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ON THE STABILITY OF MAPPINGS IN BANACH ALGEBRAS

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ABSTRACT. We obtain the stability of additive, multiplicative map -pings and derivations in Banach algebras.

1. Introduction

A mapping f from a ring G into a normed algebra A is *approximately additive* if there is a $\delta > 0$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \delta$$

for all $x, y \in G$. A mapping $f : G \longrightarrow A$ is *approximately multiplicative* if there is an $\varepsilon > 0$ such that

$$\|f(xy) - f(x)f(y)\| \leq \epsilon$$

for all $x, y \in G$. A linear mapping D from a Banach algebra A into A is an *approximate derivation* if there is $\eta > 0$ such that

$$\|D(ab) - D(a)b - aD(b)\| \leq \eta$$

for all $a, b \in A$.

S. M. Ulam [13] proposed the stability question; Give conditions in order for an additive mapping near an approximately additive mapping to exist. The case of approximately additive mappings between Banach spaces was solved by D. H. Hyers [6]. In 1968, S. M. Ulam [13] proposed the more general problem: When is it true that by changing the hypothesis of Hyers' theorem a little one can still assert that the thesis

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of the theorem remains true or approximately true! Th. M. Rassias [11] proved a substantial generalization of the result of Hyers. Taking this fact into account, the additive functional equation is said to have the Hyers-Ulam-Rassisas stability. And many authors answered the Ulam's question for several cases [2-10, 12].

In this paper, we give conditions that approximately additive, approximately multiplicative mappings, and approximate derivations are additive, multiplicative mappings, and derivations, respectively.

2. Stability of additive and multiplicative mappings

In this section we give conditions that approximately additive and also approximately multiplicative mappings from a ring into a commutative semisimple Banach algebra are additive and also multiplicative.

THEOREM 2.1. *Let G be a ring and A a commutative semisimple Banach algebra. If $f : G \longrightarrow A$ is an approximately additive and approximately multiplicative mapping, then either there exists a nonzero multiplicative linear functional ϕ on A such that*

$$|\phi f(x)| \leq \min\left\{\delta, \frac{1+\sqrt{1+4\epsilon}}{2}\right\}$$

for all $x \in G$ and for some $\delta, \varepsilon > 0$, or f is additive and multiplicative.

PROOF. Suppose that there exists $\delta > 0$, $\epsilon > 0$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \delta$$

for all $x, y \in G$ and

$$\|f(xy) - f(x)f(y)\| \leq \epsilon$$

for all $x, y \in G$.

For any nonzero multiplicative linear functional $\phi : A \longrightarrow C$,

$$\|\phi f(x+y) - \phi f(x) - \phi f(y)\| \leq \delta$$

for all $x, y \in G$ and

$$\|\phi f(xy) - \phi f(x)\phi f(y)\| \leq \epsilon$$

for every $x, y \in G$ because $\|\phi\| \leq 1$. Thus $\phi f : G \longrightarrow C$ is approximately additive and approximately multiplicative. If $\phi f = 0$ then

$$\phi f(x + y) = \phi f(x) + \phi f(y)$$

for all $x, y \in G$ and

$$\phi f(xy) = \phi f(x)\phi f(y)$$

for all $x, y \in G$.

Suppose that $|\phi f(x_0)| > \min\{\delta, \frac{1+\sqrt{1+4\epsilon}}{2}\}$ for some $x_0 \in G$. By Hyers-Ulam stability of Cauchy functional equation [4], there exists a unique additive mapping T such that

$$|\phi T(x) - \phi f(x)| \leq \|T(x) - f(x)\| \leq \delta$$

for all $x \in G$.

If $|\phi f(x_0)| > \delta$ for some $x_0 \in G$, $\phi T(x_0) \neq 0$. Thus we can choose $x_n = nx_0 \in G$ such that $|\phi T(x_n)| \rightarrow \infty$ as $n \rightarrow \infty$. Thus $|\phi f(x_n)| \rightarrow \infty$ as $n \rightarrow \infty$.

If $|\phi f(x_n)| > \eta := \frac{1+\sqrt{1+4\epsilon}}{2}$, let $p = |\phi f(x_0)| - \eta$. Note that $\eta^2 - \eta = \epsilon$ and $\eta > 1$. We have

$$\begin{aligned}
\left|\phi f(x_0^2)\right| &= \left|(\phi f(x_0))^2 + \phi f(x_0^2) - \phi f(x_0)\phi f(x_0)\right| \\
&\geq |\phi f(x_0)|^2 - \epsilon \\
&> \epsilon + 2p.
\end{aligned}$$

By induction, we have

$$\left|\phi f(x_0^{2n})\right| > \epsilon + (n+1)p$$

for all $n \in N$. Letting $x_n = x_0^{2n} \in G$, $|\phi f(x_n)| \rightarrow \infty$ as $n \rightarrow \infty$.

Now for every $x, y, z \in G$

$$\begin{aligned}
&|\phi(f(xy)f(z) - f(x)f(yz))| \\
&\leq |\phi(f(xyz) - f(x)f(yz))| + |\phi(f(xy)f(z) - f(xyz))| \\
&\leq 2\epsilon.
\end{aligned}$$

Thus

$$\begin{aligned}
&|\phi(f(xy) - f(x)f(y))\phi(f(z))| \\
&\leq |\phi(f(xy)f(z) - f(x)f(yz))| + |\phi(f(x)f(yz) - f(x)f(y)f(z))| \\
&\leq 2\epsilon + |\phi f(x)|\,\epsilon
\end{aligned}$$

for all $x, y, z \in G$. Replacing z by x_n we have

$$|\phi(f(xy) - f(x)f(y))| \leq \frac{2\epsilon + |\phi f(x)|\,\epsilon}{|\phi f(x_n)|}$$

for all $x, y \in G$. As $n \to \infty$, we have $\phi f(xy) = \phi f(x)\phi f(y)$ for every multiplicative linear functional ϕ on A, and for all $x, y \in G$. Since A is semisimple, the intersection of all multiplicative linear functionals on A is zero [2]. Thus we have

$$f(xy) = f(x)f(y)$$

for all $x, y \in G$. Therefore f is multiplicative.

Now for any $a, b \in G$, let $u = x_n a$ and $v = x_n b$. Then we have

$$\begin{aligned}
&|\phi f(x_n)|\,|\phi[f(a+b) - f(a) - f(b)]| \\
&= |\phi[f(x_n a + x_n b) - f(x_n a) - f(x_n b)]| \\
&\leq \|f(x_n a + x_n b) - f(x_n a) - f(x_n b)\| \\
&\leq \delta
\end{aligned}$$

for all $n \in N$ and all nonzero multiplicative linear functional ϕ on A.

Dividing by $|\phi f(x_n)|$ and $n \to \infty$, we have

$$\phi(f(a+b) - f(a) - f(b)) = 0$$

for every multiplicative linear functional ϕ on A. Since A is commutative semisimple, we get f is additive. □

COROLLARY 2.2. *Let G be a ring and $C(S)$ a set of all continuous functionals on a compact Hausdorff space S. Suppose that $f : G \longrightarrow C(S)$ is a mapping.*

(1) *If f is additive and approximately multiplicative, then f is multiplicative.*
(2) *If f is multiplicative and approximately additive, then either f is additive or there exists a nonzero multiplicative linear functional ϕ on A such that*

$$|\phi f(x)| \leq 1$$

for all $x \in G$.

(3) *If f is approximately additive and approximately multiplicative, then either f is additive and multiplicative or there exists a nonzero multiplicative linear functional ϕ on A such that*

$$|\phi f(x)| \leq \min\left\{\delta, \frac{1+\sqrt{1+4\epsilon}}{2}\right\}$$

for all $x \in G$ and for some $\delta, \varepsilon > 0$.

PROOF. Note that $C(S)$ is a commutative semisimple Banach algebra. (1) If f is additive, then for any $\delta > 0$ we can choose $x_0 \in G$ such that $|\phi f(x_0)| > \delta$ for all nonzero multiplicative linear functional ϕ on $C(S)$. Then $|\phi f(x_0)| > \min\{\delta, \frac{1+\sqrt{1+4\epsilon}}{2}\}$.

(2) If f is multiplicative and $|\phi f(x)| > 1$ then for any $\delta > 0$ we can choose $x_0 \in G$ such that $|\phi f(x_0)| > \delta$ for all nonzero multiplicative linear functional ϕ on $C(S)$. Then $|\phi f(x_0)| > \min\{\delta, \frac{1+\sqrt{1+4\epsilon}}{2}\}$. By Theorem 2.1, we complete the proof. □

3. Stability of derivations

In this section we give conditions that every approximate derivation is a derivation.

THEOREM 3.1. *Every approximate derivation D on a commutative semisimple Banach algebra A is zero.*

PROOF. Suppose that there exists $\delta > 0$ such that

$$\|D(ab) - aD(b) - D(a)b\| \leq \delta$$

for all $a, b \in A$. For every $a, b, c \in A$ and any nonzero linear functional ϕ on A,

$$\begin{aligned} &|\phi(c)\phi(D(ab) - D(a)b - aD(b))| \\ &\leq |\phi(cD(ab) + abD(c) - D(abc))| \\ &\quad + |\phi(-bcD(a) - aD(bc) + D(abc))| \\ &\quad + |\phi(a(D(bc) - cD(b) - bD(c)))| \\ &< 2\delta + \delta \|a\|. \end{aligned}$$

Dividing by $|\phi(c)|$ and $|\phi(c)| \to \infty$, we have

$$\phi(D(ab) - D(a)b - aD(b)) = 0$$

for all $a, b \in A$ and any multiplicative linear functional ϕ on A. Since A is semisimple,

$$D(ab) = D(a)b + aD(b)$$

for all $a, b \in A$. Thus D is a derivation. By Thomas's theorem [12], D maps into the radical of A. Since A is semisimple, the radical of A is zero. □

Since a set of all continuous functionals on a compact Hausdorff space S is a commutative semisimple Banach algebra, we have:

COROLLARY 3.2. *Every approximate derivation on a continuous function space $C(S)$ is zero.*

Now we consider noncommutative and not semisimple cases. The following theorem states that every approximate derivation on a Banach algebra with some conditions is near a zero derivation.

THEOREM 3.3. *Let A be a Banach algebra with multiplicative norm. If $D : A \longrightarrow A$ is a continuous approximate derivation such that*

$$\|D(ab) - aD(b) - D(a)b\| \leq \delta$$

for all $a, b \in A$ and $\delta > 0$ and $D(a)a = aD(a)$ for all $a \in A$, then

$$\|D\| \leq \delta.$$

PROOF. If $\|D\| > \delta$, we can choose $a \in A$ with $\|a\| = 1$ such that $\|D(a)\| > \delta$. Let $p = \|D(a)\| - \delta > 0$.

Then we have

$$\begin{aligned} \left\|D(a^2)\right\| &= \left\|2aD(a) - (2aD(a) - D(a^2))\right\| \\ &> 2\left\|D(a)\right\| - \delta = \delta + 2p. \end{aligned}$$

By induction, we get

$$\left\|D(a^{2n})\right\| > \delta + 2^n p.$$

Since $\left\|a^{2n}\right\| = 1$, it contradicts to continuity of D. Therefore $\|D\| \leq \delta$. □

REMARK 3.4. In Theorem 3.3, even if we have a condition

$$\|D(ab) - aD(b) - D(a)b\| \leq \delta \, \|a\| \, \|b\|$$

for all $a, b \in A$ instead of an approximate condition, we get $\|D\| \leq \delta$.

The following theorem states a continuous approximate derivation on a finite dimensional Banach algebra is near a derivation.

THEOREM 3.5. *Let A be a finite dimensional Banach algebra. Assume that D is a bounded linear mapping. Then there exists $m \in N$ such that if*

$$\|D(ab) - aD(b) - D(a)b\| \leq \frac{\delta}{m}$$

then there exists a derivation T on A such that for all $a \in A$

$$\|D(a) - T(a)\| < \delta.$$

PROOF. Let $\|D\| \leq K$ for some K and define as follows;

$$BD(A) = \{T \mid T : A \longrightarrow A \quad \text{is a bounded derivation}\},$$

$$d(D) = \inf \{\|D - T\| \mid T \in BD(A)\},$$

$$BL(A) = \{T \mid T : A \longrightarrow A \quad \text{is a bounded linear mapping}\},$$

$$BC = \{D \in BL(A) \mid d(D) \geq \delta \quad \text{and} \quad \|D\| \leq K\},$$

and

$$G_n = \left\{D \in BL(A) \mid \|D(ab) - aD(b) - D(a)b\| > \frac{\delta}{n}\right\}.$$

for all $n \in N$. Then G_n is an open set, $G_n \subset G_{n+1}$ and we have

$$\bigcup_{n=1}^{\infty} G_n \supset BL(A) \setminus \{D \mid D \quad \text{is a derivation on A}\} \supset BC.$$

Since BC is closed and bounded, BC is compact. Thus there exists $m \in N$ such that $BC \subset G_m$. If

$$\|D(ab) - aD(b) - D(a)b\| \leq \frac{\delta}{m}$$

then $D \in G_m$ and so $D \notin BC$. Thus we have $d(D) < \delta$. By definition of $d(D)$ there exists a bounded derivation T such that

$$\|D(a) - T(a)\| < \delta$$

for all $a \in A$. □

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