Commun. Korean Math. Soc. **18** (2003), No. 2, pp. 297–308

ON HYPERSURFACES OF MANIFOLDS EQUIPPED WITH A HYPERCOSYMPLECTIC 3-STRUCTURE

YEONG-MOO SONG, JEONG-SIK KIM AND MUKUT MANI TRIPATHI

ABSTRACT. Hypersurfaces of a Riemannian manifold equipped with a hypercosymplectic 3-structure are studied. Integrability conditions for certain distributions on the hypersurface are investigated. Geometry of leaves of certain distribution are also studied.

1. Introduction

The quaternionic analog of almost complex structures is the almost quaternion (hypercomplex) which is defined by three local (global) almost complex structures which satisfy the quaternionic relations as the imaginary quaternions satisfy ([3]). Quaternion Kähler manifolds and hyper-Kähler manifolds are special and interesting cases of Riemannian manifolds with almost quaternion and almost hypercomplex structure, respectively. Quaternion Kähler manifolds are Einstein, while hyper-Kähler manifolds are Ricci flat.

An almost contact 3-structure was defined by Kuo ([5]) and it is closely related to both almost quaternion and almost hypercomplex structures. Hypersurfaces of manifolds with almost hypercomplex structure inherit naturally three almost contact structures which constitute an almost contact 3-structure. An almost contact metric 3-structure manifold is always $(4m+3)$-dimensional. The structural group of the tangent bundle of a $(4m+3)$-dimensional manifold equipped with an almost contact 3-structure is reducible to $Sp(m) \times I_3$.

In particular, if each almost contact metric structure of an almost contact metric 3-structure is Sasakian, then this structure is called a Sasakian 3-structure. Riemannian manifolds with Sasakian 3-structure

Received June 10, 2002.
2000 Mathematics Subject Classification: 53C25, 53C40.
Key words and phrases: hypersurface and hypercosymplectic 3-structure.
The second author was supported by Korea Science and Engineering Foundation Grant(R01-2001-000-00003-0).

are called 3-Sasakian manifolds. A $(4m+3)$-dimensional sphere is a 3-Sasakian manifold, while on a $(4m+3)$-dimensional torus one gets an almost contact metric 3-structure, where each almost contact metric structure is cosymplectic ([2]). F. Martín Cabrera calls such a structure as a hypercosymplectic 3-structure ([6]).

Hypersurfaces of manifolds equipped with Sasakian 3-structures are studied by A. Bejancu in [1]. Thus motivated sufficiently, in this paper we study hypersurfaces of a manifold equipped with a hypercosymplectic 3-structure. The paper is organized as follows. Section 2 contains preliminaries. In section 3, some basic results are given. Integrability conditions for certain natural distributions on the hypersurface are investigated in section 4. In the last section, geometry of leaves of certain distribution are studied.

2. Hypercosymplectic 3-structures

Let $\bar{M}$ be a $(4m+3)$-dimensional manifold. Let $\bar{M}$ admit three almost contact structures (ϕ_a, ξ_a, η_a), $a = 1, 2, 3$, that is,

$$\phi_a^2 = -I + \eta_a \otimes \xi_a, \quad \eta_a(\xi_a) = 1, \quad \phi_a(\xi_a) = 0, \quad \eta_a \circ \phi_a = 0. \tag{1}$$

Let these three almost contact structures satisfy

$$\phi_a \circ \phi_b - \eta_b \otimes \xi_a = -\phi_b \circ \phi_a + \eta_a \otimes \xi_b = \phi_c, \tag{2}$$

$$\phi_a \xi_b = -\phi_b \xi_a = \xi_c, \tag{3}$$

$$\eta_a \circ \phi_b = -\eta_b \circ \phi_a = \eta_c, \tag{4}$$

$$\eta_a(\xi_b) = \eta_b(\xi_a) = 0, \quad a \neq b \tag{5}$$

for every cyclic permutation (a, b, c) of $(1, 2, 3)$. Then we say that $\bar{M}$ is endowed with an *almost contact 3-structure* (Kuo, [5]). If $\bar{M}$ is a Riemannian manifold, then there is always a Riemannian metric g on $\bar{M}$ such that

$$g(\phi_a X, \phi_a Y) = g(X, Y) - \eta_a(X)\eta_a(Y), \qquad a = 1, 2, 3 \tag{6}$$

$$g(X, \xi_a) = \eta_a(X), \qquad a = 1, 2, 3 \tag{7}$$

for all $X, Y \in T\bar{M}$. Then we say that $\bar{M}$ is endowed with an almost contact metric 3-structure $(\phi_a, \xi_a, \eta_a, g)$ (Kuo, [5]). From (5) and (7) it follows that ξ_1, ξ_2, ξ_3 are mutually orthogonal. We also have

$$\Omega_a(X, Y) \equiv g(X, \phi_a Y) = -g(\phi_a X, Y), \qquad a = 1, 2, 3. \tag{8}$$

We know that an almost contact metric structure (ϕ, ξ, η, g) is called a cosymplectic structure if (Blair, [2])

$$\bar{\nabla}\phi = 0, \tag{9}$$

where $\bar{\nabla}$ is Riemannian connection. From (9) it follows that

$$\bar{\nabla}\xi = 0, \qquad \bar{\nabla}\eta = 0. \tag{10}$$

If all the three almost contact metric structures $(\phi_a, \xi_a, \eta_a, g)$, $a = 1, 2, 3$, are cosymplectic structures, that is,

$$\bar{\nabla}\phi_a = 0, \tag{11}$$

$$\bar{\nabla}\xi_a = 0 \qquad \bar{\nabla}\eta_a = 0, \tag{12}$$

then the manifold $\bar{M}$ is said to have a hypercosymplectic 3-structure $(\phi_a, \xi_a, \eta_a, g)$, $a = 1, 2, 3$ (cf. Martin Cabrera, [6]).

EXAMPLE 1. We construct a simple example of a hypercosymplectic 3-structure in the 3-dimensional Euclidean space $\mathbb{R}^3$. We define $(\phi_a, \xi_a, \eta_a, g)$, $a = 1, 2, 3$ in $\mathbb{R}^3$ by their matrices as follows:

$$\phi_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \phi_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \phi_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\xi_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \xi_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \xi_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\eta_1 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \quad \eta_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad \eta_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix},$$

and

$$g = I_3.$$

By direct computations, we find that the above set provides a hypercosymplectic 3-structure on $\mathbb{R}^3$.

EXAMPLE 2. We consider a $(4m+3)$-dimensional torus $\mathbb{T}^{4m+3}$ $(m \geq 1)$ and let $\{\alpha_1, \alpha_2, \cdots, \alpha_{4m+3}\}$ be a basis for 1-forms such that each α_i is integral and closed. "α_i is integral and closed" means that α_i defines an element of the first integral cohomology group. That is, if one integrates α_i along any 1-cycle, then the result is an integral number. On $\mathbb{T}^{4m+3}$ we consider the metric tensor field given by

$$g(X, Y) = \sum_{i=1}^{4m+3} \alpha_i(X)\, \alpha_i(Y)$$

and the almost contact metric 3-structure consisting of the three $(1,1)$ tensor fields

$$\phi_a = \sum_{i=1}^{4m+3} (e_{am+i}\otimes\alpha_i - e_i\otimes\alpha_{a+i} + e_{cm+i}\otimes\alpha_{bm+i}$$
$$-e_{bm+i}\otimes\alpha_{cm+i} + e_{4m+c}\otimes\alpha_{4m+b} - e_{4m+b}\otimes\alpha_{4m+c}),$$

where $\{e_1, e_2, \cdots, e_{4m+3}\}$ is the orthonormal frame field dual of $\{\alpha_1, \alpha_2, \cdots, \alpha_{4m+3}\}$ and (a,b,c) is a cyclic permutation of $(1,2,3)$; the three 1-forms

$$\eta_1 = \alpha_{4m+1}, \quad \eta_2 = \alpha_{4m+2}, \quad \eta_3 = \alpha_{4m+3};$$

and the three vector fields

$$\xi_1 = e_{4m+1}, \quad \xi_2 = e_{4m+2}, \quad \xi_3 = e_{4m+3}.$$

Then the torus $\mathbb{T}^{4m+3}$ contains a hypercosymplectic 3-structure $(\phi_a, \xi_a, \eta_a, g)$ [6].

3. Some basic results

Let M be a hypersurface of $\bar{M}$ equipped with an almost contact metric 3-structure $(\phi_a, \xi_a, \eta_a, g)$, $a = 1,2,3$, such that the structure vector fields ξ_a, $a = 1,2,3$ are tangential to M. Let N be the unit normal to M. Since,

$$g(\phi_a N, N) = 0, \qquad a = 1,2,3, \tag{13}$$

therefore denoting

$$X_a = \phi_a N, \qquad a = 1,2,3 \tag{14}$$

we can define three 1-dimensional distributions $\mathcal{D}_a$, $a = 1,2,3$ spanned by X_a, $a = 1,2,3$, respectively:

$$\mathcal{D}_a : x \longrightarrow \mathcal{D}_{ax} \equiv \phi_a(T_x^\perp M) \subset T_x M, \qquad a = 1,2,3.$$

We denote by

$$\mathcal{E} = \{\xi_1\} \oplus \{\xi_2\} \oplus \{\xi_3\} \qquad \text{and} \qquad \mathcal{D}^\perp = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \mathcal{D}_3,$$

where $\{\xi_a\}$, $a = 1,2,3$ are the 1-dimensional distributions spanned by structure vector fields ξ_a on M. In view of (14) we have

$$\phi_a X_a = -N, \qquad a = 1,2,3, \tag{15}$$

$$\phi_a X_b = X_c = -\phi_b X_a \tag{16}$$

for any cyclic permutation (a,b,c) of $(1,2,3)$.

THEOREM 1. *Let M be a hypersurface of $\bar{M}$ equipped with an almost contact metric 3-structure $(\phi_a, \xi_a, \eta_a, g)$, $a = 1, 2, 3$, such that the structure vector fields ξ_a, $a = 1, 2, 3$ are tangential to M. Then*
(a) *$\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ are mutually orthogonal,*
(b) *$\mathcal{D}^{\perp}$ is orthogonal to $\mathcal{E}$.*
Consequently, $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \{\xi_1\}, \{\xi_2\}, \{\xi_3\}$ are mutually orthogonal.

PROOF. **(a)** Let $Y_a \in \mathcal{D}_a, Y_b \in \mathcal{D}_b$. By definition of $\mathcal{D}_a$ and $\mathcal{D}_b$ there exist differentiable functions f_a, f_b such that $Y_{a1} = f_a\phi_a N$ and $Y_b = f_b\phi_b N$. Then by using (8), (2), (7), (13) we get

$$\begin{aligned} g(Y_a, Y_b) &= f_a f_b g(\phi_a N, \phi_b N) = -f_a f_b g(N, \phi_a\phi_b N) \\ &= -f_a f_b g(N, \phi_c N + \eta_b(N)\xi_a) = 0. \end{aligned}$$

Thus $\mathcal{D}_a$ is orthogonal to $\mathcal{D}_b$ for each cyclic permutation.(a, b, c) of $(1, 2, 3)$.

(b) For $Y_a \in \mathcal{D}_a$, $a = 1, 2, 3$, we get $g(Y_a, \xi_b) = f_a g(\phi_a N, \xi_b) = -f_a g(N, \phi_a\xi_b) = 0$, which completes the proof. □

Now, we denote by $\mathcal{D}$ the orthogonal complementary distribution to $\mathcal{D}^{\perp}\oplus\mathcal{E}$ in M.

THEOREM 2. *If M is a hypersurface of $\bar{M}$ equipped with an almost contact metric 3-structure $(\phi_a, \xi_a, \eta_a, g)$, $a = 1, 2, 3$, such that the structure vector fields ξ_a, $a = 1, 2, 3$ are tangential to M, then the distribution $\mathcal{D}$ is invariant by each ϕ_a.*

PROOF. Let $X \in \mathcal{D}, Y \in \mathcal{D}^{\perp}$. Using (8), we get

$$g(\phi_a X, Y) = -g(X, \phi_a Y) = 0,$$

which implies that $\phi_a\mathcal{D} \perp \mathcal{D}^{\perp}$. Next, we get

$$g(\phi_a X, N) = -g(X, \phi_a N) = 0,$$

which implies that $\phi_a\mathcal{D} \perp T^{\perp}M$. Finally, we have

$$g(\phi_a X, \xi_b) = -g(X, \phi_a\xi_b) = 0,$$

which shows that $\phi_a\mathcal{D} \perp \mathcal{E}$. Hence $\phi_a\mathcal{D} = D$, that is, $\mathcal{D}$ is invariant by each ϕ_a. □

Denoting by U the projection operator of TM on to the invariant distribution $\mathcal{D}$, an arbitrary vector field X on M can be written as

$$X = UX + \sum_{a=1}^{3} (\eta_a(X)\xi_a + \omega_a(X)X_a), \tag{17}$$

where ω_a, $a = 1, 2, 3$ are 1-forms locally defined on M by

$$\omega_a(X) = g(X, X_a), \qquad a = 1, 2, 3. \tag{18}$$

Operating by ϕ_a to (17) and taking account of (15), (16), (1) and (3) we obtain

$$\phi_a X = \phi_a UX + \eta_b(X)\xi_c - \eta_c(X)\xi_b + \omega_b(X)X_c - \omega_c(X)X_b - \omega_a(X)N. \tag{19}$$

Now, we write the Gauss and Weingarten formulae as

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y)N, \tag{20}$$

$$\bar{\nabla}_X N = -AX, \tag{21}$$

for all $X, Y \in TM$, where ∇ is the induced Riemannian connection on M, h is the second fundamental form of the immersion and A is the fundamental tensor of Weingarten with respect to the unit normal N. It is well known that

$$h(X, Y) = g(AX, Y). \tag{22}$$

If each almost contact metric structure $(\phi_a, \xi_a, \eta_a, g)$, $a = 1, 2, 3$ is a cosymplectic structure, then we have

$$\begin{aligned}
0 &= (\overline{\nabla}_X \phi_a)Y \\
&= \overline{\nabla}_X(\phi_a UY + \eta_b(Y)\xi_c - \eta_c(Y)\xi_b + \omega_b(Y)X_c - \omega_c(Y)X_b \\
&\quad - \omega_a(Y)N) - \phi_a\left(\nabla_X Y + h(X, Y)N\right) \\
&= (\nabla_X\left(\phi_a UY + \eta_b(Y)\xi_c - \eta_c(Y)\xi_b + \omega_b(Y)X_c - \omega_c(Y)X_b\right) \\
&\quad + \omega_a(Y)(AX) - \phi_a U\nabla_X Y - \eta_b(\nabla_X Y)\xi_c + \eta_c(\nabla_X Y)\xi_b \\
&\quad - \omega_b(\nabla_X Y)X_c + \omega_c(\nabla_X Y)X_b - h(X, Y)X_a) \\
&\quad + \left(h\left(X, \phi_a UY + \eta_b(Y)\xi_c - \eta_c(Y)\xi_b + \omega_b(Y)X_c - \omega_c(Y)X_b\right)N\right. \\
&\quad \left.- ((\nabla_X\omega_a)Y)N\right).
\end{aligned}$$

Therefore, equating normal parts of both the sides we get

$$\begin{aligned}
0 &= h(X, \phi_a UY) + \eta_b(Y)h(X, \xi_c) - \eta_c(Y)h(X, \xi_b) \\
&\quad + \omega_b(Y)h(X, X_c) - \omega_c(Y)h(X, X_b) - (\nabla_X\omega_a)Y.
\end{aligned} \tag{23}$$

4. Integrability of certain distributions

LEMMA 3. *Let M be a hypersurface of $\bar{M}$ equipped with a hyper-cosymplectic 3-structure $(\phi_a, \xi_a, \eta_a, g)$, $a = 1, 2, 3$, such that the structure vector fields ξ_a, $a = 1, 2, 3$ are tangential to M. Then we have*

$$\nabla_X \xi_a = 0, \qquad X \in TM, \tag{24}$$

$$h(X,\xi_a) = 0, \qquad X \in TM. \tag{25}$$

PROOF. In view of (12) and (20), for all $X \in TM$ we get

$$\nabla_X \xi_a + h(X,\xi_a)N = \bar{\nabla}_X \xi_a = 0. \tag{26}$$

Equating tangential and normal parts in (26) we get (24) and (25) respectively. □

LEMMA 4. *Let M be a hypersurface of $\bar{M}$ equipped with a hypercosymplectic 3-structure $(\phi_a, \xi_a, \eta_a, g)$, $a = 1,2,3$, such that the structure vector fields ξ_a, $a = 1,2,3$ are tangential to M. Then for each $X \in \mathcal{D}$ and $Y \in \mathcal{D}^{\perp}$, we have*

$$g\left([X,\xi_a], Y\right) = 0, \qquad a = 1,2,3. \tag{27}$$

PROOF. Using (12) for all $X \in \mathcal{D}$ and $Y \in \mathcal{D}^{\perp}$, we get

$$g\left([X,\xi_a], Y\right) = g\left(\bar{\nabla}_X \xi_a - \bar{\nabla}_{\xi_a} X, Y\right) = -g\left(\bar{\nabla}_{\xi_a} X, Y\right) = g\left(X, \bar{\nabla}_{\xi_a} Y\right).$$

Putting $Y = X_b$, $X = \phi_b Z$, for some $Z \in \mathcal{D}$, in the above equation we get

$$\begin{aligned} g\left(\phi_b Z, \bar{\nabla}_{\xi_a} X_b\right) &= g\left(Z, -\phi_b \bar{\nabla}_{\xi_a} X_b\right) = g\left(Z, \left(\bar{\nabla}_{\xi_a} \phi_b\right) X_b - \bar{\nabla}_{\xi_a} \phi_b X_b\right) \\ &= g\left(Z, -\bar{\nabla}_{\xi_a} \phi_b X_b\right) = g\left(Z, \bar{\nabla}_{\xi_a} N\right) = g\left(Z, -A\xi_a\right) \\ &= -h\left(Z, \xi_a\right) = 0. \end{aligned}$$

Hence we get (27). □

LEMMA 5. *Let M be a hypersurface of $\bar{M}$ equipped with a hypercosymplectic 3-structure $(\phi_a, \xi_a, \eta_a, g)$, $a = 1,2,3$, such that the structure vector fields ξ_a, $a = 1,2,3$ are tangential to M. Then for each $X \in \mathcal{D}$ and $Y \in \mathcal{D}^{\perp}$, we have*

$$g\left([Y,\xi_a], X\right) = 0. \tag{28}$$

PROOF. Using (12) for all $X \in \mathcal{D}$ and $Y \in \mathcal{D}^{\perp}$, we get

$$g\left([Y,\xi_a], X\right) = g\left(\bar{\nabla}_Y \xi_a - \bar{\nabla}_{\xi_a} Y, X\right) = -g\left(\bar{\nabla}_{\xi_a} Y, X\right) = g\left(X, \bar{\nabla}_{\xi_a} Y\right).$$

Then as in Lemma 4, we get the result. □

THEOREM 6. *Let M be a hypersurface of $\bar{M}$ equipped with a hypercosymplectic 3-structure $(\phi_a, \xi_a, \eta_a, g)$, $a = 1,2,3$, such that the structure vector fields ξ_a, $a = 1,2,3$ are tangential to M. Then the distribution $\mathcal{E}$ is integrable.*

PROOF. We have

$$[\xi_a, \xi_b] = \nabla_{\xi_a}\xi_b - \nabla_{\xi_b}\xi_a = 0 - 0 = 0 \in \mathcal{E},$$

which implies the proof. □

THEOREM 7. *Let M be a hypersurface of $\bar{M}$ equipped with a hypercosymplectic 3-structure $(\phi_a, \xi_a, \eta_a, g)$, $a = 1, 2, 3$, such that the structure vector fields ξ_a, $a = 1, 2, 3$ are tangential to M. Then the distribution $\mathcal{D} \oplus D^\perp$ is integrable.*

PROOF. Let $X, Y \in \mathcal{D} \oplus D^\perp$. Then in view of (24), we have

$$\begin{aligned} g([X,Y], \xi_a) &= g(\nabla_X Y, \xi_a) - g(\nabla_Y X, \xi_a) \\ &= -g(Y, \nabla_X \xi_a) + g(X, \nabla_Y \xi_a) = 0, \end{aligned}$$

from where it follows that $[X, Y] \in \mathcal{D} \oplus D^\perp$. □

Unlike in the case of Sasakian 3-structure, where $\mathcal{D}$ and $\mathcal{D}^\perp$ are not integrable, in view of Theorem 7, we can state the following two corollaries.

COROLLARY 8. *Let M be a hypersurface of $\bar{M}$ equipped with a hypercosymplectic 3-structure $(\phi_a, \xi_a, \eta_a, g)$, $a = 1, 2, 3$, such that the structure vector fields ξ_a, $a = 1, 2, 3$ are tangential to M. Then the distribution $\mathcal{D}$ is integrable if and only if*

$$g([X,Y], Z) = 0, \qquad X, Y \in \mathcal{D},\ Z \in \mathcal{D}^\perp.$$

COROLLARY 9. *Let M be a hypersurface of $\bar{M}$ equipped with a hypercosymplectic 3-structure $(\phi_a, \xi_a, \eta_a, g)$, $a = 1, 2, 3$, such that the structure vector fields ξ_a, $a = 1, 2, 3$ are tangential to M. Then the distribution $\mathcal{D}^\perp$ is integrable if and only if*

$$g([X,Y], Z) = 0, \qquad X, Y \in \mathcal{D}^\perp,\ Z \in \mathcal{D}.$$

THEOREM 10. *Let M be a hypersurface of $\bar{M}$ equipped with a hypercosymplectic 3-structure $(\phi_a, \xi_a, \eta_a, g)$, $a = 1, 2, 3$, such that the structure vector fields ξ_a, $a = 1, 2, 3$ are tangential to M. Then the following statements are equivalent:*

(a) *the hypersurface M is $\mathcal{D}$-geodesic.*

(b) *the distribution $\mathcal{D} \oplus E$ is integrable.*

(c) *the second fundamental form h of the immersion of M satisfies*

$$h(X, \phi_a Y) = h(\phi_a X, Y), \qquad a = 1, 2, 3, \quad X, Y \in \mathcal{D}. \tag{29}$$

PROOF. First we prove (**a**) $\Rightarrow$ (**b**). Let the hypersurface M be $\mathcal{D}$-geodesic, that is,

$$h(X,Y)=0, \qquad X,Y\in\mathcal{D}. \tag{30}$$

In view of Lemma 4 and Theorem 6, to prove the integrability of $\mathcal{D}\oplus E$ it is sufficient to prove that $[X,Y]\in\mathcal{D}\oplus E$ for all $X,Y\in\mathcal{D}$. Since $\omega_a(X)$ and $\eta_a(X)$ are zero for all $X\in\mathcal{D}$, using (30) in (23), we have

$$0=-(\nabla_X\omega_a)Y \qquad \text{or} \qquad \omega_a(\nabla_XY)=0.$$

Therefore, we get

$$g([X,Y],X_a)=\omega_a([X,Y])=\omega_a(\nabla_XY)-\omega_a(\nabla_YX)=0, \quad a=1,2,3.$$

Hence $[X,Y]\in\mathcal{D}\oplus E$ for all $X,Y\in\mathcal{D}$. Thus we have (**a**) $\Rightarrow$ (**b**).

Next, we prove (**b**) $\Rightarrow$ (**c**). Assume that the distribution $\mathcal{D}\oplus E$ is integrable. Then for all $X,Y\in\mathcal{D}$ we get

$$0=g([X,Y],X_a)=\omega_a(\nabla_XY)-\omega_a(\nabla_YX).$$

Therefore, in view of (23) we get

$$h(X,\phi_aY)=-\omega_a(\nabla_XY)=-\omega_a(\nabla_YX)=h(Y,\phi_aX), \qquad a=1,2,3.$$

In last, we prove that (**c**) $\Rightarrow$ (**a**). Assuming (**c**) we have for all $X,Y\in\mathcal{D}$:

$$\begin{aligned} h(\phi_3X,Y) &= h(X,\phi_3Y)=h(X,(\phi_1\circ\phi_2)Y)\\ &= h((\phi_2\circ\phi_1)X,Y)=-h(\phi_3X,Y). \end{aligned}$$

Thus $h(\phi_3X,Y)=0$, which implies that the hypersurface M is $\mathcal{D}$-geodesic because ϕ_3 is an automorphism on $\mathcal{D}$. □

THEOREM 11. *Let M be a hypersurface of $\bar{M}$ equipped with a hyper-cosymplectic 3-structure (ϕ_a,ξ_a,η_a,g), $a=1,2,3$, such that the structure vector fields ξ_a, $a=1,2,3$ are tangential to M. Then the following statements are equivalent:*

(**a**) *the distribution $\mathcal{D}^{\perp}\oplus E$ is integrable.*

(**b**) *the hypersurface M is $(\mathcal{D},\mathcal{D}^{\perp})$-geodesic, that is,*

$$h(X,Y)=0, \qquad X\in\mathcal{D}, Y\in\mathcal{D}^{\perp}. \tag{31}$$

PROOF. For all $X,Y\in\mathcal{D}$, we have

$$\begin{aligned} g([X_1,X_2],\phi_3X) &= g(\bar\nabla_{X_1}X_2-\bar\nabla_{X_2}X_1,\phi_3X)\\ &= g(\bar\nabla_{X_1}\phi_3X_1+\bar\nabla_{X_2}\phi_3X_2,\phi_3X)\\ &= g\left(\phi_3\bar\nabla_{X_1}X_1+\phi_3\bar\nabla_{X_2}X_2,\phi_3X\right)\\ &= g(\nabla_{X_1}X_1,X)+g(\nabla_{X_2}X_2,X). \end{aligned}$$

Similarly, we obtain

$$g([X_2, X_3], \phi_1 X) = g(\nabla_{X_2} X_2, X) + g(\nabla_{X_3} X_3, X)$$

and

$$g([X_3, X_1], \phi_2 X) = g(\nabla_{X_3} X_3, X) + g(\nabla_{X_1} X_1, X).$$

By Lemma 5 and Theorem 7, the distribution $\mathcal{D}^{\perp} \oplus E$ is integrable if and only if

$$g(\nabla_{X_a} X_a, X) = 0, \qquad X \in \mathcal{D}. \tag{32}$$

Now, we get

$$\begin{aligned} g(\nabla_{X_a} X_a, X) &= g(\phi_a \bar{\nabla}_{X_a} X_a, \phi_a X) = -g(\bar{\nabla}_{X_a} N, \phi_a X) \\ &= g(AX_a, \phi_a X) = h(X_a, \phi_a X). \end{aligned} \tag{33}$$

Thus from (32) and (33) the two statements are equivalent. □

5. Geometry of leaves

In this section we find minimum sufficient condition for leaves of distribution $\mathcal{D} \oplus E$ (resp. $\mathcal{D}^{\perp} \oplus E$) to be totally geodesic immersed in $\bar{M}$ (resp. M).

THEOREM 12. *Let M be a hypersurface of $\bar{M}$ equipped with a hyper-cosymplectic 3-structure $(\phi_a, \xi_a, \eta_a, g)$, $a = 1, 2, 3$, such that the structure vector fields ξ_a, $a = 1, 2, 3$ are tangential to M. If the distribution $\mathcal{D} \oplus E$ is integrable then each leaf of $\mathcal{D} \oplus E$ is totally geodesic immersed in $\bar{M}$.*

PROOF. Let M' be a leaf of $\mathcal{D} \oplus E$. We denote by h' the second fundamental form of the immersion of M' in $\bar{M}$ and by ∇' the Riemannian connection induced by $\bar{\nabla}$ on M'. Then we get

$$\bar{\nabla}_X Y = \nabla'_X Y + h'(X, Y), \qquad X, Y \in TM'. \tag{34}$$

Since $\mathcal{D} \oplus E$ is invariant by each ϕ_a, $a = 1, 2, 3$, from (34) taking account of (11), we obtain

$$\begin{aligned} h'(X, \phi_a Y) &= \bar{\nabla}_X \phi_a Y - \nabla'_X \phi_a Y = \phi_a \bar{\nabla}_X Y - \nabla'_X \phi_a Y \\ &= \phi_a \nabla'_X Y + \phi_a h'(X, Y) - \nabla'_X \phi_a Y. \end{aligned} \tag{35}$$

Taking normal parts to TM' in (35), we get

$$\begin{aligned} h'(X, \phi_a Y) &= \phi_a h'(X, Y) = (\phi_b \circ \phi_c) h'(X, Y) = \phi_b h'(X, \phi_c Y) \\ &= h'(\phi_b X, \phi_c Y) = h'((\phi_c \circ \phi_b) X, Y) = -h'(\phi_a X, Y) \\ &= -h'(X, \phi_a Y). \end{aligned}$$

Hence we get

(36) $$h'(X,\phi_a Y)=0, \qquad X,Y\in\mathcal{D}\oplus E.$$

Since ϕ_a is an automorphism of $\mathcal{D}\oplus\{\xi_b\}\oplus\{\xi_c\}$ from (36) it follows that

(37) $$h'(X,Z)=0, \qquad X\in\mathcal{D}\oplus E,\ Z\in\mathcal{D}\oplus\{\xi_b\}\oplus\{\xi_c\}.$$

Next, (36) is valid if we replace ϕ_a by ϕ_b, ϕ_c. Thus we get

(38) $$h'(X,\xi_a)=h'(X,\phi_b\xi_c)=0, \qquad X\in\mathcal{D}\oplus E.$$

By (37) and (38) it follows that M' is totally geodesic immersed in $\bar{M}$. □

THEOREM 13. *Let M be a hypersurface of $\bar{M}$ equipped with a hypercosymplectic 3-structure (ϕ_a,ξ_a,η_a,g), $a=1,2,3$, such that the structure vector fields ξ_a, $a=1,2,3$ are tangential to M. If the distribution $\mathcal{D}^\perp\oplus E$ is integrable then each leaf of $\mathcal{D}^\perp\oplus E$ is totally geodesic immersed in M.*

PROOF. Let M^* be a leaf of $\mathcal{D}^\perp\oplus E$. We denote by h^* the second fundamental form of the immersion of M^* in M. By using (8), (11) and Gauss and Weingarten formulae we get for $X\in\mathcal{D}^\perp\oplus E$, $Z\in\mathcal{D}$ we have

$$\begin{aligned} g(\nabla_X X_1,Z) &= g\left(\bar{\nabla}_X X_1,Z\right)=g\left(\bar{\nabla}_X\phi_1 N,Z\right)=g\left(\phi_1\bar{\nabla}_X N,Z\right)\\ &= -g\left(\bar{\nabla}_X N,\phi_1 Z\right)=g\left(AX,\phi_1 Z\right)=h\left(X,\phi_1 Z\right), \end{aligned}$$

which, in view of Theorem 11 and Lemma 3, gives

$$g(\nabla_X X_1,Z)=0.$$

Therefore we have

(39) $$g(\nabla_X X_a,Z)=0, \qquad a=1,2,3$$

for all $X\in\mathcal{D}^\perp\oplus E$, $Z\in\mathcal{D}$. On the other hand for $X\in\mathcal{D}^\perp\oplus E$, $Z\in\mathcal{D}$ we get

(40) $$g(\nabla_X\xi_a,Z)=g\left(\bar{\nabla}_X\xi_a,Z\right)=0.$$

Hence from (39) and (40) by using the equation of Gauss for immersion of M^* in M we obtain

$$g(h^*(X,Y),Z)=0, \qquad X,Y\in\mathcal{D}^\perp\oplus E,\ Z\in\mathcal{D}.$$

Thus each leaf M^* of $\mathcal{D}^\perp\oplus E$ is totally geodesic immersed in M. □

References

[1] A. Bejancu, *Hypersurfaces of manifolds with a Sasakian 3-structure*, Demonstratio Mathematica **17** (1984), no. 1, 197–209.

[2] D. E. Blair, *Contact manifolds in Riemannian geometry*, Lecture Notes in Math. **509**, Springer Verlag, 1976.

[3] S. Ishihara, *Quaternion Kählerian manifolds*, J. Differential Geometry **9** (1974), 483–500.

[4] T. Kashiwada, *A note on a Riemannian space with Sasakian 3-structure*, Natural Science Report, Ochanomizu University **22** (1971), no. 1, 1–2.

[5] Y.-Y. Kuo, *On almost contact 3-structure*, Tôhoku Math. J. **22** (1970), 325–332.

[6] F. Martin Cabrera, *Almost hyper-Hermitian structures in bundle space over manifolds with almost contact 3-structure*, Czechoslovak Math. J. **48(123)** (1998), no. 3, 545–563.

[7] K. Yano and M. Kon, *Structures on a manifold*, World Scientific, 1984.

Yeong-Moo Song and Jeong-Sik Kim
Department of Mathematics Education
Sunchon National University
Chonnam 540-742, Korea
E-mail: ymsong@sunchon.ac.kr
jskim01@hanmir.com

Mukut Mani Tripathi
Department of Mathematics and Astronomy
Lucknow University
Lucknow 226 007, India
E-mail: mm_tripathi@hotmail.com