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ON SPECTRAL BOUNDEDNESS

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ABSTRACT. For linear operators between Banach algebras "spectral boundedness" is derived from ordinary boundedness by substituting spectral radius for norm. The interplay between this concept and some of its near relatives is conspicuous in a result of Curto and Mathieu.

It is familiar that a linear operator $T : X \to Y$ between Banach spaces is continuous if and only if it is "bounded" in the usual sense, that for some $k > 0$ there is inequality $\|Tx\| \le k\|x\|$ for every $x \in X$. If $X = A$ and $Y = B$ are Banach algebras, and we replace norms by spectral radii, then we are looking at something new: "spectral boundedness". In this note we look at spectral boundedness and some of its relatives. Suppose A is a complex linear algebra, with identity $1 \neq 0$ and invertible group A^{-1}: we write

$$(0.1) \qquad \sigma(x) = \{\lambda \in \mathbf{C} : x - \lambda \notin A^{-1}\}$$

for the spectrum of $x \in A$ and

$$(0.2) \qquad |x|_\sigma = \sup\{|\lambda| : \lambda \in \sigma(x)\}$$

for the spectral radius. Recall the *quasinilpotents* in A

$$(0.3) \qquad QN(A) = \{x \in A : \sigma(x) = \{0\}\} = \{x \in A : |x|_\sigma = 0\}$$

and the *radical* of A

$$(0.4) \qquad RAD(A) = \{x \in A : 1 - Ax \subseteq A^{-1}\} :$$

obviously

$$(0.5) \qquad RAD(A) \subseteq QN(A),$$

and if x and y are arbitrary in A then (cf. [8] Theorem 3.6.4)

$$(0.6) \qquad |xy|_\sigma = |yx|_\sigma.$$

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DEFINITION 1. If A and B are linear algebras write

(1.1) $$L^\sigma_\bullet(A,B) = \{T \in L(A,B) : T\,QN(A) \subseteq QN(B)\}$$

for the spectrally consistent linear mappings from A to B,

(1.2) $$L^\sigma_\infty(A,B) = \{T \in L(A,B) : \exists k > 0,\ \forall x \in A,\ |Tx|_\sigma \le k|x|_\sigma\}$$

for the spectrally bounded linear mappings from A to B,

(1.3) $$L^\sigma_0(A,B) = \{T \in L(A,B) : T(A) \subseteq QN(B)\}$$

for the spectrally infinitesimal linear mappings from A to B, and

(1.4) $$L^\sigma_{00}(A,B) = \{T \in L(A,B) : T(A) \subseteq RAD(B)\}$$

for the spectrally trivial linear mappings from A to B. If we define a "quadratic" operator $S \cdot T \in Q(A,B)$ by setting

(1.5) $$(S \cdot T)(x) = S(x) \cdot T(x) \text{ for each } x \in A,$$

then we may define for S and T in $L(A,B)$

(1.6) $$S \sim_{00} T \Longleftrightarrow (S \cdot T - T \cdot S)(A) \subseteq RAD(B)$$

to mean that each pair of images commute modulo the radical, and if K, H are subsets of $L(A,B)$ define

(1.7) $$K \mathbin{\tilde{+}_{00}} H = \{S + T : S \in K, T \in H, S \sim_{00} T\}$$

and

(1.8) $$K \mathbin{\tilde{\cdot}_{00}} H = \{S \cdot T : S \in K, T \in H, S \sim_{00} T\}.$$

Recall the *homomorphisms* from A to B,

(1.9) $$HL(A,B) = \{T \in L(A,B) : T(ab) - (Ta)(Tb) = T(1) - 1 = 0\},$$

as well as the notation

(1.10) $$L_a(b) = R_b(a) = ab \text{ for each } a, b \in A.$$

We collect some trivialities:

THEOREM 2. *If A and B are (normed) linear algebras then*

(2.1) $$L^\sigma_{00}(A,B) \subseteq L^\sigma_0(A,B) \subseteq L^\sigma_\infty(A,B) \subseteq L^\sigma_\bullet(A,B).$$

If also D is a (normed) linear algebra there are inclusions

(2.2) $$L^\sigma_\bullet(B,D) \circ L^\sigma_*(A,B) \subseteq L^\sigma_*(A,D) \text{ for each } * = \bullet, 0$$

and

(2.3) $$L^\sigma_*(B,D) \circ L(A,B) \subseteq L^\sigma_*(A,D) \text{ for each } * = 0, 00,$$

also

$$L^\sigma_\infty(B,D)\circ L^\sigma_\infty(A,B)\subseteq L^\sigma_\infty(A,D). \tag{2.4}$$

We have

$$L^\sigma_*(A,B)+L^\sigma_{00}(A,B)\subseteq L^\sigma_*(A,B) \text{ for each } *=\bullet,\infty,0,00 \tag{2.5}$$

and for Banach algebras

$$L^\sigma_*(A,B)\mathbin{\tilde{+}_{00}}L^\sigma_*(A,B)\subseteq L^\sigma_*(A,B) \text{ for each } *=\bullet,\infty,0; \tag{2.6}$$

also, recalling the "quadratic" notation of (1.5),

$$L(A,B)\cdot L^\sigma_{00}(A,B)\in Q^\sigma_{00}(A,B) \text{ and } L^\sigma_{00}(A,B)\cdot L(A,B)\in Q^\sigma_{00}(A,B) \tag{2.7}$$

and

$$L(A,B)\mathbin{\tilde{\cdot}_{00}}L^\sigma_0(A,B)\in Q^\sigma_0(A,B) \text{ and } L^\sigma_0(A,B)\mathbin{\tilde{\cdot}_{00}}L(A,B)\in Q^\sigma_0(A,B). \tag{2.8}$$

There is inclusion

$$HL(A,B)\subseteq L^\sigma_\infty(A,B), \tag{2.9}$$

and for each $a\in A$ equivalence

$$L_a\in L^\sigma_*(A,A)\Longleftrightarrow R_a\in L^\sigma_*(A,A) \text{ for each } *=\bullet,\infty,0,00; \tag{2.10}$$

in addition

$$R_a\in L^\sigma_0(A,A)\Longrightarrow a\in RAD(A)\Longrightarrow R_a\in L^\sigma_{00}(A,A). \tag{2.11}$$

Proof. For (2.1) note (0.5) $RAD(A)\subseteq QN(A)$. For (2.2) we have $|Tx|_\sigma=0\Longrightarrow|STx|_\sigma=0$ either for all $x\in A$ or if $|x|_\sigma=0$. For (2.3) note $T(A)\subseteq B$. For (2.4) argue $|STx|_\sigma\le h|Tx|_\sigma\le hk|x|_\sigma$. For (2.5) note that if $Tx\in RAD(B)$ then $\sigma(Sx+Tx)=\sigma(Sx)$, while for (2.6), (2.7) and (2.8) note that if Tx and Sx commute modulo the radical then $\sigma(Sx+Tx)\subseteq\sigma(Sx)+\sigma(Tx)$ and $\sigma(Sx\cdot Tx)\subseteq\sigma(Sx)\sigma(Tx)$. For (2.9) of course $\sigma(Tx)\subseteq\sigma(x)$, while (2.10) is (0.6). For (2.11) recall

$$\begin{aligned} &Aa\subseteq QN(A)\Longrightarrow 1-Aa\subseteq A^{-1}\\ &\Longleftrightarrow a\in RAD(A)\Longrightarrow Aa\subseteq RAD(A).\end{aligned} \tag{2.12}$$

□

Spectral boundedness can sometimes imply norm boundedness: for example using Johnson's uniqueness of norm there is an implication ([1] Theorem 5.5.2) if $RAD(B)=\{0\}$

$$T(A)=B,\ T\in L^\sigma_\infty(A,B)\Longrightarrow T\in BL(A,B). \tag{2.13}$$

Conversely not every bounded linear mapping from A to B is spectrally bounded: if $A = BL(X, X)$ for a Banach space X then for $a \in A$ there is an implication

$$L_a \in L_{\bullet}^{\sigma}(A, A) \Longrightarrow a \in \mathbf{C}I. \tag{2.14}$$

Indeed if $a \in A$ is not scalar then there is $\xi \in X$ for which $(\xi, a\xi)$ is linearly independent (cf. [1] Theorem 4.2.7). If not then $a\xi = \lambda_\xi \xi$ for all $\xi \in X$, in which case (ξ, η) linearly independent implies $\lambda_\xi = \lambda_{\xi+\eta} = \lambda_\eta$. Then $\psi \in X^\dagger$ for which $\psi(\xi) = 0$ and $\psi(a\xi) = 1$. But now

$$x = \psi \odot \xi(\eta \mapsto \psi(\eta)\xi) \Longrightarrow x^2 = 0 \text{ and } 0 \neq ax = (ax)^2. \tag{2.15}$$

Of course if $A = C_\infty(\Omega)$ for a normal Hausdorff space Ω then $L_a = R_a$ is spectrally bounded for every $a \in A$.

Derivations on a ring are linear mappings that satisfy the multiplicative property characteristic of elementary calculus:

Definition 3. If A is a ring with identity then the derivations on A form the space

$$DL(A, A) = \{T \in L(A, A) : T(ab) = (Ta)b + a(Tb)\}. \tag{3.1}$$

The left and right multiplications form the spaces

$$LL(A, A) = \{T \in L(A, A) : T(ab) = (Ta)b\} = \{L_a : a \in A\} \tag{3.2}$$

and

$$RL(A, A) = \{T \in L(A, A) : T(ab) = a(Tb)\} = \{R_a : a \in A\}; \tag{3.3}$$

more generally the generalized derivations form the space

$$GDL(A, A) = \{T \in L(A, A) : T(abc) = T(ab)c - a(Tb)c + aT(bc)\}. \tag{3.4}$$

The reader can see that derivations, left and right multiplications are all generalized derivations, and that [5]

$$DL(A, A) + LL(A, A) = DL(A, A) + RL(A, A) = GDL(A, A). \tag{3.5}$$

The *inner derivations* are the operators

$$\{L_a - R_a : a \in A\} \subseteq DL(A, A), \tag{3.6}$$

more generally we may refer to operators $L_a - R_b$ as "inner generalized derivations", or alternatively "generalized inner derivations". The

Kleinecke-Sirokov and *Singer-Wermer* theorems say that if $D \in DBL(A, A)$ is a bounded derivation on A then ([8] Theorem 11.8.2)

$$D^2 = 0 \Longrightarrow D \in L_0^\sigma(A, A) \tag{3.7}$$

and

$$D \sim_{00} I \Longrightarrow D \in L_0^\sigma(A, A). \tag{3.8}$$

There is also the result [10] that

$$DL(A, A) \cap L_0^\sigma(A, A) \subseteq L_{00}^\sigma(A, A). \tag{3.9}$$

We might notice here that ([5] Lemma 2.1) spectrally bounded derivations send two-sided ideals into their "hull-kernels" ([8] Definition 7.2.5): indeed if $D \in DL(A, A)$ then by Leibnitz' rule

$$D^n(x^n) - n!(Dx)^n \in xA + Ax \tag{3.10}$$

for each $n \in \mathbf{N}$ and each $x \in A$, so that if J is a two-sided ideal of A then

$$D \in DL(A, A) \cap L_\infty^\sigma(A, A),\ x \in J \implies Dx + J \in RAD(A/J). \tag{3.11}$$

Also

$$D + L_a \in GDL(A, A) \cap L_\infty^\sigma(A, A) \Longrightarrow (D + L_a)RAD(A) \subseteq RAD(A), \tag{3.12}$$

and more generally ([5] Lemma 2.7) the full analogue of (3.11) holds with $D + L_a$ in place of D. For generalized inner derivations we can say more [6]:

LEMMA 4. *If $a \in A$ and $b \in A$ are arbitrary there is implication*

$$L_a - R_b \in L_\infty^\sigma(A, A) \Longrightarrow L_b - R_b \in L_{00}^\sigma(A, A). \tag{4.1}$$

In particular if $b \in A$ then

$$L_b - R_b \in L_\infty^\sigma(A, A) \Longrightarrow L_b - R_b \in L_{00}^\sigma(A, A) \tag{4.2}$$

and

$$R_b \in L_\infty^\sigma(A, A) \Longrightarrow L_b - R_b \in L_{00}^\sigma(A, A). \tag{4.3}$$

Proof. This is the argument of Curto and Mathieu ([6] Theorem B), and uses the Jacobson density theorem and its modification due to Sinclair. If $(L_b - R_b)(A) \not\subseteq RAD(A)$ then there must be a maximal left ideal $J \subseteq A$ and an element $c \in A$ for which

$$(c, bc) \text{ is linearly independent modulo } J \text{ in } A: \tag{4.4}$$

for if $(b-\lambda)A \subseteq J$ then $(bx-xb)c \in J$ for all $c, x \in A$. By the Jacobson density theorem ([1] Theorem 4.2.5) there exists $x \in A$ for which

$$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} c \\ bc \end{pmatrix} - \begin{pmatrix} 0 \\ c \end{pmatrix} \in \begin{pmatrix} J \\ J \end{pmatrix}, \tag{4.5}$$

and then for arbitrary $n \in \mathbf{N}$ there exists by Sinclair's modification ([1] Corollary 4.2.6) invertible $u \in A^{-1}$ for which

$$\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} c \\ bc \end{pmatrix} - \begin{pmatrix} -nc \\ bc \end{pmatrix} \in \begin{pmatrix} J \\ J \end{pmatrix}. \tag{4.6}$$

But now

$$(auxu^{-1} - uxu^{-1}b)c - nc \in J \tag{4.7}$$

and hence

$$|auxu^{-1} - uxu^{-1}b|_\sigma \geq n > k|x|_\sigma = k|uxu^{-1}|_\sigma \tag{4.8}$$

by the choice of $n \in \mathbf{N}$. This means that $L_a - R_b$ is not spectrally bounded and proves (4.1), and then (4.2) and (4.3) follow by specializing respectively to $a = b$ and $a = 0$. $\square$

By an extension of this argument, with a little automatic continuity, Bresar and Mathieu ([5] Theorem 2.8) have the following improvement of (3.9):

$$DL(A,A) \cap L^\sigma_\infty(A,A) \subseteq L^\sigma_{00}(A,A). \tag{4.9}$$

The rest of Theorem B in Curto and Mathieu [6] is just the combination of Lemma 4 and the trivialities of Theorem 2:

THEOREM 5. *If $a \in A$ and $b \in A$ are arbitrary then the following are equivalent*:

$$L_a - R_b \in L^\sigma_\infty(A,A); \tag{5.1}$$

$$L_b - R_a \in L^\sigma_\infty(A,A); \tag{5.2}$$

$$\text{(i) } L_a - R_a \in L^\sigma_{00}(A,A) \text{ and (ii) } L_b - R_b \in L^\sigma_{00}(A,A); \tag{5.3}$$

$$\{L_a, R_b\} \subseteq L^\sigma_\infty(A,A); \tag{5.4}$$

$$\{L_b, R_a\} \subseteq L^\sigma_\infty(A,A). \tag{5.5}$$

Proof. The equivalence of (5.4) and (5.5) is just (2.10). Implication (5.3) $\Longrightarrow$ (5.4) follows from (2.6): if (5.3) holds then for arbitrary $x \in A$ the spectral mapping theorem in two variables ([8] Theorem 11.3.4) gives

$$\sigma(ax) \subseteq \sigma(a)\sigma(x) \text{ and } \sigma(xb) \subseteq \sigma(b)\sigma(x);$$

alternatively remember that $|x|_\sigma = \lim_n \|x^n\|^{1/n}$. We claim that there is an implication

$$(5.6) \qquad (5.1) \text{ and } (5.3)(\text{ii}) \Longrightarrow (5.2):$$

for we have

$$L_a - L_b = (L_a - R_b) - (L_b - R_b) \in L^\sigma_\infty(A, A) + L^\sigma_{00}(A, A)$$

in $L^\sigma_\infty(A, A)$ by (2.5), hence $R_a - R_b \in L^\sigma_\infty(A, A)$ by (2.10), and finally

$$L_b - R_a = (L_b - R_b) - (R_a - R_b) \in L^\sigma_{00}(A, A) + L^\sigma_\infty(A, A)$$

in $L^\sigma_\infty(A, A)$ by (2.5) again. We have proved the implication (5.6): but this combines with (4.1) from Lemma 4 to give the implication (5.1) $\Longrightarrow$ (5.2). Interchanging a and b in (4.1) shows that (5.2) implies (5.3)(i): thus (5.1) implies the whole of (5.3). Similarly of course (5.2) implies both (5.1) and (5.3). In particular, recalling (4.3) from Lemma 4, we now also know that (5.4) $\Longrightarrow$ (5.3). Finally to see that (5.3) $\Longrightarrow$ (5.1) note that if (5.3) holds then also

$$L_{a-b} - R_{a-b} = (L_a - R_a) - (L_b - R_b) \in L^\sigma_{00}(A, A) + L^\sigma_{00}(A, A)$$

is in $L^\sigma_{00}(A, A)$ by (2.5), and hence also (implication (5.3) $\Longrightarrow$ (5.4)) $R_{a-b} \in L^\sigma_\infty(A, A)$. But now

$$L_a - R_b = R_{a-b} + (L_a - R_a) \in L^\sigma_\infty(A, A) + L^\sigma_{00}(A, A)$$

is in $L^\sigma_\infty(A, A)$ by (2.5) again. □

The special case of Theorem 5 in which either $a = 0$ or $b = 0$ is (4.3) equivalence (5.3) $\Longleftrightarrow$ (5.4), which is Theorem A of Curto and Mathieu [6], proved originally by Ptak using the subharmonicity of the spectral radius. The special case $a = b$ is equivalence (4.2), which is Bresar's result [4]. In the special case $A = C_\infty(\Omega)$ it is clear that each of the conditions (5.1)-(5.5) holds for arbitrary $a \in A$; if instead $A = BL(X, X)$ then, generalizing (2.14), (5.1) holds if and only if a and b are both scalars: this is obvious for each of the conditions (5.3)-(5.5). (2.5) and essentially (4.9) also give ([5] Theorem 2.8) for $D \in DL(A, A)$ and $a \in A$

$$(5.7) \qquad D + L_a \in L^\sigma_\infty(A, A) \Longleftrightarrow \{D, L_a\} \subseteq L^\sigma_\infty(A, A).$$

In another direction the condition (5.4), together with (2.2), gives

$$L_a R_b \in L^\sigma_\infty(A, A), \tag{5.8}$$

which suggests looking for extensions to "elementary operators".

When $L_a - R_b$ is in $L^\sigma_0(A, A)$ then it must ([6] Propositions 1,2) be in $L^\sigma_{00}(A, A)$:

THEOREM 6. *If $a \in A$ and $b \in A$ the followings are equivalent:*

$$L_a - R_b \in L^\sigma_0(A, A); \tag{6.1}$$

$$R_a - L_b \in L^\sigma_0(A, A); \tag{6.2}$$

$$\{L_a - R_a, L_b - R_b, L_{a-b}, R_{a-b}\} \subseteq L^\sigma_{00}(A, A); \tag{6.3}$$

$$L_a - R_b \in L^\sigma_{00}(A, A); \tag{6.4}$$

$$L_b - R_a \in L^\sigma_{00}(A, A). \tag{6.5}$$

Proof. If (6.1) holds then so does (5.1) and hence (5.3): thus the first two operators in (6.3) are spectrally trivial. Also

$$L_a - L_b = (L_a - R_b) - (L_b - R_b) \in L^\sigma_0(A, A) + L^\sigma_{00}(A, A)$$

is in $L^\sigma_0(A, A)$ and hence by (2.11) in $L^\sigma_{00}(A, A)$. Similarly $R_a - R_b$, so that the whole of (6.3) holds: but now

$$L_a - R_b = (L_a - R_a) + R_{a-b} \in L^\sigma_{00}(A, A) + L^\sigma_{00}(A, A)$$

is in $L^\sigma_{00}(A, A)$, giving (6.4), and similarly (6.5). Trivially of course (6.4) $\Longrightarrow$ (6.1) and (6.5) $\Longrightarrow$ (6.2). □

In the special case $A = C_\infty(\Omega)$ it is clear that each of the conditions (6.1)-(6.5) holds if and only if $a = b$; if $A = BL(X, X)$ then (6.3) holds if and only if $a = b = 0$. Analagous to (5.7), we also have

$$D + L_a \in L^\sigma_0(A, A) \Longleftrightarrow \{D, L_a\} \subseteq L^\sigma_0(A, A). \tag{6.6}$$

Complementary to "spectral boundedness" would be "spectral boundedness below":

DEFINITION 7. Write

$$L^\sigma_+(A, B) = \{T \in L(A, B) : T^{-1}QN(B) \subseteq QN(A)\} \tag{7.1}$$

for the linear mappings from A to B which are spectrally one and

$$L^\sigma_{++}(A, B) = \{T \in L(A, B) : \exists k > 0,\ \forall x \in A,\ |x|_\sigma \le k|Tx|_\sigma\} \tag{7.2}$$

for the linear mappings from A to B which are spectrally bounded below.

Analagous to Theorem 2,

THEOREM 8. *If A and B are linear algebras then*

$$\begin{aligned}(8.1)\qquad & L^\sigma_{++}(A,B)\cap L^\sigma_{00}(A,B)\subseteq L^\sigma_{++}(A,B)\cap L^\sigma_0(A,B)\\ &\subseteq L^\sigma_+(A,B)\cap L^\sigma_0(A,B)=\emptyset.\end{aligned}$$

If also D is a linear algebra then

$$(8.2)\qquad L^\sigma_*(B,D)\circ L^\sigma_*(A,B)\subseteq L^\sigma_*(A,D)\ \textit{for each}\ *=+,++;$$

conversely there is an implication

$$(8.3)\qquad S\in L^\sigma_\bullet(B,D)\ \textit{and}\ S\circ T\in L^\sigma_+(A,D)\Longrightarrow T\in L^\sigma_+(A,B)$$

and

$$(8.4)\qquad S\in L^\sigma_\infty(B,D)\ \textit{and}\ S\circ T\in L^\sigma_{++}(A,D)\Longrightarrow T\in L^\sigma_{++}(A,B).$$

There is an inclusion

(8.5)

$$L^\sigma_*(A,B)+L^\sigma_{00}(A,B)\subseteq L^\sigma_*(A,B)\ \textit{and}\ L^\sigma_*(A,B)\tilde{+}_{00}L^\sigma_0(A,B)\subseteq L^\sigma_*(A,B)$$

for each $*=+,++$.

Proof. Most of this can be left to the reader: for (8.1) observe that if $T\in L^\sigma_+(A,B)\cap L^\sigma_0(A,B)$ then $|1|_\sigma=0$, and for (8.2) recall $T(A)\subseteq B$. For (8.3) argue

$$|Tx|_\sigma=0\Longrightarrow|STx|_\sigma=0\Longrightarrow|x|_\sigma=0,$$

and for (8.4)

$$|x|_\sigma\le\ell|STx|_\sigma\le\ell\|S\|\ |Tx|_\sigma.$$

For (8.5) note that $|Tx+Sx|_\sigma=|Tx|_\sigma$ either if $Sx\in RAD(B)$, or if $Sx\in QN(B)$ commutes with $Tx\in B$ modulo $RAD(B)$. □

If for example $A=C_\infty(\Omega)$ with normal Hausdorff Ω and $a\in A$ then $L_a=R_a$ is spectrally bounded below if and only if $a\in A^{-1}$ is invertible, and is spectrally consistent if and only if a is not a zero-divisor, so that $a^{-1}(0)$ has empty interior in Ω. If $A=BL(X,X)$ for a Banach space X then only (non-zero) scalars can be spectrally consistent: for if $a\in A$ is not scalar and $(\xi,a\xi)$ is linearly independent in X then there is $\varphi\in X^\dagger$ for which $\varphi(\xi)=1$ and $\varphi(a\xi)=0$, and then

$$(8.6)\qquad y=\varphi\odot\xi\Longrightarrow 0\ne y=y^2\ \text{and}\ (ay)^2=0.$$

Zelazko [11], [12] showed that in commutative algebras A an element $a\in A$ for which L_a is not spectrally bounded below cannot be a topological zero divisor, and conversely if the algebra is "regular"; the extension to systems of elements ([8] Theorem 11.5.4) gives rise to points of the Silov boundary of A [7]. Of course the spectrum preserving maps of Jafarian and Sourour [9] are spectrally bounded below: it is sufficient [2],[3] that the "connected hull" of the spectrum is preserved. We note here a connection between spectral boundedness below and the "commutative closure" of the invertibles:

THEOREM 9. *If $a \in A$ there is an implication*

$$a \in \mathrm{cl}_{\mathrm{comm}} A^{-1} \text{ and } L_a \in L^{\sigma}_{++}(A, A) \Longrightarrow a \in A^{-1}, \tag{9.1}$$

where if $K \subseteq A$ we write $a \in \mathrm{cl}_{\mathrm{comm}} K$ to mean $a \in \mathrm{cl}(K \cap \mathrm{comm}(a))$, so that there is (a_n) in A for which

$$a_n \in K \text{ and } aa_n = a_n a \text{ and } \|a - a_n\| \to 0. \tag{9.2}$$

Proof. If

$$\|a - a_n\| \to 0 \text{ with } a_n b_n = 1 = b_n a_n \tag{9.3}$$

then either $a \in A^{-1}$ or

$$|(a - a_n)b_n|_\sigma \geq 1 \text{ for each } n \in \mathbf{N} : \tag{9.4}$$

because if (9.4) fails with $n = m$ then ab_m is invertible, and hence also a if $b_m = a_m^{-1}$ is invertible. If now a is in the commutative closure of A^{-1} then (9.4) holds with in addition

$$aa_n = a_n a : \tag{9.5}$$

but then

$$|ab_n|_\sigma = |1 + (a - a_n)b_n|_\sigma \leq 1 + |(a - a_n)b_n|_\sigma \leq 2\|a - a_n\| \, |b_n|_\sigma, \tag{9.6}$$

noting that b_n commutes with $a - a_n$ and remembering (2.8). □

If we specialize to scalar perturbations $a_n = a - \lambda_n$ then the analogue of (9.1) holds separately for left and right invertibility ((3.8) in [7]).

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