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ON PROJECTIVE BCI-ALGEBRAS

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ABSTRACT. In this paper, we obtain $Hom(P,\_\ )$ is an exact functor if P is a p-projective BCI-algebra.

1. Introduction

C. S. Hoo and P. V. R. Murty ([5]) and E. Y. Deeba and S. K. Goal ([3]) independently showed that $Hom(X)$ may not, in general, be a BCI-algebra for an arbitrary BCI-algebra X. In view of this result we can also see that $Hom(X, Y)$, the set of all homomorphisms of a BCI-algebra X into an arbitrary BCI-algebra Y may not be a BCI-algebra in general. However, E. Y. Deeba and S. K. Goal ([3]) proved that if X is a BCI-algebra and Y is a BCK-algebra, then $Hom(X, Y)$ is a BCK-algebra and hence a BCI-algebra. Y. Liu ([8]) showed that if X is a BCI-algebra and Y is a p-semisimple BCI-algebra then $Hom(X, Y)$ is a p-semisimple BCI-algebra. In [6] and [7], Y. B. Jun et al. investigated some properties of $Hom(-,-)$ as BCK/BCI-algebras. The present authors ([1]) defined a hom functor $Hom(\_,\_\ )$ in BCK/BCI-algebras and discussed the exactness of $Hom(\_,\_\ )$ in BCK/BCI-algebras, and obtained some properties of $Hom(\_,\_\ )$. In this paper, we discuss the projectivity in this sense of p-semisimple BCI-algebras and show $Hom(P,\_\ )$ is an exact functor if P is a p-projective BCI-algebra.

Recall that a *BCI-algebra* is a non-empty set X with a binary operation "$*$" and a constant 0 satisfying the axioms:
(1) $\{(x * y) * (x * z)\} * (z * y) = 0$,
(2) $\{x * (x * y)\} * y = 0$,

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(3) $x * x = 0$,
(4) $x * y = 0$ and $y * x = 0$ imply that $x = y$
for any $x, y, z \in X$. Furthermore, if it satisfies (5) $0 * x = 0$ for any $x \in X$, then the algebra is called a *BCK-algebra.* A partial ordering " $\leq$ " on X can be defined by $x \leq y$ if and only if $x * y = 0$. Let X_+ be the BCK-part of a BCI-algebra, i.e., X_+ is the set of all $x \in X$ such that $x \geq 0$. If $X_+ = \{0\}$, then X is called a *p-semisimple BCI-algebra.* A mapping $f : X \to Y$ from BCK/BCI-algebra X into a BCK/BCI-algebra Y is called a *BCK/BCI-homomorphism* if $f(x*y) = f(x)*f(y)$ for all $x, y \in X$. Define the zero homomorphism 0 as $0(x) = 0$ for all $x \in X$. Denote by $Hom(X, Y)$ the set of all homomorphisms from a BCK/BCI-algebra X into a BCK/BCI-algebra Y.

DEFINITION 1.1. Let X and Y be BCI-algebras. A BCI-homomor phism $f : X \to Y$ is said to be *regular* if Imf is an ideal of Y.

We call an ideal A of X *regular* in case A is a subalgebra of X. In the literature, Imf need not be an ideal of Y.

DEFINITION 1.2. Let $f : A \to B$ and $g : B \to C$ be BCI-homomor -phisms. The sequence $A \to B \to C$ is said to be *exact* at B if $Kerg = Imf$. A sequence $A_0 \to A_1 \to \cdots \to A_n \to A_{n+1}$ is said to be *exact* if it is exact at $A_1, \cdots, A_n$.

EXAMPLE 1.3 ([2]). 1. $0 \to A \xrightarrow{f} B$ is exact (at A) if and only if f is injective.
2. $A \xrightarrow{f} B \to 0$ is exact (at B) if and only if f is surjective.
3. The sequence $0 \to A' \xrightarrow{\mu} A \xrightarrow{\varepsilon} A'' \to 0$ is exact (at A', A, A'') if and only if μ induces an isomorphism $A' \xrightarrow{\cong} \mu A'$ and ε induces an isomorphism $A/Ker\varepsilon = A/\mu A' \xrightarrow{\cong} A''$. Essentially A' is then a regular ideal of A and A'' the corresponding quotient algebra, such a sequence is called a *short exact* sequence.

2. Hom functors

The present authors ([1]) defined a hom functor $Hom(\_,\_)$ in BCK/BCI-algebras and discussed the exactness of $Hom(\_,\_)$ in BCK/BCI-algebras. Let $\mathbb{C}$ and $\mathbb{D}$ be any categories of BCK/BCI-algebras. A *functor* from $\mathbb{C}$ to $\mathbb{D}$ is a triple $(\mathbb{C}, F, \mathbb{D})$, where F is a function from the class of BCK/BCI-homomorphisms of $\mathbb{C}$ to the class of BCK/BCI-homomorphisms of $\mathbb{D}$ (i.e., $F : Hom(\mathbb{C}) \to Hom(\mathbb{D})$) satisfying the following conditions:

(i) F preserves identities: if e is a $\mathbb{C}$-identity, then $F(e)$ is a $\mathbb{D}$-identity

(ii) F preserves composition: i.e., $F(f \circ g) = F(f) \circ F(g)$; i.e., whenever $dom(f) = cod(g)$, then $dom(F(f)) = cod(F(g))$ and the above equality holds.

Let $u : A' \to A$ and $v : B \to B'$ be BCI-homomorphisms, where A, B and B' are BCK-algebras. We define a mapping

$$Hom(u, v) : Hom(A, B) \to Hom(A', B')$$

by requiring that $f \in Hom(A, B)$ is to be mapped into $vfu \in Hom(A', B')$. Clearly, $Hom(u, v)$ is a BCI-homomorphism and if u, v are identity maps, then $Hom(u, v)$ is also an identity map.

Again if $u' : A'' \to A'$ and $v' : B' \to B''$ are BCI-homomorphisms, where A', B'' are BCK-algebras, then

$$Hom(uu', v'v) = Hom(u', v')Hom(u, v).$$

In fact, $Hom(\_, \_)$ is a functor. This functor is called a *hom functor.*

Note that $Hom(u, B) = Hom(u, 1_B)$ and $Hom(A, v) = Hom(1_A, v)$. Let X and Y be BCI-algebras and let

$$X \oplus Y = \{(x, y) \,|\, x \in X, y \in Y\}.$$

We define the operation $*$ on $X \oplus Y$ by

$$(x, y) * (x', y') = (x * x', y * y')$$

for all $(x, y), (x', y') \in X \oplus Y$. Then $(X \oplus Y, *, (0, 0))$ is a BCI-algebra, which is called the *direct sum* of X and Y (see [3]). If $u_1, u_2 : A' \to A$ and $v_1, v_2 : B \to B'$ are BCI-homomorphisms, where A, B' are BCK-algebras, then the mappings $u_1 \oplus u_2 : A' \oplus A' \to A \oplus A$ defined by $u_1 \oplus u_2(x_1, x_2) = (u_1(x_1), u_2(x_2))$ for any $(x_1, x_2) \in A' \oplus A'$, and $v_1 \oplus v_2 : B \oplus B \to B' \oplus B'$ defined by $v_1 \oplus v_2(x_1, x_2) = (v_1(x_1), v_2(x_2))$ for any $(x_1, x_2) \in B \oplus B$, are BCI-homomorphisms.

PROPOSITION 2.1 ([1]). *If $u_1, u_2 : A' \to A$ and $v_1, v_2 : B \to B'$ are BCI-homomorphisms, where A, B and B' are BCK-algebras, then*

$$Hom(u_1 \oplus u_2, B) = Hom(u_1, B) \oplus Hom(u_2, B)$$

and

$$Hom(A, v_1 \oplus v_2) = Hom(A, v_1) \oplus Hom(A, v_2)$$

i.e., a hom functor is additive.

LEMMA 2.2. *Let $f : X \to Y$ be a homomorphism, where X is a BCI-algebra and Y is a p-semisimple BCI-algebra. Then Imf is an ideal of Y.*

PROOF. For any $y_1, y_2 \in Y$, let $y_1 * y_2 \in Imf$ and $y_2 \in Imf$. Then there exist $x_1, x_2 \in X$ such that $f(x_1) = y_1 * y_2$ and $f(x_2) = y_2$. Hence $f(x_1) = y_1 * y_2 = y_1 * f(x_2)$. Thus we have

$$\begin{aligned} f(x_1) * (0 * f(x_2)) &= (y_1 * f(x_2)) * (0 * (f(x_2)) \\ f(x_1) * (f(0) * f(x_2)) &= (y_1 * 0) * (f(x_2) * f(x_2)) \\ f(x_1) * (f(0 * x_2)) &= (y_1 * 0) * 0 \\ f(x_1 * (0 * x_2)) &= y_1. \end{aligned}$$

Therefore there is an element $x_1 * (0 * x_2) \in X$ such that $f(x_1 * (0 * x_2)) = y_1$, i.e., $y_1 \in Imf$. □

THEOREM 2.3 ([1]). *If σ, π are regular BCI-homomorphisms in the exact sequence of BCI-algebras:*

$$M' \xrightarrow{\sigma} M \xrightarrow{\pi} M'' \to 0$$

then, for any BCK-algebra N, the sequence

$$0 \to Hom(M'', N) \xrightarrow{Hom(\pi, N)} Hom(M, N) \xrightarrow{Hom(\sigma, N)} Hom(M', N)$$

is exact.

Using Lemma 2.2 we know that if $f : X \to Y$ is a BCI-homomorphi -sm from a BCI-algebra X into a p-semisimple BCI-algebra Y, then it is a regular BCI-homomorphism.

Y. Liu ([8]) showed that if X is a BCI-algebra and Y is a p-semisimple BCI-algebra then $Hom(X, Y)$ is a p-semisimple BCI-algebra. With this concept we generalize Theorem 2.3 as follows:

THEOREM 2.3′. *If $M' \xrightarrow{\sigma} M \xrightarrow{\pi} M'' \to 0$ is an exact sequence of BCI-algebras, where M and M'' are p-semisimple BCI-algebras, then, for any p-semisimple BCI-algebra N, the sequence*

$$0 \to Hom(M'', N) \xrightarrow{Hom(\pi, N)} Hom(M, N) \xrightarrow{Hom(\sigma, N)} Hom(M', N)$$

is exact.

THEOREM 2.4 ([1]). *If ϕ, ψ are regular BCI-homomorphisms in the exact sequence of p-semisimple BCI-algebras,*

$$0 \to B' \xrightarrow{\phi} B \xrightarrow{\psi} B'',$$

then for all BCI-algebra A, the sequence

$$0 \to Hom(A, B') \xrightarrow{Hom(A,\phi)} Hom(A, B) \xrightarrow{Hom(A,\psi)} Hom(A, B'')$$

is exact.

As we generalized Theorem 2.3, we generalize Theorem 2.4 as follows:

THEOREM 2.4′. *If $0 \to B' \xrightarrow{\phi} B \xrightarrow{\psi} B''$ is an exact sequence of p-semisimple BCI-algebras, then for all BCI-algebra A, the sequence*

$$0 \to Hom(A, B') \xrightarrow{Hom(A,\phi)} Hom(A, B) \xrightarrow{Hom(A,\psi)} Hom(A, B'')$$

is exact.

3. Characterization of projective BCI-algebras

In this section, we show that $Hom(P, \_\,)$ is an exact functor if P is a p-projective BCI-algebra. Recall that a BCI-algebra P is called *projective* if for every BCI-homomorphism $f : P \to Y$ and every BCI-epimorphism $g : X \to Y$ there exists a BCI-homomorphism $h : P \to X$ satisfying $gh = f$. A BCI-algebra P is said to be *p-projective* if for every BCI-homomorphism $f : P \to Y$ and every BCI-epimorphism $g : X \to Y$ of p-semisimple BCI-algebras there exists a BCI-homomorphism $h : P \to X$ satisfying $gh = f$. Clearly every projective BCI-algebra is p-projective.

LEMMA 3.1 ([9]). *Let*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\
0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0
\end{array}$$

be a commutative diagram of BCI-algebras and BCI-homomorphisms such that each row is a short exact sequence and B' is p-semisimple. Then

(i) α, γ *are monomorphisms* $\Longrightarrow \beta$ *is a monomorphism;*
(ii) α, γ *are epimorphisms* $\Longrightarrow \beta$ *is an epimorphism;*
(iii) α, γ *are isomorphisms* $\Longrightarrow \beta$ *is an isomorphism.*

Let $X \oplus Y := \{(x,y) \mid x \in X, y \in Y\}$ be the direct sum of BCI-algebras X and Y. Then the mappings $i_X : X \to X \oplus Y$ and $i_Y : Y \to X \oplus Y$ given by $i_X(x) = (x, 0)$ and $i_Y(y) = (0, y)$ are clearly BCI-monomorphisms.

THEOREM 3.2 ([9]). *Let $0 \to A_1 \xrightarrow{f} B \xrightarrow{g} A_2 \to 0$ be a short exact sequence of p-semisimple BCI-algebras. Then the following conditions are equivalent:*

(i) *There is a BCI-homomorphism $h : A_2 \to B$ with $gh = 1_{A_2}$;*
(ii) *There is a BCI-homomorphism $k : B \to A_1$ with $kf = 1_{A_1}$;*
(iii) *The given sequence is isomorphic (with identity maps on A_1 and A_2) to the direct sum short exact sequence*

$$0 \to A_1 \xrightarrow{i_1} A_1 \oplus A_2 \xrightarrow{\pi_2} A_2 \to 0;$$

in particular $B \cong A_1 \oplus A_2$.

A short exact sequence $0 \to A_1 \xrightarrow{f} B \xrightarrow{g} A_2 \to 0$ satisfies one of the equivalent conditions (i), (ii), and (iii) of Theorem 3.2 is called a *split* (*exact sequence*).

THEOREM 3.3. *Let $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ be a short exact sequence of p-semisimple BCI-algebras. Then the followings are equivalent on BCI-algebras:*

(i) $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ *is split;*

(ii) $0 \longrightarrow Hom(D,A) \xrightarrow{Hom(D,\varphi)} Hom(D,B) \xrightarrow{Hom(D,\psi)} Hom(D,C) \longrightarrow 0$ *is a split exact sequence of p-semisimple* BCI*-algebras;*

(iii) $0 \longrightarrow Hom(C,D) \xrightarrow{Hom(\psi,D)} Hom(B,D) \xrightarrow{Hom(\varphi,D)} Hom(A,D) \longrightarrow 0$ *is a split exact sequence of* BCI*-algebras, where* D *is a p-semisimple* BCI*-algebra.*

PROOF. (i) $\Rightarrow$ (iii): Assume that the sequence is split. Then, by Theorem 3.2, there exists a BCI-homomorphisms $\alpha : B \longrightarrow A$ such that $\alpha\varphi = 1_A$. It induces a BCI-homomorphism $Hom(\alpha, D) : Hom(A,D) \longrightarrow Hom(B,D)$. We claim that $Hom(\varphi,D)Hom(\alpha,D) = 1_{Hom(A,D)}$. In fact, $Hom(\varphi,D)Hom(\alpha,D) = Hom(\varphi,1_D)\ Hom(\alpha,1_D) = Hom(\alpha\varphi, 1_D) = Hom(1_A,1_D) = 1_{Hom(A,D)}$. By applying Theorem 2.3′ and Theorem 3.2 the $Hom(\_,\_)$ sequence is split.
(iii)$\Rightarrow$ (i): Assume that the $Hom(\_,\_)$ sequence is split. Then, by Theorem 3.2, there exists a BCI-homomorphism $Hom(\alpha,D) : Hom(A,D) \longrightarrow Hom(B,D)$ such that $Hom(\varphi,D)\ Hom(\alpha,D) = 1_{Hom(A,D)}$, i.e., $\alpha\varphi = 1_A$. It follows from Theorem 3.2 that the exact sequence $0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$ is split.
(i) $\Leftrightarrow$ (ii): This can be proved similarly by using Theorem 3.2 and Theorem 2.4′, and we omit the proof. □

THEOREM 3.4. *The following conditions on* BCI*-algebra are equivalent:*

(i) P *is p-projective;*
(ii) *if* $\psi : B \longrightarrow C$ *is a* BCI*-epimorphism of p-semisimple* BCI*-algebras, then* $Hom(P,\psi) : Hom(P,B) \longrightarrow Hom(P,C)$ *is a* BCI*-epimorphism;*
(iii) *if* $0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$ *is any short exact sequence of p-semisimple* BCI*-algebras, then* $0 \longrightarrow Hom(P,A) \longrightarrow Hom(P,B) \longrightarrow Hom(P,C) \longrightarrow 0$ *is an exact sequence of* BCI*-algebras.*

PROOF. (i) $\Rightarrow$ (ii): Let $\psi : B \longrightarrow C$ be a BCI-epimorphism of p-semisimple BCI-algebras. Then the hom functor $Hom(P,\psi) : Hom(P, B) \longrightarrow Hom(P,C)$ is a BCI-homomorphism. We claim that $Hom(P,\psi)$ is onto. In fact, for any $g \in Hom(P,C)$, since P is p-projective, there exists a BCI-homomorphism $h : P \longrightarrow B$ such that $g = \psi \circ h = \psi \circ h \circ 1_P = Hom(1_P,\psi)(h) = Hom(P,\psi)(h)$. Hence $Hom(P,\psi)$ is a BCI-epimorphism.

(ii) $\Rightarrow$ (i): Let $f : P \to C$ be a BCI-homomorphism and let $\psi : B \to C$ be a BCI-epimorphism of p-semisimple BCI-algebras. It follows from the condition (ii) that $Hom(P, \psi) : Hom(P, B) \to Hom(P, C)$ is a BCI-epimorphism. Since $f \in Hom(P, C)$, there exists $h \in Hom(P, B)$ such that $Hom(P, \psi)(h) = f$, i.e., $\psi \circ h = f$. This means that there exists a BCI-homomorphism $h : P \to B$ such that $\psi \circ h = f$, i.e., P is p-projective.

(ii) $\Rightarrow$ (iii): By Theorem 2.4$'$.

(iii) $\Rightarrow$ (ii): Let $\psi : B \to C$ be a BCI-homomorphism and let $A := Ker\psi$. Then $0 \to A = \text{Ker}\psi \xrightarrow{i} B \xrightarrow{\psi} C \to 0$ is a short exact sequence of p-semisimple BCI-algebras. It follows from the condition (iii) that $0 \to Hom(P, A) \xrightarrow{Hom(P,i)} Hom(P, B) \xrightarrow{Hom(P,\psi)} Hom(P, C) \to 0$ is an exact sequence of BCI-algebras, and hence $Hom(P, \psi) : Hom(P, B) \to Hom(P, C)$ is a BCI-epimorphism, completing the proof. □

The hom functor $Hom(P, \_\ )$ discussed in Theorem 3.4 is said to be *exact* if it satisfies the condition (iii) of Theorem 3.4. With this concept we conclude that *$Hom(P, \_\ )$ is exact if P is a p-projective BCI-algebra.*

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