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CORRELATION DIMENSIONS OF CANTOR-LIKE SETS

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Abstract. In this paper, we calculate the lower and upper bound of the correlation dimension([4], [6]) for a Cantor-like set([5]).

1. Introduction

The most basic tool to characterize fractal sets or attractors is their dimensions, such as the Hausdorff dimension, correlation dimension and box dimension, etc. To explain fractal sets or attractors induced by a dynamics in statistical mechanics or physics, the correlation dimension is usually used rather than the Hausdorff dimension or box dimension([4], [6]).

So far, considerable theories of the dimension are usually studied on the invariant set which is generated by an iterated function system of a family of contractions(≡IFS) or by the Moran construction. However, Pesin and Weiss([5]) have defined Cantor-like sets by a symbolic construction without regarding to IFS and the Moran construction. They have obtained the lower bound and upper bound of the Hausdorff dimension and box dimension of a Cantor-like set F. They have extended many of the known results on the dimensions of the set.

In this paper, we get the lower bound and upper bound of the correlation dimension of F calculating the energy with respect to an equilibrium measure on F. We notice that our method is a little simpler than one in Pesin and Weiss to have the upper bound of the Hausdorff dimension of F(see the Remark 3.5). In addition, we obtained a sufficient condition

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on which the correlation dimension is exactly equal to the Hausdorff dimension. Finally in this paper, we introduce an example of a Cantor-like set generated by an infinite contractive maps.

2. Preliminaries

Fix $p \geq 2$.

Denote $\Sigma_p \equiv \{1, 2, \cdots, p\}^{\mathbb{N}} = \{(i_1, i_2, \cdots) : i_j \in \{1, \cdots, p\}, j \geq 1\}$. Consider a compact set $Q \subset \Sigma_p$ which is invariant under the shift map σ(i.e. $\sigma(Q) = Q$) and a family of compact sets $\{\Delta_{i_1, i_2, \cdots, i_k}\}$ in $\mathbb{R}^d$ $(1 \leq i_j \leq p)$ satisfy that the k-tuples $(i_1, i_2, \cdots, i_k)$ are Q-admissible $(k \geq 1)$ (i.e., there exists $w = (i'_1, i'_2, \cdots) \in Q$ such that $i'_j = i_j$ for $1 \leq j \leq k$) and

$$\lim_{k \to \infty} \max_{(i_1, i_2, \cdots, i_k)} \operatorname{diam}(\Delta_{i_1, i_2, \cdots, i_k}) = 0,$$

where $\operatorname{diam}(A) = \sup\{|x - y| : x, y \in A\}$ for a subset A of $\mathbb{R}^d$.

We assume that for any admissible sequences $(i_1, i_2, \cdots, i_k) \in \{1, 2, \cdots, p\}^k$, $\Delta_{i_1, \cdots, i_{k+1}} \subset \Delta_{i_1, \cdots, i_k}$ and $\Delta_{i_1, \cdots, i_k} \cap \Delta_{i'_1, \cdots, i'_k} = \emptyset$ $(i_j \neq i'_j$ for some $j)$.

Set

$$F = \bigcap_{k=1}^{\infty} \bigcup_{(i_1, \cdots, i_k): Q\text{-admissible}} \Delta_{i_1, i_2, \cdots, i_k}.$$

We call F *a Cantor-like set* of $\mathbb{R}^d$([5]). For $x \in F$ and $n > 0$, write $\Delta_{i_1, i_2, \cdots, i_n}(x)$ for $\Delta_{i_1, \cdots, i_n}$ containing x. We define a bijective map Π from F to Q as $\Pi(x) = (i_1, i_2, \cdots, i_n, \cdots)$, where $x = \cap_{n=1}^{\infty} \Delta_{i_1, \cdots, i_n}(x)$.

Given a p-tuple $\dot{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_p)$ with $0 < \alpha_i < 1$, there exists a unique number s_α such that $P(s_\alpha \log \alpha_{i_1}) = 0$, where P is the topological pressure([1], [5]). Let μ_α be an equilibrium measure([1], [5]) for the function $\phi(i_1, i_2, \cdots) = \log \alpha_{i_1}$ on Q, and let m_α be the *pull back measure* on F under the map Π(i.e., $m_\alpha = \mu_\alpha \circ \Pi$).

We recall the following definition of the *correlation dimension* of $A(\subset \mathbb{R}^d)$ with respect to a probability measure η on A([6]);

$$D_2(A, \eta) = \sup\{s \geq 0 : I_s(\eta) < \infty\},$$

where $I_s(\eta) = \int_A \int_A |x - y|^{-s} d\eta(x) d\eta(y)$ is the s-energy of A with respect to η.

REMARK 2.1. In the definition of $D_2(A,\eta)$, we can easily see that

$$D_2(A,\eta) = \inf\{s \geq 0 : I_s(\eta) = \infty\} = \inf\{s \geq 0 : I_s(\eta) > 0\}.$$

In order to calculate the correlation dimension of the Cantor-like set F, we have moderate control over the spacing or sizes of the basic sets constructing F.

Given $0 < r < 1$, a vector of numbers $\gamma = (\gamma_1, \cdots, \gamma_p)$ with each $0 < \gamma_i < 1$, and any $x \in F$. we can find an integer $n(x)$ such that $\gamma_{i_1} \cdots \gamma_{i_{n(x)+1}} \leq r < \gamma_{i_1} \cdots \gamma_{i_{n(x)}}$ where $\Pi(x) = (i_1, i_2, \cdots)$. Plainly, $x \in \Delta_{i_1,i_2,\cdots,i_{n(x)}}$. We note that if $y \in \Delta_{i_1,i_2,\cdots,i_{n(x)}}$ and $n(y) \geq n(x)$, then $\Delta_{i_1,\cdots,i_{n(y)}} \subset \Delta_{i_1,\cdots,i_{n(x)}}$.

Let $\Delta(x)$ be the largest basic set containing x with the property that $\Delta(x) = \Delta_{i_1,\cdots,i_{n(z)}}$ for some $z \in \Delta(x)$ and $\Delta_{i_1,\cdots,i_{n(y)}} \subset \Delta(x)$ for any $y \in \Delta(x)$. Then the sets $\Delta(x)$ corresponding to different $x \in F$ either coincide or are disjoint. We denote these sets by $\Delta_r^{(j)}$, $j = 1, 2, \cdots, N_r$. These sets form a disjoint cover of F([5]).

Let $N(x,r)$ denote the number of sets $\Delta_r^{(j)}$ that have non-empty intersection with the open ball $B(x,r)$ centered x and radius r.

DEFINITION 2.2. ([5]) A vector γ is called an *l-estimating* if $N(x,r)$ is less than some constant, uniformly in x and r. We call a symbolic construction *regular* if it admits an *l*-estimating vector.

REMARK 2.3. ([5]) If $\gamma = (\gamma_1, \cdots, \gamma_p)$ is an *l*-estimating vector for a regular symbolic construction, then any vector $\alpha = (\alpha_1, \cdots, \alpha_p)$ for each $\alpha_i \leq \gamma_i$, $i = 1, \cdots, p$ is also an *l*-estimating.

In order to obtain upper estimates for the correlation dimension, we use that the diameters of the basic sets decrease exponentially.

DEFINITION 2.4. ([5]) A vector $\lambda = (\lambda_1, \cdots, \lambda_p)$, $0 < \lambda_i < 1$ is called a *u-estimating vector* for a construction if $\operatorname{diam}(\Delta_{i_1,\cdots,i_n}) \leq C\Pi_{j=1}^n \lambda_{i_j}$ for a constant $C > 0$. The symbolic construction is called *bounded* if there exists a *u*-estimating vector.

REMARK 2.5. (1) If $\lambda = (\lambda_1, \cdots, \lambda_p)$ is a *u*-estimating vector for a bounded symbolic construction, then any vector $\beta = (\beta_1, \cdots, \beta_p)$ for each $\beta_i \geq \lambda_i$, $i = 1, \cdots, p$ is also *u*-estimating.

(2) If the diameter of $\Delta_{i_1,\cdots,i_n}$ decreases very slowly to 0, then the symbolic construction is not bounded. For example, consider diam $\Delta_{i_1,\cdots,i_n} = \frac{1}{n+1}$, $n \in \mathbb{N}$, then there is no such constant C.

3. Results

Since the shift map on a general symbolic space is expansive, the existence of an equilibrium measure on the space is known in [7]. However, in general, this equilibrium measure is not always a Gibbs measure. For our construction, if the Gibbs measure exists, then we can easily get the following results by the boundedness of the Gibbs measure.

Throughout this paper, let F, Q, Π, μ, m and P be as in Section 2.

LEMMA 3.1. ([5]) *Let F be a Cantor-like set for a regular symbolic construction and let μ_γ be an equilibrium measure for any l-estimating vector γ. Let the number s_γ be satisfying $P(s_\gamma \log \gamma_{i_1}) = 0$. Then for $r > 0$, $\alpha > 0$ and all $x \in \Pi^{-1}(Q_k)$,*

$$m_\gamma(B(x,r) \cap \Pi^{-1}(Q_k) \leq L \cdot r^{s_\gamma - \alpha},$$

where the constant $L = L(k) > 0$ for sufficiently large number k and the set $Q = \cup_{i=1}^{\infty} Q_i$ (μ_γ-almost everywhere).

THEOREM 3.2. *Let F be as in the Lemma 3.1. Then $s_\gamma \leq D_2(F, m_\gamma)$ for the pull-back measure m_γ on F.*

PROOF. Given a vector $\gamma = (\gamma_1, \cdots, \gamma_p)$, $0 < \gamma_i < 1 (i = 1, 2, \cdots, p)$. Let μ_γ be an equilibrium measure on Q corresponding to the function $\log \gamma_{i_1}$. For each $w = (i_1, \cdots, i_n, \cdots) = \Pi(x) \in Q_k$. Put $k_0 \equiv \min\{k : \mu_\gamma(Q_k) > 0\}$. Fix $k \geq k_0$ and $0 < r < 1$.

Now, in order to calculate the energy of F with respect to the measure m_γ, we put $\phi_t(x) = \int_F |x-y|^{-t} dm_\gamma(y)$. We have, using the Theorem in [3] and the Lemma 3.1,

$$\begin{aligned}
\phi_t(x) &= \int_{F \cap \Pi^{-1}(Q_k)} |x-y|^{-t} dm_\gamma(y) \\
&= \int_0^\infty m_\gamma(\{y \in F \cap \Pi^{-1}(Q_k) : |x-y|^{-t} \geq r\}) dr \\
&= \int_0^\infty m_\gamma(B(x, r^{-1/t}) \cap \Pi^{-1}(Q_k)) dr \\
&= t \int_0^\infty \epsilon^{-t-1}\, m_\gamma(B(x,\epsilon) \cap \Pi^{-1}(Q_k)) d\epsilon \\
&< t\left[\int_0^1 \epsilon^{-t-1} m_\gamma(B(x,\epsilon) \cap \Pi^{-1}(Q_k)) d\epsilon \;+\; \int_1^\infty \epsilon^{-t-1} m_\gamma(F) d\epsilon\right] \\
&\leq Lt \int_0^1 \epsilon^{s_\gamma - \alpha - t - 1}\, d\epsilon \;+\; m_\gamma(F) \;< \infty
\end{aligned}$$

for all $0 \leq t < s_\gamma - \alpha$. Hence, for all $t < s_\gamma - \alpha$,

$$I_t(m_\gamma) = \int_F \phi_t(x) dm_\gamma(x) < \infty,$$

which implies $D_2(F, m_\gamma) \geq s_\gamma - \alpha$. Since α was arbitrarily small, $D_2(F, m_\gamma) \geq s_\gamma$. □

COROLLARY 3.3. *Let F be the Cantor-like set by a symbolic construction with exponentially large gaps ; there exists a number $0 < \beta < 1$ such that $d(\Delta_{i_1,\cdots,i_n}, \Delta_{j_1,\cdots,j_n}) \geq C\beta^n$ where $C > 0$ and $d(A, B) = \inf\{|x - y| : x \in A,\ y \in B\}$. Then*
(1) the construction is regular with l-estimating vector $\gamma = (\beta, \cdots, \beta)$.
(2) $s_\gamma \leq D_2(F, m_\gamma)$, where m_γ is an equilibrium measure on F of the vector γ.

THEOREM 3.4. *Let F be the Cantor-like set for a bounded symbolic construction and let m_λ be an equilibrium measure on F for any u-estimating vector λ. Then $D_2(F, m_\lambda) \leq s_\lambda$.*

PROOF. In [4], we have the upper bound of the Hausdorff dimension of F, $\dim_H F \leq s_\lambda$, for the number s_λ satisfying $P(s_\lambda \log \lambda_{i_1}) = 0$. Owing to the relation of the correlation dimension and Hausdorff dimension, $D_2(F, m_\lambda) \leq \dim_H F \leq s_\lambda$. □

REMARK 3.5. In the proof of Theorem 3.4, we can obtain that $D_2(F, m_\lambda) \leq s_\lambda$ by calculating the energy, *i.e.* for all $t > s_\lambda$,

$$I_t(m_\lambda) = \iint_F |x - y|^{-t}\ dm_\lambda(x)\ dm_\lambda(y) = \infty.$$

So we can get $\dim_H F \leq s_\lambda$(see [2]).

COROLLARY 3.6. *Let F be the Cantor-like set by Moran symbolic* construction ; each basic set $\Delta_{i_1,\cdots,i_n}$ satisfies

$$B(x, C_1 \prod_{j=1}^{n} \lambda_{i_j}) \subset \Delta_{i_1,\cdots,i_n} \subset B(x, C_2 \prod_{j=1}^{n} \lambda_{i_j}),$$

where $0 < \lambda_i < 1 (i = 1, \cdots, p)$ and constants $C_1, C_2 > 0$. Then
(1) the construction is regular and bounded with l-estimating vector and u-estimating vector equal to $\lambda = (\lambda_1, \cdots, \lambda_p)$.

(2) $s_\lambda = D_2(F, m_\lambda)$ *where* m_λ*is an equilibrium measure on* F *of the vector* λ. *Moreover, owing to the result in* [5], *we can get* $\dim_H F = s_\lambda = D_2(F, m_\lambda)$.

The following Theorem is an immediate consequence from above Theorems and the results of [5].

THEOREM 3.7. *Let* F *be the Cantor-like set for a regular and bounded symbolic construction. Then* $s_\gamma \leq D_2(F, m_\gamma)$ *and* $D_2(F, m_\lambda) \leq s_\lambda$, *where* s_γ *and* s_λ *are the unique solutions of the equations* $P(s_\gamma \log \gamma_{i_1}) = 0$ *and* $P(s_\lambda \log \lambda_{i_1}) = 0$, *respectively, for an l-estimating vector* γ *and u-estimating* λ. *Moreover, if* $\gamma_i = \lambda_i$ *for* $i = 1, \cdots, p$, *then*

$$s_\gamma = s_\lambda = D_2(F, m_\gamma) = D_2(F, m_\lambda) = \dim_H F.$$

For the case of simple construction([5]), $Q = \Sigma_p$, using some operator theories in [5], we have the following results for the correlation dimension which is similar to results in [5].

COROLLARY 3.8. *Let* F *be the set for a simple regular and bounded construction and let* μ_γ *and* μ_λ *be equilibrium measures for any l-estimating vector* γ *and any u-estimating vector* λ *respectively. Then, for the pull-back measures* m_γ *and* m_λ *on* F,
(1) $s_\gamma \leq D_2(F, m_\gamma)$ *and* $D_2(F, m_\lambda) \leq s_\lambda$, *where* s_γ *and* s_λ *are the unique solutions of the equations* $\sum_{i=1}^p \gamma_i^t = 1$ *and* $\sum_{i=1}^p \lambda_i^t = 1$, *respectively.*
(2) *In particular, if the measures* μ_γ *and* μ_λ *are Gibbs measures in simple construction, then the measures satisfy the following equations*

$$\mu_\gamma(C_{i_1,\cdots,i_n}) = \prod_{j=1}^n \gamma_{i_j}^{s_\gamma} \quad \text{and} \quad \mu_\lambda(C_{i_1,\cdots,i_n}) = \prod_{j=1}^n \lambda_{i_j}^{s_\lambda},$$

where $C_{i_1,\cdots,i_n}$ *is a cylinder set.*

4. Examples

EXAMPLE 4.1. Consider the middle third Cantor set F constructed by repeated removal of the middle third of intervals on $[0, 1]$. Then, for a vector $\gamma = (\frac{1}{3}, \frac{1}{3})$,
(1) F is an regular and bounded set of the construction for the vector γ.
(2) $\dim_H F = \frac{\log 2}{\log 3} = D_2(F, m_\gamma)$ about the Gibbs measure m_γ on F.

PROOF. Let $\Sigma_3 = \{1,2,3\}^{\mathbb{N}}$. Put $Q = \{1,2\}^{\mathbb{N}}$. Then $Q \subset \Sigma_3$. Since the diameter of basic intervals, $\operatorname{diam}(\Delta_{i_1,\cdots,i_n}) = (\frac{1}{3})^n$ for all $(i_1,\cdots,i_n) \in \{1,2\}^n$, each $i_j \in \{1,2\}$, we can easily see that the given vector $\gamma = (\frac{1}{3},\frac{1}{3})$ is an l-estimating and u-estimating vector of the construction.

Put s_γ satisfying $P(s_\gamma \cdot \log \gamma_{i_1}) = 0$. Define a probability measure m_γ on F by for any $(i_1,\cdots,i_n) \in \{1,2\}^n$ and $n \geq 1$,

$$m_\gamma(\Delta_{i_1,\cdots,i_n}) = \left(\frac{1}{3}\right)^{ns_\gamma}$$

We can see that the probability measure m_γ on F is an equilibrium measure and the Gibbs measure on F. Also we obtain that $s_\gamma = \frac{\log 2}{\log 3}$. In [1], we know that $\dim_H F = s_\gamma$. We can easily check that $D_2(F, m_\gamma) = s_\gamma$ by calculating the potential energy. Hence $\dim_H F = \frac{\log 2}{\log 3} = D_2(F, m_\gamma)$ for the Gibbs measure m_γ on F. □

The following example is a generalization of an loosely self-similar set ([2]) which is constructed by an infinite contractive maps.

EXAMPLE 4.2. Consider a sequence of contraction maps $\{\psi_{i_1,\cdots,i_n}\}$ for $(i_1,\cdots,i_n) \in \{1,2\}^n$ $(n = 1,2,\cdots)$ as follows; for all $x, y \in [0,1]$,

$$\psi_{i_1,\cdots,i_n} = \begin{cases} (r_{i_1,\cdots,i_n}x,\ r_{i_1,\cdots,i_n}y) & (i_n = 1) \\ (r_{i_1,\cdots,i_n}x + \frac{8}{9},\ r_{i_1,\cdots,i_n}y) & (i_1 = 1,\ i_n = 2) \\ (r_{i_1,\cdots,i_n}x + \frac{8}{9},\ r_{i_1,\cdots,i_n}y + \frac{8}{9}) & (i_1 = 2,\ i_n = 2), \end{cases}$$

where $r_{i_1,\cdots,i_{n-1},1} \in (\frac{1}{9},\frac{4}{9}]$ and $r_{i_1,\cdots,i_{n-1},2} \in (0,\frac{1}{9}]$ for all $n = 1,2,\cdots$. Put

$$\Delta_{i_1,i_2,\cdots,i_n} = \psi_{i_1} \circ \psi_{i_1,i_2} \circ \cdots \circ \psi_{i_1,\cdots,i_n}([0,1] \times [0,1])$$

and set

$$F = \bigcup_{n=1}^{\infty} \bigcap_{(i_1,\cdots,i_n)\in\{1,2\}^n} \Delta_{i_1,i_2,\cdots,i_n}.$$

Then (1) the set F is a Cantor-like set for simple regular symbolic construction for an l-estimating vector $\gamma = (\frac{1}{9},\frac{1}{9})$.
(2) The set F is a Cantor-like set for simple bounded symbolic construction for an u-estimating vector $\lambda = (\frac{4}{9},\frac{1}{9})$.
(3) $\frac{\log 2}{2\log 3} = s_\gamma \leq D_2(F, m_\gamma)$ and $D_2(F, m_\lambda) \leq s_\lambda = \frac{1}{2}$ for the Gibbs measures m_γ and m_λ, for the given vectors γ and λ.

REMARK 4.3. In Example 4.2, we can take $r_{i_1,\cdots,i_n} = r_{i_n}$ for each $i_j \in \{1,2\}$ $(j = 1,2,\cdots,n)$. Then the set F is a loosely self-similar set([2]). In particular, if we take $r_{i_n} = \frac{4}{9}$ $(i_n = 1)$ and $r_{i_n} = \frac{1}{9}$ $(i_n = 2)$ for $n = 1,2,\cdots$, then a vector $\gamma = (\frac{4}{9}, \frac{1}{9})$ is an l-estimating and u-estimating vector for the set F. Moreover, owing to the fact $\dim_H F = \frac{1}{2}$([2]), we have $D_2(F, m_\gamma) = \frac{1}{2} = \dim_H F$ for the Gibbs measure m_γ on F.

References

[1] K. Falconer, *Techniques in Fractal Geometry*, Mathematical Foundations and Applications, John Wiley & Sons (1997).
[2] S. Ikeda, *On loosely self-similar sets*, Hiroshima Math. J. **25** (1995), 527–540.
[3] Pertti Mattila, *Geometry of Sets and Measures in Euclidean spaces*, Cambridge Univ. Press, 1995.
[4] Y. Pesin, *On rigorous definitions of correlation dimension and generalized spectrum for dimensions*, J. Statist. Phys. **71** (1993), 529–547.
[5] Y. Pesin and H. Weiss, *On the dimension of deterministic and random Cantor-like sets, symbolic dynamics, and the Eckmann-Ruelle conjecture*, (preprint).
[6] T. D. Sauer and J. A. Yorke, *Are the dimensions of a set and its images equal under typical smooth functions ?*, Erg. Th. & Dyn. Syst. **17** (1997), 941–956.
[7] P. Walters, *Introduction to Ergodic Theory*, Springer-Verlag, 1982.

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