in [t-\tau_{\max}, t]} |x_j(\xi)|$, which renders the reachability condition. To see this, rewrite the sliding function behavior as

$$\begin{aligned} \dot{s} &= G\dot{x} \\ &= GAx + GA_d x(t-\tau) + B_2 u + GBw. \end{aligned} \tag{32}$$

Then, for the Lyapunov functional candidate $V_s = \frac{1}{2} s^T s$, it can be shown that $\dot{V}_s < -\beta \|s\|^2$ using the fact

$$\begin{aligned} s^T GA_d x(t-\tau) &= \sum_{i=1}^{m} s_i \sum_{j=1}^{n} (GA_d)_{ij} x_j(t-\tau) \\ &\le \sum_{i=1}^{m} |s_i| \sum_{j=1}^{n} |(GA_d)_{ij}| \bar{x}_j(t). \end{aligned} \tag{33}$$

For addressing the existence of sliding modes, we rely upon the following lemma.

Lemma 1: The system

$$\dot{x} = A_c x + A_d x(t-r) \tag{34}$$

is quadratically stable if there exist some $P > 0$ and $H > 0$ satisfying

$$\begin{bmatrix} A_c^T P + PA_c + H & PA_d \\ A_d^T P & -H \end{bmatrix} < 0. \tag{35}$$

The above lemma has been well known in the literature to handle the systems with uncertain state delay even though it may cause conservatism since the maximum amount of delay is not reflected in the inequality constraint. Note that the quadratic stability of (34) can be shown by using the Lyapunov-Krasovskii functional (see [9] and the original reference therein):

$$V = x^T Px + \int_{t-\tau}^{t} x(\theta)^T Hx(\theta) d\theta \tag{36}$$

for some $P > 0$ and $H > 0$.

We can now present the result:

Theorem 4: There exist some sliding modes if there exist some $P > 0$, $H > 0$ and $K \in \Re^{m \times x}$ satisfying

$$\begin{bmatrix} (A-BK)^T P + P(A-BK) + H + Q & PA_d \\ A_d^T P & -H \end{bmatrix} < 0 \tag{37}$$

for a matrix $Q \ge 0$. Moreover, the sliding function coefficient is given by $S = P_{22}^{-1} P_{12}^T$ for any feasible P in (37). With the sliding function coefficient S, it holds that

$$J(S) < x_1(t_s)^T P_r x_1(t_s) \tag{38}$$

where $P_r = P_{11} - P_{12} P_{22}^{-1} P_{12}^T$.

Proof: For T_r (as defined in (13)), define the augmented matrix $T := \begin{bmatrix} T_r & 0 \\ 0 & T_r \end{bmatrix}$. Pre and post multiplying (37) by T and T^T, respectively, it follows that

$$\begin{bmatrix} A_s^T P_r + P_r A_s + T_r(Q+H)T_r^T & P_r A_{d,s} \\ A_{d,s}^T P_r & -T_r H T_t^T \end{bmatrix} < 0 \tag{39}$$

where $A_s = A_{11} - A_{12} P_{22}^{-1} P_{12}^T$ and $A_{d,s} = A_{d,11} - A_{d,12} P_{22}^{-1} P_{12}$ which implies, from Lemma 1, the stability of the system

$$\dot{\xi} = A_s \xi + A_{d,s} \xi(t-r). \tag{40}$$

Note that (40) is equivalent to the system behavior in the sliding mode determined by $S = P_{22}^{-1} P_{12}^T$.

Now, the performance issue can be addressed according to the similar argument through (16)-(17). That is, using the fact that $B^T Px = 0$ in the sliding mode, one may show

$$\begin{aligned} \dot{V} &= \eta^T \begin{bmatrix} A_K^T P + PA_K + H & PA_d \\ A_d^T P & -H \end{bmatrix} \eta \\ &< -x^T Qx \end{aligned} \tag{41}$$

where $A_K = A - BK$ and $\eta = [x(t)^T, x(t-r)^T]^T$. Then integrating both sides with respect to the time results in (38). This completes the proof. □

It is noted that the inequality (37) can be rewritten by an LMI using the change of variables $Y := P^{-1}$, $L := KP^{-1}$ and $\hat{H} := PHP$ as follows:

$$\begin{bmatrix} AY+YA^T-BL-L^TB^T+\hat{H} & YC_q & A_dY \\ C_q^TY & -I & 0 \\ YA_d^T & 0 & -\hat{H} \end{bmatrix} < 0 \quad (42)$$

Hence, combining with (21) in order to bound the quadratic performance index, the sliding mode can be designed based on the guaranteed cost control through the convex search.

2.6 Parametric uncertain systems

Sliding mode design for parametric uncertain systems has been addressed in [7] based on the Riccati approach. However, we restate the issue here for the completeness of the paper and the introduction of the recently developed quadratic stability condition, which uses the symmetric and the skew symmetric scales.

Consider the uncertain system

$$\dot{x} = (A+\Delta A)x + B(u+w) \quad (43)$$

where ΔA represents the real parametric uncertainties of the form

$$\Delta A = MF(t)N \quad (44)$$

where $M, N^T \in \Re^{n\times h}$ and $F(t) = diag[\delta_1(t),\cdots,\delta_p(t)]$ for the Lebesgue measurable functions δ_i such that $|\delta_i(t) \le 1|$, $\forall t \ge 0$. As to the reachability issue, it has been shown in [7] that the reachability condition is met by the control (4) with

$$z_i = \sum_{j=1}^{n} \left|(GM)_{ij}(Nx)_j\right| + \sum_{k=1}^{l} \left|(GB)_{ik}\right| \overline{w}_k. \quad (45)$$

This can be proven by showing that the derivative of the quadratic function $V = \frac{1}{2}s^Ts$ is made to be negative.

Now, to address the existence of sliding modes, the quadratic stability should be considered *a priori*.

Lemma 2: The system

$$\dot{x} = (A_c + \Delta A)x \quad (46)$$

is quadratically stable if there exist some $P > 0$, $X \in S_{sym}$ and $U \in S_{skew}$ satisfying,

$$\begin{aligned} &A_c^TP + PA_c + PMXM^TP \\ &+(N+PMU^T)X^{-1}(N+PMU^T)^T < 0 \end{aligned} \quad (47)$$

where the sets for the scales are defined as follows:

$$\begin{cases} S_{sym} \triangleq \{X \mid XF = FX, X > 0\}, \\ S_{skew} \triangleq \{U \mid UF = FU, U = -U^T\}. \end{cases} \quad (48)$$

The condition has been derived in [17] based on the S-procedure and the realness of uncertainties, and applied to the L_2 disturbance attenuation problem [18]. See Appendix for a proof using the quadratic bounding technique which is simpler than those in the references. It is noted that the usage of the skew symmetric scales (as well as the symmetric scales) effectively reduces the design conservatism in the presence of the multi-rank uncertain parameters (*i.e.*, the repeated uncertain parameters in $F(\bullet)$).

Theorem 5: There exist sliding modes if there exist some $P > 0$, $K \in \Re^{m\times n}$, $X \in S_{sym}$ and $U \in S_{skew}$ satisfying

$$\begin{aligned} &(A-BK)^TP + P(A-BK) + Q + PMXM^TP \\ &+(N+PMU^T)X^{-1}(N+PMU^T)^T < 0. \end{aligned} \quad (49)$$

for a matrix $Q \ge 0$. Moreover, the matrix $S = P_{22}^{-1}P_{12}^T$, for any feasible matrix P, is the stabilizing sliding function coefficient and over bounds the quadratic performance index as follows:

$$\int_{t_s}^{\infty} x^TQxdt < x_1(t_s)^TP_rx_1(t_s) \quad (50)$$

where $P_r = P_{11} - P_{12}P_{22}^{-1}P_{12}^T$.

Proof: In order to save space, we refer to [7] for the detailed procedures. The rough sketch of the proof is as follows. First, pre and post multiply T_r and T_r^T (defined in (13)) by (49), respectively. Then, through some manipulations, one may show the quadratic stability of the reduced order uncertain system by choosing $S = P_{22}^{-1}P_{12}^T$. Also, the relation (50) can be shown by following the similar steps done in the proof of Theorem 2. This completes the proof. □

It is noted that Theorem 5 is an extension of Theorems 1 and 2 to uncertain systems. It can be observed that Theorem 5 would be equivalent to the results of Theorems 1 and 2 in the absence of uncertainties, *i.e.*, $M = N = 0$. Also, in practice, the inequality (49) can be rewritten by an LMI using the change of variables and the Schur complement [9] as follows:

$$\begin{bmatrix} YA^T+AY-BL-L^TB^T+M\times M^T & * & * \\ C_q^TY & -I & * \\ N^TY+UM^T & 0 & -X \end{bmatrix} < 0 \quad (51)$$

where $Y = P^{-1}$, $L = KP^{-1}$, $Q = C_qC_q^T$. Consequently, the concept of the quadratic performance optimization in the sliding mode (discussed in §2.3) can be extended to the guaranteed cost optimization [13] by replacing (20) with (51) as follows:

Guaranteed cost optimization: *Given* $Q \ge 0$, *minimize* γ *with respect to* $Y > 0$, L, $X \in S_{sym}$ *and* $U \in S_{skew}$ *satisfying (51) and (21).*

2.7. Further remarks

Through the previous discussions, we have shown the common nature of the sliding mode structure for several types of problems. That is, as long as the full state feedback problem is formulated by the parametric approach (utilizing the Lyapunov matrix) and solvable, the sliding mode can be immediately designed by manipulating certain parts of the Lyapunov matrix. To be specific, suppose that there exist $P>0$ and K satisfying

$$(A-BK)^T P+P(A-BK)+F(P)<0 \quad (52)$$

where F constrains the desired objective. Then, the sliding mode can be given by $S=P_{22}^{-1}P_{12}^T$ with no loss of generality. Hence, a variety of the results developed in the linear control theory can be adopted for the sliding mode design. Also, the problem description gets significantly simplified by handling the original full order system instead of the reduced one. In addition, the above property enables us to apply the multiobjective approach (e.g., see [16], [19]) to the sliding mode design with relative ease. To this end, consider, for example, the quadratic performance optimization with the pole placement constraint. Combining the results in Theorems 2 and 3 with the assumption that the Lyapunov matrices (P 's of (7) and (24)) are common, it is easy to have the result:

Corollary 1: Given some c, $\rho>0$ and $Q\geq 0$, there exist some sliding modes S such that

$$(i)\ \lambda(A_{11}-A_{12}S)\subset Z(c,\rho),$$

$$(ii)\ \int_{t_s}^{\infty} x^T Qx dt<\gamma,$$

if there exist some $Y>0$ and L satisfying inequalities

$$\begin{bmatrix} \rho Y & YA^T-L^T B^T+cY \\ AY-BL+cY & \rho Y \end{bmatrix}>0, \quad (53)$$

$$\begin{bmatrix} AY+YA^T-BL-L^T B^T & YC_q \\ C_q^T Y & -I \end{bmatrix}>0, \quad (54)$$

$$\begin{bmatrix} \gamma & x_1(t_s)^T \\ x_1(t_s) & U_1 Y U_1^T \end{bmatrix}>0. \quad (55)$$

Then, the admissible sliding function coefficient is given by $S=P_{22}^{-1}P_{12}^T$ for any feasible $Y\ (:=P^{-1})$.

The above shows how the various objectives can be effectively combined in a design problem. First, note that the formulation has the convexity for the design parameters, which can be solved using the LMIs technique. Also, the proposed approach does not require handling the reduced order system that has made the design complicated. As a result, a variety of results in the parametric approaches (using the Lyapunov matrices) can be adopted for the sliding mode design in the framework of the multi-objective approach. Further applications remain as an active area of research.

3. CONCLUSIONS

In this manuscript, explicit formulas for sliding mode design have been newly proposed based on the parametric approaches that use the Lyapunov matrices of full order. It has been shown that the sliding mode can be designed by combining the partitions of the Lyapunov matrix that constrains the design objective. Taking advantage of the construction technique, many of results that have been developed for the full state feedback synthesis in the linear control theory are shown to be applicable to sliding mode design. Through the paper, we addressed the issues on the quadratic performance optimization, the state delayed systems, parametric uncertain systems, and the pole placement problem.

APPENDIX

Proof of Lemma 2

Proof: For the quadratic function $V=x^T Px$, where $P>0$, the quadratic stability can be proven by showing $\dot{V}<0$. To do this, the following bounding technique is crucial:

$$PMFN+N^T FM^T P=PMFH+H^T FM^T P \quad (56)$$

$$\leq PMXM^T P+H^T X^{-1}H \quad (57)$$

where $H=N+UM^T P$. Note that (56) is established thanks to the commuting property and the skew symmetricity, i.e., $FU=UF=-U^T F$. Also, (57) is the standard bounding technique for the block-diagonal uncertainties (e.g. see [7]). This completes the proof. □

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