Commun. Korean Math. Soc. 18 (2003), No. 2, pp. 341–353

FUZZY STRONGLY r-SEMICONTINUOUS MAPS

Seok Jong Lee and Eun Pyo Lee

ABSTRACT. As a generalizaton of the concepts of fuzzy strongly semiopen sets and fuzzy strongly semicontinuous maps, we introduce the concepts of fuzzy strong-ly r-semiopen sets and fuzzy strongly r-semicontinuous maps in fuzzy topology. Also we introduce fuzzy r-semiinterior and fuzzy r-semiclosure. By these concept, we characterize fuzzy strongly r-semicontinuous, fuzzy strongly r-semiopen and fuzzy strongly r-semiclosed maps.

1. Introduction and preliminaries

Chang [3] introduced fuzzy topological spaces and several other authors continued the investigation of such spaces. Some authors [4], [5], [7]-[10] introduced other definitions of fuzzy topology as a generalization of Chang's fuzzy topology. In this paper, as a generalizaton of the concepts of fuzzy strongly semiopen sets and fuzzy strongly semicontinuous maps of Shi-Zhong Bai [2], we introduce the concepts of fuzzy strongly r-semiopen sets and fuzzy strongly r-semicontinuous maps in fuzzy topology. Also we introduce fuzzy r-semiinterior and fuzzy r-semiclosure. By these concept, we characterize fuzzy strongly r-semicontinuous, fuzzy strongly r-semiopen and fuzzy strongly r-semiclosed maps.

We will denote the unit interval $[0,1]$ of the real line by I and $I_0 = (0,1]$. A member μ of I^X is called a *fuzzy set* in X. For any $\mu \in I^X$, μ^c denotes the complement $1-\mu$. By $\tilde{0}$ and $\tilde{1}$ we denote constant maps on X with value 0 and 1, respectively.

A *Chang's fuzzy topology* on X is a family T of fuzzy sets in X which satisfies the following properties:

Received April 11, 2002.

2000 Mathematics Subject Classification: 54A40, 54C08.

Key words and phrases: fuzzy strongly r-semiopen, fuzzy strongly r-semicontinuous.

This work was supported by grant No. R01-2002-000-00315-0 from the Basic Research Program of the Korea Science & Engineering Foundation.

(1) $\tilde{0}, \tilde{1} \in T$.
(2) If $\mu_1, \mu_2 \in T$ then $\mu_1 \wedge \mu_2 \in T$.
(3) If $\mu_i \in T$ for each i, then $\bigvee \mu_i \in T$.

The pair (X, T) is called a *Chang's fuzzy topological space.*

A *fuzzy topology* on X is a map $\mathcal{T} : I^X \to I$ which satisfies the following properties:

(1) $\mathcal{T}(\tilde{0}) = \mathcal{T}(\tilde{1}) = 1$.
(2) $\mathcal{T}(\mu_1 \wedge \mu_2) \geq \mathcal{T}(\mu_1) \wedge \mathcal{T}(\mu_2)$.
(3) $\mathcal{T}(\bigvee \mu_i) \geq \bigwedge \mathcal{T}(\mu_i)$.

The pair $(X, \mathcal{T})$ is called a *fuzzy topological space.*

DEFINITION 1.1. Let μ be a fuzzy set in a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then μ is said to be

(1) *fuzzy r-preopen* if $\mu \leq \operatorname{int}(\operatorname{cl}(\mu, r), r)$,
(2) *fuzzy r-preclosed* if $\operatorname{cl}(\operatorname{int}(\mu, r), r) \leq \mu$.

DEFINITION 1.2. Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map from a fuzzy topological space X to a fuzzy topological space Y and $r \in I_0$. Then f is called a *fuzzy r-precontinuous* map if $f^{-1}(\mu)$ is a fuzzy r-preopen set in X for each fuzzy r-open set μ in Y.

The notions of fuzzy semiopen, semiclosed sets and the weaker forms of fuzzy continuity which are related to our discussion, can be found in [1, 11]. All the other nonstandard definitions and notations can be found in [6].

2. Fuzzy strongly r-semiopen sets

We are going to define fuzzy strongly r-semiopen sets and fuzzy strongly r-semiclosed sets, and then investigate some of their properties.

DEFINITION 2.1. Let μ be a fuzzy set in a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then μ is said to be

(1) *fuzzy strongly r-semiopen* if there is a fuzzy r-open set ρ in X such that
$$\rho \leq \mu \leq \operatorname{int}(\operatorname{cl}(\rho, r), r),$$
(2) *fuzzy strongly r-semiclosed* if there is a fuzzy r-closed set ρ in X such that
$$\operatorname{cl}(\operatorname{int}(\rho, r), r) \leq \mu \leq \rho.$$

THEOREM 2.2. *Let μ be a fuzzy set in a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then the following statements are equivalent:*

(1) *μ is fuzzy strongly r-semiopen.*
(2) *μ^c is fuzzy strongly r-semiclosed.*
(3) $\mu \leq \text{int}(\text{cl}(\text{int}(\mu, r), r), r)$.
(4) $\mu^c \geq \text{cl}(\text{int}(\text{cl}(\mu^c, r), r), r)$.
(5) *μ is fuzzy r-semiopen and fuzzy r-preopen.*
(6) *μ^c is fuzzy r-semiclosed and fuzzy r-preclosed.*

PROOF. (1) $\Leftrightarrow$ (2), (3) $\Leftrightarrow$ (4) and (5) $\Leftrightarrow$ (6) are trivial.

(1) $\Rightarrow$ (3). Let μ be a fuzzy strongly r-semiopen set in X. Then there is a fuzzy r-open set ρ in X such that $\rho \leq \mu \leq \text{int}(\text{cl}(\rho, r), r)$. Since $\mathcal{T}(\rho) \geq r$ and $\mu \geq \rho$, $\rho = \text{int}(\rho, r) \leq \text{int}(\mu, r)$. Thus we have

$$\mu \leq \text{int}(\text{cl}(\rho, r), r) \leq \text{int}(\text{cl}(\text{int}(\mu, r), r), r).$$

(3) $\Rightarrow$ (1). Let $\text{int}(\text{cl}(\text{int}(\mu, r), r), r) \geq \mu$ and take $\rho = \text{int}(\mu, r)$. Since $\mathcal{T}(\text{int}(\mu, r)) \geq r$, ρ is a fuzzy r-open set. Also,

$$\rho = \text{int}(\mu, r) \leq \mu \leq \text{int}(\text{cl}(\text{int}(\mu, r), r), r) = \text{int}(\text{cl}(\rho, r), r).$$

Hence μ is a fuzzy strongly r-semiopen set.

(1) $\Rightarrow$ (5). It is obvious.

(5) $\Rightarrow$ (3). Let μ be fuzzy r-semiopen and fuzzy r-preopen. Then $\mu \leq \text{cl}(\text{int}(\mu, r), r)$ and $\mu \leq \text{int}(\text{cl}(\mu, r), r)$. Thus

$$\begin{aligned} \mu &\leq \text{int}(\text{cl}(\mu, r), r) \leq \text{int}(\text{cl}(\text{cl}(\text{int}(\mu, r), r), r), r) \\ &= \text{int}(\text{cl}(\text{int}(\mu, r), r), r). \end{aligned}$$

□

THEOREM 2.3. (1) *Any union of fuzzy strongly r-semiopen sets is fuzzy strongly r-semiopen.*
(2) *Any intersection of fuzzy strongly r-semiclosed sets is fuzzy strongly r-semiclosed.*

PROOF. (1) Let $\{\mu_i\}$ be a collection of fuzzy strongly r-semiopen sets. Then for each i, there is a fuzzy r-open set ρ_i such that $\rho_i \leq \mu_i \leq \text{int}(\text{cl}(\rho_i, r), r)$. Since $\mathcal{T}(\bigvee \rho_i) \geq \bigwedge \mathcal{T}(\rho_i) \geq r$, $\bigvee \rho_i$ is a fuzzy r-open set. Also

$$\bigvee \rho_i \leq \bigvee \mu_i \leq \bigvee \text{int}(\text{cl}(\rho_i, r), r) \leq \text{int}(\text{cl}(\bigvee \rho_i, r), r).$$

Hence $\bigvee \mu_i$ is a fuzzy strongly r-semiopen set.

(2) It follows from (1) by Theorem 2.2. □

REMARK 2.4. It is obvious that every fuzzy r-open set is fuzzy strongly r-semiopen and every fuzzy strongly r-semiopen set is not only a fuzzy r-semiopen set but also a fuzzy r-preopen set. All of the converses need not be true as shown in the following example.

EXAMPLE 2.5. Let $X = I$ and μ_1, μ_2 and μ_3 be fuzzy sets in X defined by

$$\mu_1(x) = \begin{cases} 0 & \text{if } 0 \le x \le \frac{1}{2}, \\ 2x-1 & \text{if } \frac{1}{2} \le x \le 1; \end{cases}$$

$$\mu_2(x) = \begin{cases} 1 & \text{if } 0 \le x \le \frac{1}{4}, \\ -4x+2 & \text{if } \frac{1}{4} \le x \le \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} \le x \le 1; \end{cases}$$

and

$$\mu_3(x) = \begin{cases} 0 & \text{if } 0 \le x \le \frac{1}{4}, \\ \frac{1}{3}(4x-1) & \text{if } \frac{1}{4} \le x \le 1. \end{cases}$$

Define $\mathcal{T}_1 : I^X \to I$, $\mathcal{T}_2 : I^X \to I$ and $\mathcal{T}_3 : I^X \to I$ by

$$\mathcal{T}_1(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1, \\ 0 & \text{otherwise}; \end{cases}$$

$$\mathcal{T}_2(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1, \mu_2, \mu_1 \vee \mu_2, \\ 0 & \text{otherwise}; \end{cases}$$

and

$$\mathcal{T}_3(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_3, \\ 0 & \text{otherwise}. \end{cases}$$

Then clearly $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$ are fuzzy topologies on X. The fuzzy set $\mu_3(\mu_3^c)$ is fuzzy strongly $\frac{1}{2}$-semiopen ($\frac{1}{2}$-semiclosed) which is not fuzzy $\frac{1}{2}$-open ($\frac{1}{2}$-closed) in $(X, \mathcal{T}_1)$. Also $\mu_3(\mu_3^c)$ is fuzzy $\frac{1}{2}$-semiopen ($\frac{1}{2}$-semiclosed) but not fuzzy strongly $\frac{1}{2}$-semiopen ($\frac{1}{2}$-semiclosed) in $(X, \mathcal{T}_2)$. It can be also seen that $\mu_1(\mu_1^c)$ is fuzzy $\frac{1}{2}$-preopen ($\frac{1}{2}$-preclosed) which is not fuzzy strongly $\frac{1}{2}$-semiopen ($\frac{1}{2}$-semiclosed) in $(X, \mathcal{T}_3)$. The example further shows that μ_1 in $(X, \mathcal{T}_3)$ is a fuzzy $\frac{1}{2}$-preopen set which is not fuzzy $\frac{1}{2}$-semiopen and μ_3 in $(X, \mathcal{T}_2)$ is a fuzzy $\frac{1}{2}$-semiopen set which is not fuzzy $\frac{1}{2}$-preopen.

DEFINITION 2.6. Let $(X, \mathcal{T})$ be a fuzzy topological space. For each $r \in I_0$ and for each $\mu \in I^X$, the *fuzzy strong r-semiclosure* is defined by

$$\mathrm{sscl}(\mu, r) = \bigwedge\{\rho \in I^X : \mu \leq \rho,\ \rho \text{ is fuzzy strongly } r\text{-semiclosed}\}$$

and the *fuzzy strong r-semiinterior* is defined by

$$\mathrm{ssint}(\mu, r) = \bigvee\{\rho \in I^X : \mu \geq \rho,\ \rho \text{ is fuzzy strongly } r\text{-semiopen}\}.$$

Obviously $\mathrm{sscl}(\mu, r)$ is the smallest fuzzy strongly r-semiclosed set which contains μ and $\mathrm{ssint}(\mu, r)$ is the greatest fuzzy strongly r-semiopen set which is contained in μ. Moreover, $\mathrm{sscl}(\mu, r) = \mu$ for any fuzzy strongly r-semiclosed set μ and $\mathrm{ssint}(\mu, r) = \mu$ for any fuzzy strongly r-semiopen set μ. Also we have

$$\mathrm{int}(\mu, r) \leq \mathrm{ssint}(\mu, r) \leq \mu \leq \mathrm{sscl}(\mu, r) \leq \mathrm{cl}(\mu, r).$$

Moreover, we have the following results.

(1) $\mathrm{ssint}(\tilde{0}, r) = \tilde{0}, \mathrm{ssint}(\tilde{1}, r) = \tilde{1}$.
(2) $\mathrm{ssint}(\mu, r) \leq \mu$.
(3) $\mathrm{ssint}(\mu \wedge \rho, r) \leq \mathrm{ssint}(\mu, r) \wedge \mathrm{ssint}(\rho, r)$.
(4) $\mathrm{ssint}(\mathrm{ssint}(\mu, r), r) = \mathrm{ssint}(\mu, r)$.
(5) $\mathrm{sscl}(\tilde{0}, r) = \tilde{0}, \mathrm{sscl}(\tilde{1}, r) = \tilde{1}$.
(6) $\mathrm{sscl}(\mu, r) \geq \mu$.
(7) $\mathrm{sscl}(\mu \vee \rho, r) \geq \mathrm{sscl}(\mu, r) \vee \mathrm{sscl}(\rho, r)$.
(8) $\mathrm{sscl}(\mathrm{sscl}(\mu, r), r) = \mathrm{sscl}(\mu, r)$.

THEOREM 2.7. *For a fuzzy set μ in a fuzzy topological space X and $r \in I_0$,*

(1) $\mathrm{ssint}(\mu, r)^c = \mathrm{sscl}(\mu^c, r)$.
(2) $\mathrm{sscl}(\mu, r)^c = \mathrm{ssint}(\mu^c, r)$.

PROOF. (1) Since $\mathrm{ssint}(\mu, r) \leq \mu$ and $\mathrm{ssint}(\mu, r)$ is fuzzy strongly r-semiopen, $\mu^c \leq \mathrm{ssint}(\mu, r)^c$ and $\mathrm{ssint}(\mu, r)^c$ is fuzzy strongly r-semiclosed in X. Thus $\mathrm{sscl}(\mu^c, r) \leq \mathrm{ssint}(\mu, r)^c$. Conversely, since $\mu^c \leq \mathrm{sscl}(\mu^c, r)$ and $\mathrm{sscl}(\mu^c, r)$ is fuzzy strongly r-semiclosed in X, $\mathrm{sscl}(\mu^c, r)^c \leq \mu$ and $\mathrm{sscl}(\mu^c, r)^c$ is fuzzy strongly r-semiopen in X. Thus $\mathrm{sscl}(\mu^c, r)^c \leq \mathrm{ssint}(\mu, r)$ and hence $\mathrm{ssint}(\mu, r)^c \leq \mathrm{sscl}(\mu^c, r)$.

(2) Similar to (1). □

Let $(X, \mathcal{T})$ be a fuzzy topological space. For an r-cut $\mathcal{T}_r = \{\mu \in I^X \mid \mathcal{T}(\mu) \geq r\}$, it is obvious that $(X, \mathcal{T}_r)$ is a Chang's fuzzy topological space for all $r \in I_0$.

Let (X, T) be a Chang's fuzzy topological space and $r \in I_0$. Recall [4] that a fuzzy topology $T^r : I^X \to I$ is defined by

$$T^r(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ r & \text{if } \mu \in T - \{\tilde{0}, \tilde{1}\}, \\ 0 & \text{otherwise.} \end{cases}$$

THEOREM 2.8. *Let μ be a fuzzy set in a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then μ is fuzzy strongly r-semiopen (r-semiclosed) in $(X, \mathcal{T})$ if and only if μ is fuzzy strongly semiopen (semiclosed) in $(X, \mathcal{T}_r)$.*

PROOF. Straightforward. □

THEOREM 2.9. *Let μ be a fuzzy set in a Chang's fuzzy topological space (X, T) and $r \in I_0$. Then μ is fuzzy strongly semiopen (semiclosed) in (X, T) if and only if μ is fuzzy strongly r-semiopen (semiclosed) in (X, T^r).*

PROOF. Straightforward. □

REMARK 2.10. By the above two theorems, we know that the concept of fuzzy strongly r-semiopen (r-semiclosed) is a generalization of the concept of fuzzy strongly semiopen (semiclosed).

3. Fuzzy strongly r-semicontinuous maps

We introduce the notions of fuzzy strongly r-semicontinuous maps, fuzzy strongly r-semiopen maps and fuzzy strongly r-semiclosed maps, and then investigate some of their properties.

DEFINITION 3.1. Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map from a fuzzy topological space X to a fuzzy topological space Y and $r \in I_0$. Then f is said to be

(1) *fuzzy strongly r-semicontinuous* if $f^{-1}(\mu)$ is a fuzzy strongly r-semiopen set of X for each fuzzy r-open set μ in Y, or equivalently, $f^{-1}(\mu)$ is a fuzzy strongly r-semiclosed set in X for each fuzzy r-closed set μ in Y,
(2) *fuzzy strongly r-semiopen* if $f(\rho)$ is a fuzzy strongly r-semiopen set in Y for each fuzzy r-open set ρ in X,
(3) *fuzzy strongly r-semiclosed* if $f(\rho)$ is a fuzzy strongly r-semiclosed set in Y for each fuzzy r-closed set ρ in X.

REMARK 3.2. Clearly, every fuzzy r-continuous map is also a fuzzy strongly r-semicontinuous map and every fuzzy strongly r-semicontinuous map is not only a fuzzy r-semicontinuous map but also a fuzzy r-precontinuous map. All of the converses need not be true as shown in the following example.

EXAMPLE 3.3. Let $X = \{x\}$ and μ_1, μ_2 and μ_3 be fuzzy sets in X defined by

$$\mu_1(x) = \tfrac{1}{2}, \quad \mu_2(x) = \tfrac{1}{3}, \quad \mu_3(x) = \tfrac{1}{4}.$$

Define $\mathcal{T}_1 : I^X \to I$, $\mathcal{T}_2 : I^X \to I$ and $\mathcal{T}_3 : I^X \to I$ by

$$\mathcal{T}_1(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1, \mu_3, \\ 0 & \text{otherwise;} \end{cases}$$

$$\mathcal{T}_2(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_2, \\ 0 & \text{otherwise;} \end{cases}$$

and

$$\mathcal{T}_3(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_3, \\ 0 & \text{otherwise.} \end{cases}$$

Then clearly $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ are fuzzy topologies on X.

(1) Consider the map $f : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ defined by $f(x) = x$. Then $f^{-1}(\tilde{0}) = \tilde{0}$, $f^{-1}(\tilde{1}) = \tilde{1}$ and $f^{-1}(\mu_2) = \mu_2$ are fuzzy strongly $\frac{1}{2}$-semiopen sets in $(X, \mathcal{T}_1)$ and hence f is fuzzy strongly $\frac{1}{2}$-semicontinuous. On the other hand, $f^{-1}(\mu_2) = \mu_2$ is not fuzzy $\frac{1}{2}$-open in $(X, \mathcal{T}_1)$ and hence f is not fuzzy $\frac{1}{2}$-continuous.

(2) Consider the map $f : (X, \mathcal{T}_2) \to (X, \mathcal{T}_3)$ defined by $f(x) = x$. Then $f^{-1}(\tilde{0}) = \tilde{0}$, $f^{-1}(\tilde{1}) = \tilde{1}$ and $f^{-1}(\mu_3) = \mu_3$ are fuzzy $\frac{1}{2}$-preopen sets in $(X, \mathcal{T}_2)$ and hence f is fuzzy $\frac{1}{2}$-precontinuous. On the other hand, $f^{-1}(\mu_3) = \mu_3$ is not fuzzy strongly $\frac{1}{2}$-semiopen in $(X, \mathcal{T}_2)$ and hence f is not fuzzy strongly $\frac{1}{2}$-semicontinuous.

(3) Consider the map $f : (X, \mathcal{T}_3) \to (X, \mathcal{T}_2)$ defined by $f(x) = x$. Then $f^{-1}(\tilde{0}) = \tilde{0}$, $f^{-1}(\tilde{1}) = \tilde{1}$ and $f^{-1}(\mu_2) = \mu_2$ are fuzzy $\frac{1}{2}$-semiopen sets in $(X, \mathcal{T}_3)$ and hence f is fuzzy $\frac{1}{2}$-semicontinuous. On the other hand, $f^{-1}(\mu_2) = \mu_2$ is not fuzzy strongly $\frac{1}{2}$-semiopen in $(X, \mathcal{T}_3)$ and hence f is not fuzzy strongly $\frac{1}{2}$-semicontinuous.

The next theorem provides alternative characterizations of a fuzzy strongly r-semicontinuous map by fuzzy r-closure and fuzzy r-interior.

THEOREM 3.4. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map and $r \in I_0$. Then the following statements are equivalent:*

(1) *f is a fuzzy strongly r-semicontinuous map.*
(2) $\mathrm{cl}(\mathrm{int}(\mathrm{cl}(f^{-1}(\mu), r), r), r) \leq f^{-1}(\mathrm{cl}(\mu, r))$ *for each fuzzy set μ in Y.*
(3) $f(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(\rho, r), r), r)) \leq \mathrm{cl}(f(\rho), r)$ *for each fuzzy set ρ in X.*

PROOF. (1) $\Rightarrow$ (2). Let f be a fuzzy strongly r-semicontinuous map and μ a fuzzy set in Y. Then $\mathrm{cl}(\mu, r)$ is a fuzzy r-closed set in Y. Since f is fuzzy strongly r-semicontinuous, $f^{-1}(\mathrm{cl}(\mu, r))$ is a fuzzy strongly r-semiclosed set in X. By Theorem 2.2,

$$\begin{aligned} f^{-1}(\mathrm{cl}(\mu, r)) &\geq \mathrm{cl}(\mathrm{int}(\mathrm{cl}(f^{-1}(\mathrm{cl}(\mu, r)), r), r), r) \\ &\geq \mathrm{cl}(\mathrm{int}(\mathrm{cl}(f^{-1}(\mu), r), r), r). \end{aligned}$$

(2) $\Rightarrow$ (3). Let ρ be a fuzzy set in X. Then $f(\rho)$ is a fuzzy set in Y. By (2),

$$f^{-1}(\mathrm{cl}(f(\rho), r)) \geq \mathrm{cl}(\mathrm{int}(\mathrm{cl}(f^{-1}f(\rho), r), r), r) \geq \mathrm{cl}(\mathrm{int}(\mathrm{cl}(\rho, r), r), r).$$

Hence we have

$$\mathrm{cl}(f(\rho), r) \geq ff^{-1}(\mathrm{cl}(f(\rho), r)) \geq f(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(\rho, r), r), r)).$$

(3) $\Rightarrow$ (1). Let μ be a fuzzy r-closed set in Y. Then $f^{-1}(\mu)$ is a fuzzy set in X. By (3),

$$f(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(f^{-1}(\mu), r), r), r)) \leq \mathrm{cl}(ff^{-1}(\mu), r) \leq \mathrm{cl}(\mu, r) = \mu.$$

Hence we have

$$\begin{aligned} \mathrm{cl}(\mathrm{int}(\mathrm{cl}(f^{-1}(\mu), r), r), r) &\leq f^{-1}f(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(f^{-1}(\mu), r), r), r)) \\ &\leq f^{-1}(\mu). \end{aligned}$$

Thus $f^{-1}(\mu)$ is a fuzzy strongly r-semiclosed set in X and hence f is a fuzzy strongly r-semicontinuous map. □

A fuzzy strongly r-semicontinuous map can be characterized as follows.

THEOREM 3.5. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map and $r \in I_0$. Then the following statements are equivalent:*

(1) *f is a fuzzy strongly r-semicontinuous map.*
(2) $f(\mathrm{sscl}(\rho, r)) \leq \mathrm{cl}(f(\rho), r)$ *for each fuzzy set ρ in X.*
(3) $\mathrm{sscl}(f^{-1}(\mu), r) \leq f^{-1}(\mathrm{cl}(\mu, r))$ *for each fuzzy set μ in Y.*

(4) $f^{-1}(\mathrm{int}(\mu, r)) \leq \mathrm{ssint}(f^{-1}(\mu), r)$ *for each fuzzy set* μ *in* Y.

PROOF. (1) $\Rightarrow$ (2). Let ρ be a fuzzy set in X. Since $\mathrm{cl}(f(\rho), r)$ is a fuzzy r-closed set in Y, $f^{-1}(\mathrm{cl}(f(\rho), r))$ is a fuzzy strongly r-semiclosed set in X. Thus

$$\begin{aligned}\mathrm{sscl}(\rho, r) \leq \mathrm{sscl}(f^{-1}f(\rho), r) &\leq \mathrm{sscl}(f^{-1}(\mathrm{cl}(f(\rho), r)), r)\\ &= f^{-1}(\mathrm{cl}(f(\rho), r)).\end{aligned}$$

Hence

$$f(\mathrm{sscl}(\rho, r)) \leq ff^{-1}(\mathrm{cl}(f(\rho), r)) \leq \mathrm{cl}(f(\rho), r).$$

(2) $\Rightarrow$ (3). Let μ be a fuzzy set in Y. By (2),

$$f(\mathrm{sscl}(f^{-1}(\mu), r)) \leq \mathrm{cl}(ff^{-1}(\mu), r) \leq \mathrm{cl}(\mu, r).$$

Thus

$$\mathrm{sscl}(f^{-1}(\mu), r) \leq f^{-1}f(\mathrm{sscl}(f^{-1}(\mu), r)) \leq f^{-1}(\mathrm{cl}(\mu, r)).$$

(3) $\Rightarrow$ (4). Let μ be a fuzzy set in Y. Then μ^c is a fuzzy set in Y. By (3),

$$\mathrm{sscl}(f^{-1}(\mu)^c, r) = \mathrm{sscl}(f^{-1}(\mu^c), r) \leq f^{-1}(\mathrm{cl}(\mu^c, r)).$$

By Theorem 2.7,

$$f^{-1}(\mathrm{int}(\mu, r)) = f^{-1}(\mathrm{cl}(\mu^c, r))^c \leq \mathrm{sscl}(f^{-1}(\mu)^c, r)^c = \mathrm{ssint}(f^{-1}(\mu), r).$$

(4) $\Rightarrow$ (1). Let μ be a fuzzy r-open set in Y. Then $\mathrm{int}(\mu, r) = \mu$. By (4),

$$f^{-1}(\mu) = f^{-1}(\mathrm{int}(\mu, r)) \leq \mathrm{ssint}(f^{-1}(\mu), r) \leq f^{-1}(\mu).$$

So $f^{-1}(\mu) = \mathrm{ssint}(f^{-1}(\mu), r)$ and hence $f^{-1}(\mu)$ is a fuzzy strongly r-semiopen set in X. Thus f is fuzzy strongly r-semicontinuous. □

THEOREM 3.6. *Let* $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ *be a bijection and* $r \in I_0$. *Then* f *is a fuzzy strongly* r*-semicontinuous map if and only if* $\mathrm{int}(f(\rho), r) \leq f(\mathrm{ssint}(\rho, r))$ *for each fuzzy set* ρ *in* X.

PROOF. Let f be a fuzzy strongly r-semicontinuous map and ρ a fuzzy set in X. Since $\mathrm{int}(f(\rho), r)$ is fuzzy r-open in Y, $f^{-1}(\mathrm{int}(f(\rho), r))$ is fuzzy strongly r-semiopen in X. Since f is one-to-one, we have

$$f^{-1}(\mathrm{int}(f(\rho), r)) \leq \mathrm{ssint}(f^{-1}f(\rho), r) = \mathrm{ssint}(\rho, r).$$

Since f is onto, we have

$$\mathrm{int}(f(\rho), r) = ff^{-1}(\mathrm{int}(f(\rho), r)) \leq f(\mathrm{ssint}(\rho, r)).$$

Conversely, let μ be a fuzzy r-open set in Y. Then $\text{int}(\mu, r) = \mu$. Since f is onto,

$$f(\text{ssint}(f^{-1}(\mu), r)) \geq \text{int}(ff^{-1}(\mu), r) = \text{int}(\mu, r) \;=\; \mu.$$

Since f is one-to-one, we have

$$f^{-1}(\mu) \leq f^{-1}f(\text{ssint}(f^{-1}(\mu), r)) = \text{ssint}(f^{-1}(\mu), r) \leq f^{-1}(\mu).$$

So $f^{-1}(\mu) = \text{ssint}(f^{-1}(\mu), r)$ and hence $f^{-1}(\mu)$ is a fuzzy strongly r-semiopen set in X. Thus f is fuzzy strongly r-semicontinuous. □

The next theorem provides alternative characterizations of a fuzzy strongly r-semiopen map.

THEOREM 3.7. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map and $r \in I_0$. Then the following statements are equivalent:*

(1) *f is a fuzzy strongly r-semiopen map.*
(2) *$f(\text{int}(\rho, r)) \leq \text{ssint}(f(\rho), r)$ for each fuzzy set ρ in X.*
(3) *$\text{int}(f^{-1}(\mu), r) \leq f^{-1}(\text{ssint}(\mu, r))$ for each fuzzy set μ in Y.*

PROOF. (1) $\Rightarrow$ (2). Let ρ be a fuzzy set in X. Clearly $\text{int}(\rho, r)$ is a fuzzy r-open set in X. Since f is a fuzzy strongly r-semiopen map, $f(\text{int}(\rho, r))$ is a fuzzy strongly r-semiopen set in Y. Thus

$$f(\text{int}(\rho, r)) = \text{ssint}(f(\text{int}(\rho, r)), r) \leq \text{ssint}(f(\rho), r).$$

(2) $\Rightarrow$ (3). Let μ be a fuzzy set in Y. Then $f^{-1}(\mu)$ is a fuzzy set in X. By (2),

$$f(\text{int}(f^{-1}(\mu), r)) \leq \text{ssint}(ff^{-1}(\mu), r) \leq \text{ssint}(\mu, r).$$

Thus we have

$$\text{int}(f^{-1}(\mu), r) \;\leq\; f^{-1}f(\text{int}(f^{-1}(\mu), r)) \leq f^{-1}(\text{ssint}(\mu, r)).$$

(3) $\Rightarrow$ (1). Let ρ be a fuzzy r-open set in X. Then $\text{int}(\rho, r) = \rho$ and $f(\rho)$ is a fuzzy set in Y. By (3),

$$\rho \;=\; \text{int}(\rho, r) \leq \text{int}(f^{-1}f(\rho), r) \;\leq\; f^{-1}(\text{ssint}(f(\rho), r)).$$

Hence we have

$$f(\rho) \leq ff^{-1}(\text{ssint}(f(\rho), r)) \leq \text{ssint}(f(\rho), r) \leq f(\rho).$$

Thus $f(\rho) = \text{ssint}(f(\rho), r)$ and hence $f(\rho)$ is a fuzzy strongly r-semiopen set in Y. Therefore f is fuzzy strongly r-semiopen. □

A fuzzy strongly r-semiclosed map can be characterized as follows.

THEOREM 3.8. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map and $r \in I_0$. Then the following statements are equivalent:*

(1) *f is a fuzzy strongly r-semiclosed map.*
(2) $\mathrm{sscl}(f(\rho), r) \leq f(\mathrm{cl}(\rho, r))$ *for each fuzzy set ρ in X.*

PROOF. (1) $\Rightarrow$ (2). Let ρ be a fuzzy set in X. Clearly $\mathrm{cl}(\rho, r)$ is a fuzzy r-closed set in X. Since f is a fuzzy strongly r-semiclosed map, $f(\mathrm{cl}(\rho, r))$ is a fuzzy strongly r-semiclosed set of Y. Thus

$$\mathrm{sscl}(f(\rho), r) \leq \mathrm{sscl}(f(\mathrm{cl}(\rho, r)), r) = f(\mathrm{cl}(\rho, r)).$$

(2) $\Rightarrow$ (1). Let ρ be a fuzzy r-closed set in X. Then $\mathrm{cl}(\rho, r) = \rho$. By (2),

$$\mathrm{sscl}(f(\rho), r) \leq f(\mathrm{cl}(\rho, r)) = f(\rho) \leq \mathrm{sscl}(f(\rho), r).$$

Thus $f(\rho) = \mathrm{sscl}(f(\rho), r)$ and hence $f(\rho)$ is a fuzzy strongly r-semiclosed set in Y. Therefore f is fuzzy strongly r-semiclosed. □

THEOREM 3.9. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a bijection and $r \in I_0$. Then f is a fuzzy strongly r-semiclosed map if and only if $f^{-1}(\mathrm{sscl}(\mu, r)) \leq \mathrm{cl}(f^{-1}(\mu), r)$ for each fuzzy set μ in Y.*

PROOF. Let f be a fuzzy strongly r-semiclosed map and μ a fuzzy set in Y. Then $f^{-1}(\mu)$ is a fuzzy set in X. Since f is onto,

$$\mathrm{sscl}(\mu, r) = \mathrm{sscl}(ff^{-1}(\mu), r) \leq f(\mathrm{cl}(f^{-1}(\mu), r).$$

Since f is one-to-one, we have

$$f^{-1}(\mathrm{sscl}(\mu, r)) \leq f^{-1}f(\mathrm{cl}(f^{-1}(\mu), r)) = \mathrm{cl}(f^{-1}(\mu), r).$$

Conversely, let ρ be a fuzzy r-closed set in X. Then $\mathrm{cl}(\rho, r) = \rho$. Since f is one-to-one,

$$f^{-1}(\mathrm{sscl}(f(\rho), r)) \leq \mathrm{cl}(f^{-1}f(\rho), r) = \mathrm{cl}(\rho, r) = \rho.$$

Since f is onto, we have

$$f(\rho) \geq ff^{-1}(\mathrm{sscl}(f(\rho), r)) = \mathrm{sscl}(f(\rho), r) \geq f(\rho).$$

Thus $f(\rho) = \mathrm{sscl}(f(\rho), r)$ and hence $f(\rho)$ is a fuzzy strongly r-semiclosed set in Y. Therefore f is fuzzy strongly r-semiclosed. □

THEOREM 3.10. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map from a fuzzy topological space X to a fuzzy topological space Y and $r \in I_0$. Then f is fuzzy strongly r-semicontinuous (r-semiopen, r-semiclosed) if and only if $f : (X, \mathcal{T}_r) \to (Y, \mathcal{U}_r)$ is fuzzy strongly semicontinuous (semiopen, semiclosed).*

PROOF. Straightforward. □

THEOREM 3.11. *Let $f : (X, T) \to (Y, U)$ be a map from a Chang's fuzzy topological space X to a Chang's fuzzy topological space Y and $r \in I_0$. Then f is fuzzy strongly semicontinuous (semiopen, semiclosed) if and only if $f : (X, T^r) \to (Y, U^r)$ is fuzzy strongly r-semicontinuous (r-semiopen, r-semiclosed).*

PROOF. Straightforward. □

REMARK 3.12. By the above two theorems, we know that the concept of a fuzzy strongly r-semicontinuous(r-semiopen, r-semiclosed, respectively) map is a generalization of the concept of a fuzzy strongly semicontinuous(semiopen, semiclosed, respectively) map.

References

[1] K. K. Azad, *On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity*, J. Math. Anal. Appl. **82** (1981), 14–32.
[2] S.-Z. Bai, *Fuzzy pre-semiopen sets and fuzzy pre-semicontinuity*, Proc. ICIS'92 (1992), 918–920.
[3] C. L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl. **24** (1968), 182–190.
[4] K. C. Chattopadhyay, R. N. Hazra, and S. K. Samanta, *Gradation of openness : Fuzzy topology*, Fuzzy Sets and Systems **49** (1992), 237–242.
[5] R. N. Hazra, S. K. Samanta, and K. C. Chattopadhyay, *Fuzzy topology redefined*, Fuzzy Sets and Systems **45** (1992), 79–82.
[6] S. J. Lee and E. P. Lee, *Fuzzy r-continuous and fuzzy r-semicontinuous maps*, International Journal of Mathematics and Mathematical Sciences **27** (2001), no. 1, 53–63.
[7] R. Lowen, *Fuzzy topological spaces and fuzzy compactness*, J. Math. Anal. Appl. **56** (1976), 621–633.
[8] A. A. Ramadan, *Smooth topological spaces*, Fuzzy Sets and Systems **48** (1992), 371–375.
[9] A. P. Šostak, *On a fuzzy topological structure*, Suppl. Rend. Circ. Matem. Janos Palermo, Sr. II **11** (1985), 89–103.
[10] ———, *Two decades of fuzzy topology: basic ideas, notions, and results*, Russian Mathematical Surveys **44** (1989), no. 6, 125–186, (tranlated from Russian).
[11] T. H. Yalvaç, *Semi-interior and semi-closure of a fuzzy set*, J. Math. Anal. Appl. **132** (1988), 356–364.

Seok Jong Lee
Department of Mathematics
Chungbuk National University
Cheongju 361-763, Korea
E-mail: sjlee@chungbuk.ac.kr

Eun Pyo Lee
Department of Mathematics
Seonam University
Namwon 590-711, Korea
E-mail: eplee@tiger.seonam.ac.kr