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MONOMIAL CHARACTERS OVER FINITE GROUPS

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ABSTRACT. Parks [7] showed that there is an one to one correspondence between good pairs of subgroups in G and irreducible monomial characters of G. This provides a useful criterion for a group to be monomial. In this paper, we study relative monomial groups by defining triples in G, and find relationships between the triples and irreducible relative monomial characters.

1. Introduction

Let G be a finite group, F be the field of complex numbers and $\mathrm{Irr}(G)$ be the set of irreducible characters of G over F. An irreducible character $\chi \in \mathrm{Irr}(G)$ is said to be relative monomial with respect to a normal subgroup N if there is a subgroup H of G containing N and an irreducible character ψ on H such that $\psi^G = \chi$ and ψ_N is irreducible. If every irreducible character of G is relative monomial with respect to N then G is called a relative monomial group with respect to N. Due to relationships between group characters and representations, an irreducible representation of G can be referred as a relative monomial representation by substituting character by representation. If N is a trivial group then a relative monomial character with respect to N is a monomial character, that is, an irreducible character χ of G is said to be monomial if there is a subgroup H of G and an irreducible character ψ on H such that $\psi^G = \chi$ and ψ is linear (a one-dimensional character on H, i.e., $\psi(1) = 1$). A group on which every irreducible character is monomial is called a monomial group.

Most researches about monomial groups were carried out using the theory of group representation ([2], [6]). It was Isaacs ([4] and [5, p.67])

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who first considered purely group theoretic characterization for monomial groups. Later, Parks [7] introduced a good pair of subgroups in G and showed that there is an one to one correspondence between good pairs in G and irreducible monomial characters of G. This gives a useful criterion for monomial groups.

In this paper we will extend the Parks result to relative monomial groups. We define a good triple of groups that will be a counterpart of relative monomial characters. Our main results can be described that there is a correspondence between irreducible relative monomial characters on G and good triples in G (Theorem 6). Since the good pairs afforded by a character is not unique, we study a relation on good triples that are related to an irreducible relative monomial character (Theorem 7).

In what follows, let G be a finite group and let H be a subgroup of G. For $g, x \in G$, we denote the commutator element in G by $[g, x]$, the set of commutator elements $[g, h]$ with $h \in H$ and $H^g = g^{-1}Hg$ by $[g, H]$. We denote the set of irreducible characters of G by $\mathrm{Irr}(G)$. For $\chi \in \mathrm{Irr}(G)$ and $\psi \in \mathrm{Irr}(H)$, we denote the restriction of χ to H by χ_H, the induction of ψ to G by ψ^G, and the conjugate of χ by x χ^x, i.e., $\chi^x(g) = \chi(x^{-1}gx)$ for $g, x \in G$. Let $\langle\ ,\ \rangle_G$ denote the inner product of characters of G.

2. Monomial characters and good pairs

For a normal subgroup M of H, (H, M) is called a pair in G if the quotient group H/M is cyclic. We recall the following two results due to Parks.

LEMMA 1. [7] *For any $x \in G$, the set of commutators $[x, H \cap H^{x^{-1}}]$ is contained in H. Moreover if (H, M) is a pair in G and if $x \in H$ then $[x, H \cap H^{x^{-1}}] \subseteq M$.*

A pair (H, M) in G which satisfies $[x, H \cap H^{x^{-1}}] \not\subseteq M$ for any $x \in G - H$ is called a good pair. Thus (H, M) is a good pair provided $[x, H \cap H^{x^{-1}}] \subseteq M$ if and only if $x \in H$.

THEOREM 2. *A monomial character of G yields a good pair in G, and vice versa.*

We skip the proof of Theorem 2 at this moment because the main idea needed for the proof are explicitly or implicitly contained in the proofs of Proposition 1.1 and 1.2 in [7], and this theorem will be improved in next section. In Theorem 3, we give good pairs of quotient groups.

THEOREM 3. *If (H, M) is a good pair in G then $(H/N, MN/N)$ is a good pair in G/N for a normal subgroup N of G contained in H.*

PROOF. Since N is normal in both H and MN, we may consider the object $(H/N, MN/N)$. Clearly MN is normal in H and MN/N is normal in H/N. Since $(H/N)/(MN/N) \cong H/MN \cong (H/M)/(MN/M)$ and H/M is cyclic, the quotient group $(H/N)/(MN/N)$ is cyclic. This shows that $(H/N, MN/N)$ is a pair in G/N.

We now choose any element $zN \in G/N - H/N$ for $z \in G$. Since $z \notin H$, we have $[z, H \cap H^{z^{-1}}] \not\subseteq M$ because (H, M) is a good pair in G. Moreover since $(H/N) \cap (H/N)^{z^{-1}N} = (H/N) \cap (H^{z^{-1}}/N) = (H \cap H^{z^{-1}})/N$, it follows that

$$\begin{aligned}[zN, (H/N) \cap (H/N)^{z^{-1}N}] &= [zN, (H \cap H^{z^{-1}})/N] \\ &= [z, H \cap H^{z^{-1}}]/N \not\subseteq MN/N,\end{aligned}$$

hence $(H/N, MN/N)$ is a good pair in G/N. □

3. Relative monomial characters and good triples

For subgroups H, M and L in G, we say that (H, M, L) is a good triple in G if L is a normal subgroup of G contained in H and (H, M) is a good pair. If $L = 1$ then a good triple is a good pair.

THEOREM 4. *Let (H, M, L) be a good triple in G. Then there is an irreducible relative monomial character on G with respect to N where N is any normal subgroup of G containing L.*

PROOF. Since (H, M) is a pair, H/M is cyclic so there is a faithful irreducible linear character θ on H/M. Let ψ be the character lift from θ to H with $\ker\psi = M$. Then ψ is linear and irreducible over H.

Let $\chi = \psi^G$ be the induced character on G, and we will show that

$$\chi \in \mathrm{Irr}(G) \quad \text{and} \quad \psi_L \in \mathrm{Irr}(L).$$

We suppose that the inner product $\langle \psi^x|_{H^x \cap H}, \psi|_{H^x \cap H} \rangle = 1$ for any $x \in G - H$. Then we may assume $\psi^x|_{H^x \cap H} = \psi|_{H^x \cap H}$ as $\psi^x(g) = \psi(x^{-1}gx) = \psi(g)$ for all $g \in H^x \cap H$. Since ψ is linear, it follows that $1 = \psi(x^{-1}gxg^{-1}) = \psi([x, g^{-1}])$. Thus $[x, g^{-1}] \in M = \ker\psi$ for all $g \in H^x \cap H$ and $[x, H \cap H^{x^{-1}}] \subseteq M$. This leads a contradiction because (H, M) is a good pair. Therefore we have $\langle \psi^x|_{H^x \cap H}, \psi|_{H^x \cap H} \rangle \neq 1$.

Moreover since both ψ^x and ψ are irreducible characters of $H^x \cap H$ which are distinct, we conclude that $\langle \psi^x|_{H^x\cap H}, \psi|_{H^x\cap H}\rangle = 0$.

For $x \in G - H$, let T be an H,H-transversal in G containing 1 and x. By the Mackey theorem ([3, (21.6)]), we have

$$\begin{aligned}
\langle \chi, \chi \rangle &= \langle \psi^G, \psi^G \rangle = \sum_{g\in T} \langle \psi^g|_{H^g\cap H}, \psi|_{H^g\cap H}\rangle \\
&= \sum_{g\in T, g\in G-H} \langle \psi^g|_{H^g\cap H}, \psi|_{H^g\cap H}\rangle + \sum_{g\in T, g\in H} \langle \psi^g|_{H^g\cap H}, \psi|_{H^g\cap H}\rangle \\
&= \langle \psi^x|_{H^x\cap H}, \psi|_{H^x\cap H}\rangle + \langle \psi|_H, \psi|_H\rangle = \langle \psi, \psi\rangle = 1,
\end{aligned}$$

hence it follows that $\chi \in \mathrm{Irr}(G)$.

Consider the restriction ψ_L of ψ to L. Since $\ker\psi_L = L \cap M$, the function $\hat{\psi}_L$ on $L/(L\cap M)$ defined by

$$\hat{\psi}_L(y(L\cap M)) = \psi_L(y) = \psi(y) \quad \text{for } \ y \in L$$

is a character. Since H/M is cyclic, LM/M and $L/(L\cap M)$ are cyclic too. Hence $\hat{\psi}_L$ is linear and irreducible. And the irreducibility of $\hat{\psi}_L$ guarantees the irreducibility of ψ_L (refer to [5, (2.22)], hence $\psi_L \in \mathrm{Irr}(L)$.

Now we may take $N = L$. More generally we can choose any normal subgroup N of G such that $L \subseteq N \subseteq H$, because $\psi_L = (\psi_N)|_L \in \mathrm{Irr}(L)$ and $\psi_N \in \mathrm{Irr}(N)$. □

For the converse of Theorem 4, we will construct certain triples of subgroups in G for a given irreducible relative monomial character of G.

THEOREM 5. *Let χ be an irreducible relative monomial character on G with respect to a normal subgroup N of G. Then there exist subgroups H and M of G satisfying the followings.*

(1) *M is normal in H and the center $Z(H/M)$ of H/M is cyclic.*

(2) *$[x, H\cap H^{x^{-1}}] \not\subseteq M$ for all $x \in G - H$.*

PROOF. Since χ is an irreducible relative monomial character on G with respect to N, there exist a group H with $N \subseteq H \subseteq G$ and an irreducible character $\psi \in \mathrm{Irr}(H)$ such that $\psi^G = \chi$ and $\psi_N \in \mathrm{Irr}(N)$.

Let R be an irreducible representation of H that affords the character ψ. We let

$$M = \ker R = \{h \in H|\ \psi(h) = \psi(1)\} \quad \text{and} \quad W = \{h \in H|\ |\psi(h)| = \psi(1)\}.$$

Then W is a subgroup of H, M is a normal subgroup of both H and W, and W/M is a cyclic subgroup of the center $Z(H/M)$. Moreover since $\psi \in \mathrm{Irr}(H)$, the center $Z(H/M)$ is equal to W/M (refer to [5, (2.27)]) so that $Z(H/M)$ is cyclic.

Let x be any element in $G-H$ and let T be an H,H-transversal in G containing 1 and x. Since χ is irreducible on G, it follows from Mackey Theorem that

$$\begin{aligned} 1 &= \langle \chi, \chi \rangle = \langle \psi^G, \psi^G \rangle = \sum_{g \in T} \langle \psi^g|_{H^g \cap H}, \psi|_{H^g \cap H} \rangle \\ &\geq \langle \psi, \psi \rangle + \langle \psi^x|_{H^x \cap H}, \psi|_{H^x \cap H} \rangle = 1 + \langle \psi^x|_{H^x \cap H}, \psi|_{H^x \cap H} \rangle \end{aligned}$$

thus $\langle \psi^x|_{H^x \cap H}, \psi|_{H^x \cap H} \rangle = 0$ for $x \in G - H$.

Since $N \subseteq H$ and N is normal in G, we have $N = N^x \subseteq H^x$ and $N \subseteq H^x \cap H$. Thus $(\psi|_{H^x \cap H})|_N = \psi_N$. But since ψ_N is irreducible, both $\psi|_{H^x \cap H}$ and $\psi^x|_{H^x \cap H}$ are irreducible ([5, (2.26)]). Thus the fact $\langle \psi^x|_{H^x \cap H}, \psi|_{H^x \cap H} \rangle = 0$ implies

$$\psi^x|_{H^x \cap H} \neq \psi|_{H^x \cap H}.$$

To finish the proof, we suppose that $[x, H \cap H^{x^{-1}}] \subseteq M = \ker\psi$. Then for any $h \in H \cap H^{x^{-1}}$, $\psi([x,h]) = \psi(1)$. That is, $R(x^{-1}h^{-1}xh) = I_n$ for $n = \psi(1)$, hence

$$R^x(h^{-1})R(h) = I_n \quad \text{and} \quad R^x(h) = R(h).$$

This implies that $\psi^x(h) = \psi(h)$ and $\psi^x|_{H^x \cap H} = \psi|_{H^x \cap H}$, which is a contradiction. Thus we conclude that $[x, H \cap H^{x^{-1}}] \not\subseteq M = \ker\psi$. □

A difference of relative monomial character χ with respect to a normal subgroup of G from a monomial character θ is that θ is induced from an irreducible character on a subgroup of G which is linear (i.e., a homomorphism) while χ is induced from an irreducible character which need not linear. We now have analog of Theorem 2 with respect to relative monomial characters.

THEOREM 6. *There is an irreducible relative monomial character of G with respect to a normal subgroup if and only if there is a good triple in G.*

PROOF. One direction is clear from Theorem 4.

Let χ be an irreducible relative monomial character on G with respect to a normal subgroup N. Then there exists a subgroup H of G containing N and an irreducible character ψ on H such that $\psi^G = \chi$ and $\psi_N \in \mathrm{Irr}(N)$. And due to Theorem 5, there is a subgroup M of G such that M is normal in H, and the center $Z(H/M)$ of H/M is cyclic. Moreover $[x, H \cap H^{x^{-1}}] \not\subseteq M$ for all $x \in G - H$. In fact, $Z(H/M) = W/M$ where W and M were defined by $M = \ker\psi$ and $W = \{h \in H | \ |\psi(h)| = \psi(1)\}$.

We will show that $[x, W \cap W^{x^{-1}}] \not\subseteq M$ for all $x \in G - W$. Let T be a W,W-transversal in G containing 1 and x. Since $\chi = \psi^G$ is irreducible, we have

$$\begin{aligned} 1 &= \langle \chi, \chi \rangle = \langle \psi^G, \psi^G \rangle = \sum_{g \in T} \langle \psi^g|_{W^g \cap W}, \psi|_{W^g \cap W} \rangle \\ &\geq \langle \psi_W, \psi_W \rangle + \langle \psi^x|_{W^x \cap W}, \psi|_{W^x \cap W} \rangle. \end{aligned}$$

Due to [5, (2.27)], there exists a linear character λ of W such that $\psi_W = \psi(1)\lambda$. Thus

$$\begin{aligned} \langle \psi_W, \psi_W \rangle &= \frac{1}{|W|} \sum_{w \in W} \psi_W(w) \psi_W(w^{-1}) \\ &= \frac{1}{|W|} \sum_{w \in W} \psi(1)^2 \lambda(w) \lambda(w^{-1}) \;=\; \psi(1)^2. \end{aligned}$$

Since $\psi(1)$ is the degree of R where R is the representation of H which affords ψ, both $\psi(1)$ and $\langle \psi_W, \psi_W \rangle$ are positive integers. Therefore the above two equations give rise to

$$1 \geq \langle \psi_W, \psi_W \rangle + \langle \psi^x|_{W^x \cap W}, \psi|_{W^x \cap W} \rangle = \psi(1)^2 + \langle \psi^x|_{W^x \cap W}, \psi|_{W^x \cap W} \rangle,$$

so that $\langle \psi^x|_{W^x \cap W}, \psi|_{W^x \cap W} \rangle = 0$ for all $x \in G - W$.

If $[x, W \cap W^{x^{-1}}] \subseteq M$ then $[x, w] \in M$ for any $w \in W \cap W^{x^{-1}}$. Since $\psi([x, w]) = \psi(1)$ and $R([x, w]) = I_n = R(1)$ with an identity matrix I_n for $n = \psi(1)$, we have $R^x(w) = R(w)$. This implies that $\psi^x(w) = \psi(w)$ and $\langle \psi^x|_{W \cap W^{x^{-1}}}, \psi|_{W \cap W^{x^{-1}}} \rangle \neq 0$, but this is not. Hence $[x, W \cap W^{x^{-1}}] \not\subseteq M$, as is required.

Now we consider subgroups WN, MN and N of G. For any $n \in N$ and $w \in W$, $\psi(n^{-1}mn) = \psi(m)$ (since ψ is a class function on H) so $M^n = M$. Thus $(wn)^{-1}MN(wn) = n^{-1}Mn = n^{-1}MNn = M^n N^n MN$, hence MN is normal in WN. Moreover WN/MN is cyclic because

$$(WN)/(MN) \cong W/(W \cap MN) \cong (W/M)/((W \cap MN)/M).$$

And finally we will show that the set $[x, (WN) \cap (WN)^{x^{-1}}]$ is not contained in MN for all $x \in G - WN$. Indeed, since $(WN) \cap (WN)^{x^{-1}} = (W \cap W^{x^{-1}})N$, any element in $[x, (WN) \cap (WN)^{x^{-1}}]$ can be written as $[x, wn]$ with $w \in W \cap W^{x^{-1}}$ and $n \in N$. Since $x \in G - WN$, $x \in G - W$ and $[x, W \cap W^{x^{-1}}] \not\subseteq M$, as above. Thus

$$[x, wn] = [x, n][x, w]^n \in N[x, W \cap W^{x^{-1}}]^n \not\subseteq NM^n = NM = MN.$$

Therefore we conclude that (WN, MN, N) is a good triple in G afforded by a relative monomial character χ of G with respect to N. □

REMARK. We remark a distinction between good pairs and good triples. For a given irreducible monomial character χ of G, the subgroups W and M defined above form a good pair (W, M) in G. Since N is not necessarily in W, (W, M, N) may not be a good triple in H, but (WN, MN, N) is considered as a good triple in G afforded by χ of G.

We also notice that the good triple (WN, MN, N) is not determined uniquely by the relative monomial character χ. In fact, with the same reason above, $(W, M, W \cap N)$ is a good triple in G afforded by χ. Hence it is reasonable to ask relationships between two good triples produced from the same character.

Let (H_1, M_1, L_1) and (H_2, M_2, L_2) be two good triples in G. We say that they are related in G if (H_1, M_1) and (H_2, M_2) are good pairs satisfying $H_1^g \cap M_2 = H_2 \cap M_1^g$ for some $g \in G$.

THEOREM 7. *Let χ be an irreducible relative monomial character of G with respect to N. Let H, W and M be as in the proof of Theorem 6. Then the two good triples (WN, MN, N) and (W, M, L) with $L = W \cap N \subset M$ produced from χ (in particular, $L = M \cap N$) are related.*

PROOF. It suffices to show that $(WN)^x \cap M = (MN)^x \cap W$ for some $x \in G$. Consider $x \in H \subset G$. Since M is normal in H and N is normal in G, we have $M^x = M$ and $N^x = N$. Hence it follows from $N \cap W \subset M$ that

$$(WN)^x \cap M = W^x N \cap M = M^x = M = M(N \cap W) = (MN)^x \cap W.$$

□

Corollary 8 is a special case of Theorem 6 that contains Theorem 2.

COROLLARY 8. *The following statements are equivalent.*

i) *There is a monomial character $\chi \in \mathrm{Irr}(G)$.*
ii) *There is a relative monomial character $\chi \in \mathrm{Irr}(G)$ with respect to the trivial group 1.*
iii) *There is a good triple $(W, M, 1)$ in G.*
iv) *There is a good pair (W, M) of G.*

THEOREM 9. *Let N be a normal subgroup of G. Any irreducible relative monomial character of G with respect to N produces an irreducible monomial character of G/N, and vice versa.*

PROOF. Let χ be an irreducible relative monomial character of G with respect to N. Then there is a good triple (WN, MN, N) in G, where W, M are as in Theorem 5. Since N is normal in both WN and MN, we can show that $(WN/N, MN/N)$ is a good pair in G/N (refer to Theorem 3). In fact, $(WN/N)/(MN/N) \cong WN/MN$, a cyclic group, and if $xN \in G/N - WN/N$ then $[x, W \cap W^{x^{-1}}] \not\subseteq M$. Since

$$\Big[xN, (WN/N) \cap (WN/N)^{x^{-1}N}\Big] = \Big[xN, (W \cap W^{x^{-1}})N/N\Big],$$

any element in $[xN, (W \cap W^{x^{-1}})N/N]$ forms $[xN, wnN]$ for $w \in W \cap W^{x^{-1}}$ and $n \in N$ and $[xN, wnN] = [xN, wN] = [x, w]N \in [x, W \cap W^{x^{-1}}]N$, hence it is not contained in MN/N due to $[x, W \cap W^{x^{-1}}] \not\subseteq M$. This proves that $(WN/N, MN/N)$ is a good pair in G/N, so that there exists an irreducible monomial character of G/N.

Conversely, let λ be an irreducible monomial character of G/N. Then due to Theorem 6, there is a good pair $(H/N, M/N)$ in G/N where H/M is cyclic, $N \subseteq H$ and N is a normal subgroup of G. Also $[xN, (H/N) \cap (H/N)^{x^{-1}N}] \not\subseteq M/N$ for all $xN \in G/N - H/N$. Since $[x, H \cap H^{x^{-1}}] \not\subseteq M$, (H, M, N) is a good triple in G. Hence there is an irreducible relative monomial character χ on G with respect to N. □

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