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ON SECOND ORDER NECESSARY OPTIMALITY CONDITIONS FOR VECTOR OPTIMIZATION PROBLEMS

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Abstract. Second order necessary optimality condition for properly efficient solutions of a twice differentiable vector optimization problem is given. We obtain a nonsmooth version of the second order necessary optimality condition for properly efficient solutions of a nondifferentiable vector optimization problem. Furthermore, we prove a second order necessary optimality condition for weakly efficient solutions of a nondifferentiable vector optimization problem.

1. Introduction and preliminaries

Vector optimization problems are those where two or more objectives are to be minimized on some set of feasible solutions. In such problems we deal with conflicts amongst objectives. Thus such problems have important applications in economics, game theory and statistical decision theory (see [1]-[5]). The objective functions in the problems may be differentiable (smooth) or nondifferentiable (nonsmooth). In most cases we can not find a feasible solution which is optimal in the sense that it minimizes all the objectives simultaneously. So, in vector optimization we should use concepts of solutions different from the just mentioned requirement of optimality. For vector optimization problems, there are three kinds of solutions, that is, properly efficient solution, efficient solution and weakly efficient solution (see [5, 6]).

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The concept of vector variational inequality was introduced by Giannessi [7] in 1980. Also, he [8] gave first order necessary optimality conditions, which are described by vector variational inequality, for efficient solutions or weakly efficient solutions of a differentiable convex vector optimization problem. Some authors have tried to improve Giannessi's idea on necessary optimality conditions (for example, see [9]-[12]).

Recently, nonsmooth analysis for treating nondifferentiable scalar optimization problems and nondifferentiable vector optimization problems has been greatly developed. Using generalized directional derivatives, generalized subgradients, derivatives for multifunctions, normal cones and tangent cones appeared in nonsmooth analysis, many authors have improved necessary optimality conditions for optimization problems (see [11]-[23] and therein references). In particular, using tangent cones and generalized directional derivatives, Ward and Lee [11, 12] obtained first order necessary optimality conditions for properly efficient solutions or weakly efficient solutions of nondifferentiable vector optimization problems.

The purpose of this paper is that following proofs of first order necessary optimality theorems of Ward and Lee [11, 12] for nondifferentiable vector optimization problems, we obtain second order necessary optimality conditions for the problems. We give a second order necessary optimality condition for properly efficient solutions of a twice differentiable vector optimization problem, and then obtain a nonsmooth version of the second order necessary optimality condition for properly efficient solutions of a nondifferentiable vector optimization problem. Furthermore, we prove a second order necessary optimality condition for weakly efficient solutions of a nondifferentiable vector optimization. Our main results can be regarded as second order versions of recent ones in Ward and Lee [11, 12].

Let S be a nonempty subset of $\mathbb{R}^n$ and $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, p$, functions.

Consider the following vector optimization problem (VP):

$$\text{(VP)} \qquad \text{Minimize} \quad f(x) := (f_1(x), \dots, f_p(x)) \quad \text{subject to} \quad x \in S.$$

Solving (VP) means finding the efficient solutions which are defined as follows:

DEFINITION 1.1. (1) $y \in S$ is said to be *an efficient solution* of (VP) if for any $x \in S$, $(f_1(x) - f_1(y), \dots, f_p(x) - f_p(y)) \notin -\mathbb{R}^p_+ \setminus \{0\}$, where $\mathbb{R}^p_+$ is the nonnegative orthant of $\mathbb{R}^p$.

(2) $y \in S$ is called *a properly efficient solution* of (VP) if $y \in S$ is efficient for (VP) and if there exists $M > 0$ such that for each $i = 1, \ldots, p$, we have

$$\frac{f_i(y) - f_i(x)}{f_j(x) - f_j(y)} \leqq M$$

for some j such that $f_j(x) > f_j(y)$ whenever $x \in S$ and $f_i(x) < f_i(y)$.

(3) $y \in S$ is said to be *a weakly efficient solution* of (VP) if for any $x \in S$, $(f_1(x) - f_1(y), \ldots, f_p(x) - f_p(y)) \notin -\text{int}\mathbb{R}^p_+$, where $\text{int}\mathbb{R}^p_+$ is the interior of $\mathbb{R}^p_+$.

The quantity $\frac{f_i(y)-f_i(x)}{f_j(x)-f_j(y)}$ may be interpreted as the marginal trade-off for the objective functions f_i and f_j between y and x. Geoffrion [24] defined the concept of the proper efficiency to eliminate the unbounded trade-off between the objective functions of (VP).

Now we introduce the normal cone and the singular approximate subdifferential studied by Mordukhovich [17].

DEFINITION 1.2. (1) Let S be a nonempty subset of $\mathbb{R}^n$ and $x \in \mathbb{R}^n$. Define

$$P(S, x) := \{w \in \text{cl}S \mid \| x - w \| = \inf_{z \in S} \| x - z \|\}.$$

Let $\bar{x} \in \text{cl}S$, where $\text{cl}S$ is the closure of the set S. The *normal cone to S at $\bar{x}$* is defined by

$$N(S, \bar{x}) := \{y \in \mathbb{R}^n \mid \exists \{y_n\} \to y, \ \{x_n\} \to \bar{x}, \ \{t_n\} \subset (0, +\infty), \\ \{c_n\} \subset \mathbb{R}^n \text{ with } c_n \in P(S, x_n) \text{ and } y_n = t_n(x_n - c_n)\}.$$

(2) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function and $x \in \mathbb{R}^n$. The *singular approximate subdifferential of f at x* is defined by

$$\partial^\infty f(x) := \{x^* \in \mathbb{R}^n \mid (x^*, 0) \in N(\text{epi}f, (x, f(x)))\},$$

where $\text{epi}f := \{(x, r) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leqq r\}$.

If S is a convex set, then $N(S, x)$ is the usual normal cone.

REMARK 1.1 ([17, Proposition 2.3]). A lower semicontinuous function f is Lipschitz near x if and only if $\partial^\infty f(x) = \{0\}$.

DEFINITION 1.3. (1) Let S be a nonempty closed subset of $\mathbb{R}^n$ and $x \in S$. *The adjacent cone to* S *at* x is defined by

$$\begin{aligned} T(S,x) &:= \{v \in \mathbb{R}^n \mid \forall \delta > 0, \exists \lambda > 0 \text{ s.t. } \forall t \in (0,\lambda), \exists y \in B(v,\delta) \\ &\qquad \text{with } x + ty \in S\} \\ &:= \{v \in \mathbb{R}^n \mid \forall \{t_n\} \downarrow 0,\ \exists \{v_n\} \to v \text{ with } x + t_n v_n \in S\}, \end{aligned}$$

where $B(v,\delta) = \{y \in \mathbb{R}^n \mid \|y - v\| < \delta\}$.

The *contingent cone to the set* S *at* x is the set

$$\begin{aligned} K(S,x) &:= \{v \in \mathbb{R}^n \mid \forall \delta > 0, \exists t \in (0,\delta), y \in B(v,\delta) \text{ with } x + ty \in S\} \\ &:= \{v \in \mathbb{R}^n \mid \exists \{t_n\} \to 0^+,\ \{v_n\} \to v \text{ with } x + t_n v_n \in S\}. \end{aligned}$$

(2) Let $f : \mathbb{R}^n \to \overline{\mathbb{R}} := [-\infty, +\infty]$ be finite at $x \in \mathbb{R}^n$. For the adjacent cone T, we define the T *directional derivative of* f *at* x in the direction $y \in \mathbb{R}^n$ by

$$f^T(x;y) := \inf\{r \in \mathbb{R} \mid (y,r) \in T(\text{epi}f, (x, f(x)))\}.$$

We can prove an expression of $f^T(x;y)$; for any $y \in \mathbb{R}^n$,

$$\begin{aligned} f^T(x;y) &:= \limsup_{t \to 0^+} \inf_{v \to y} \frac{f(x+tv) - f(x)}{t} \\ &:= \sup_{\epsilon > 0} \inf_{\delta > 0} \sup_{t \in (0,\delta)} \inf_{v \in B(y,\epsilon)} \frac{f(x+tv) - f(x)}{t}, \end{aligned}$$

where $B(y,\epsilon) := \{z \in \mathbb{R}^n \mid \| z - y \| < \epsilon\}$.

Also, we can check that if f is differentiable at $x \in \mathbb{R}^n$, then

$$f^T(x;y) = \langle \nabla f(x), y \rangle$$

for any $y \in \mathbb{R}^n$, where $\nabla f(x)$ is the gradient of f at x and $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathbb{R}^n$. Also, if f is Lipschitz near x, $f^T(x;y) = f^+(x;y)$ (see, [22]), where

$$f^+(x;y) = \limsup_{t \to 0^+} \frac{f(x+ty) - f(x)}{t}.$$

Now we give second order tangent sets, which are found in [20, 21];

DEFINITION 1.4. Let S be a nonempty closed subset of $\mathbb{R}^n$, and let $x \in S$ and $v \in \mathbb{R}^n$.
(a) The *second order contingent set* is defined by

$$
\begin{aligned}
&K^2(S,x,v)\\
&:= \{y \in \mathbb{R}^n \mid \forall \delta > 0, \exists t \in (0,\delta), w \in B(y,\delta) \text{ with } x + tv + \frac{1}{2}t^2 w \in S\}\\
&:= \{y \in \mathbb{R}^n \mid \exists \{t_n\} \to 0^+,\ \{y_n\} \to y \text{ with } x + t_n v + \frac{1}{2}t_n^2 y_n \in S\}.
\end{aligned}
$$

(b) The *second order adjacent set* is defined by

$$
\begin{aligned}
&T^2(S,x,v)\\
&:= \{y \in \mathbb{R}^n \mid \forall \delta > 0, \exists \lambda > 0 \text{ s.t. } \forall t \in (0,\lambda), \exists w \in B(y,\delta)\\
&\qquad \text{with } x + tv + \frac{1}{2}t^2 w \in S\}\\
&:= \{y \in \mathbb{R}^n \mid \forall \{t_n\} \to 0^+,\ \exists \{y_n\} \to y \text{ with } x + t_n v + \frac{1}{2}t_n^2 y_n \in S\}.
\end{aligned}
$$

$K^2(S,x,v)$ and $T^2(S,x,v)$ are (possibly empty) closed sets, but not necessarily cones. $T^2(S,x,v) \subset K^2(S,x,v)$, but the converse inclusion may not be true.

REMARK 1.2. Let $S \subset \mathbb{R}^n$ and $x \in S$. Then $K^2(S,x,0) = K(S,x)$ and $T^2(S,x,0) = T(S,x)$.

Now we will define second order directional derivative with the second order adjacent set.

DEFINITION 1.5. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be finite at x, and suppose that $f^T(x;v)$ is finite. The T^2 *directional derivative of* f *at* x *with respect to* $v \in \mathbb{R}^n,\ y \in \mathbb{R}^n$ is defined by

$$
d^2 f^T(x;v,y) := \inf \{r \in \mathbb{R} \mid (y,r) \in T^2(\text{epi}f, (x, f(x)), (v, f^T(x;v)))\}.
$$

The following proposition can be found in [21]:

PROPOSITION 1.1. *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be finite at $x \in \mathbb{R}^n$. If $f^T(x; v)$ is finite, then for all $y \in R^n$,*

$$
\begin{aligned}
& d^2 f^T(x; v, y) \\
& := \limsup_{t \to 0^+} \inf_{w \to y} 2(f(x + tv + t^2 w/2) - f(x) - t f^T(x; v))/t^2 \\
& := \sup_{\epsilon > 0} \inf_{\lambda > 0} \sup_{t \in (0,\lambda)} \inf_{w \in B(y,\epsilon)} 2(f(x + tv + t^2 w/2) - f(x) - t f^T(x; v))/t^2.
\end{aligned}
$$

REMARK 1.3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice differentiable at $x \in \mathbb{R}^n$. Then for all $y \in \mathbb{R}^n$,

$$d^2 f^T(x; v, y) = \langle \nabla f(x), y \rangle + \langle v, \nabla^2 f(x) v \rangle.$$

REMARK 1.4 ([20]). Suppose $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is finite at $x \in \mathbb{R}^n$ and $f^T(x; y)$ is finite. Then we have
(1) if $v = 0$, then $d^2 f^T(x; v, y) = f^T(x; y)$,
(2) $\operatorname{epi} d^2 f^T(x; v, \cdot) = T^2(\operatorname{epi} f, (x, f(x)), (v, f^T(x; v)))$.

REMARK 1.5 ([23]). If f is Lipschitz near x, then $d^2 f^T(x; v, y) = d^2 f^+(x; v, y)$, where

$$d^2 f^+(x; v, y) := \limsup_{t \to 0^+} \frac{2(f(x + tv + t^2 y/2) - f(x) - t f^+(x; v))}{t^2}.$$

2. Second order necessary optimality conditions

Now we will give a second order necessary optimality conditions for properly efficient solutions of the vector optimization problem (VP) introduced in Section 1.

THEOREM 2.1. *Suppose that $f_i, i = 1, \dots, p$, are twice differentiable. Let $\bar{x} \in S$ be a properly efficient solution of* (VP). *If $\langle \nabla f_k(\bar{x}), v \rangle = 0$, $k = 1, \dots, p$ and $v \in K(S, \bar{x})$, then for each $i \in \{1, \dots, p\}$,*

$$
\begin{aligned}
& \{y | \langle \nabla f_i(\bar{x}), y \rangle + \langle v, \nabla^2 f_i(\bar{x}) v \rangle < 0\} \\
& \cap \{y \mid \langle \nabla f_j(\bar{x}), y \rangle + \langle v, \nabla^2 f_j(\bar{x}) v \rangle \leqq 0, j \neq i\} \\
& \cap K^2(S, \bar{x}, v) = \emptyset,
\end{aligned}
$$

equivalently,

$$\{y \mid (\langle \nabla f_1(\bar{x}), y\rangle + \langle v, \nabla^2 f_1(\bar{x})v\rangle, \ldots, \langle \nabla f_p(\bar{x}), y\rangle + \langle v, \nabla^2 f_p(\bar{x})v\rangle) \in -\mathbb{R}^p_+ \setminus \{0\}\} \cap K^2(S, \bar{x}, v) = \emptyset.$$

Proof. Following the approach used in the proof of Theorem 3.16 in [6], we will prove this theorem. Let $\bar{x} \in S$ be a properly efficient solution of (VP). Let $v \in K(S, \bar{x})$ be such that $\langle \nabla f_k(\bar{x}), v\rangle = 0,\ k = 1, \ldots, p$. Suppose to the contrary that there exist $i \in \{1, \ldots, p\}$ and $y \in K^2(S, \bar{x}, v)$ such that $\langle \nabla f_i(\bar{x}), y\rangle + \langle v, \nabla^2 f_i(\bar{x})v\rangle < 0$ and $\langle \nabla f_j(\bar{x}), y\rangle + \langle v, \nabla^2 f_j(\bar{x})v\rangle \leqq 0,\ j \neq i$. Since $y \in K^2(S, \bar{x}, v)$, there exist sequences $\{t_n\} \to 0^+,\ \{y_n\} \to y$ such that $\bar{x} + t_n v + \frac{1}{2}{t_n}^2 y_n \in S$. By the twice differentiability of f_i at $\bar{x}$, we have

$$\begin{aligned} & f_i(\bar{x} + t_n v + \frac{1}{2}{t_n}^2 y_n) - f_i(\bar{x}) \\ &= \langle \nabla f_i(\bar{x}), t_n v + \frac{1}{2}{t_n}^2 y_n\rangle + \frac{1}{2}\langle t_n v + \frac{1}{2}{t_n}^2 y_n, \nabla^2 f_i(\bar{x})(t_n v + \frac{1}{2}{t_n}^2 y_n)\rangle \\ &\quad + \|t_n v + \frac{1}{2}{t_n}^2 y_n\|^2 \cdot \beta_i(\bar{x}; t_n v + \frac{1}{2}{t_n}^2 y_n), \end{aligned}$$

where $\lim_{n\to\infty} \beta_i(\bar{x}; t_n v + \frac{1}{2}{t_n}^2 y_n) = 0$. Since $\langle \nabla f_i(\bar{x}), v\rangle = 0$, we have

$$\begin{aligned} & \frac{1}{{t_n}^2}\left[f_i(\bar{x} + t_n v + \frac{1}{2}{t_n}^2 y_n) - f_i(\bar{x})\right] \\ &= \frac{1}{2}\langle \nabla f_i(\bar{x}), y_n\rangle + \frac{1}{2}\langle v + \frac{1}{2}t_n y_n, \nabla^2 f_i(\bar{x})(v + \frac{1}{2}t_n y_n)\rangle \\ &\quad + \|v + \frac{1}{2}t_n y_n\|^2 \cdot \beta_i(\bar{x}; t_n v + \frac{1}{2}{t_n}^2 y_n). \end{aligned}$$

Then, we have

$$\begin{aligned} & \lim_{n\to\infty} \frac{1}{{t_n}^2}\left[f_i(\bar{x} + t_n v + \frac{1}{2}{t_n}^2 y_n) - f_i(\bar{x})\right] \\ &= \frac{1}{2}\langle \nabla f_i(\bar{x}), y\rangle + \frac{1}{2}\langle v, \nabla^2 f_i(\bar{x})v\rangle < 0. \end{aligned}$$

So, there exists a natural number $\mathbb{N}$ such that for all $n \geqq \mathbb{N}$,

$$\frac{1}{{t_n}^2}\left[f_i(\bar{x} + t_n v + \frac{1}{2}{t_n}^2 y_n) - f_i(\bar{x})\right] < 0.$$

Thus we have

$$f_i(\bar{x} + t_n v + \frac{1}{2}{t_n}^2 y_n) < f_i(\bar{x}) \text{ for all } n \geqq \mathbb{N}.$$

Since $\bar{x}$ is an efficient solution of (VP), by choosing a subsequence of $\{\bar{x}+t_n v+\frac{1}{2}t_n^2 y_n\}$, if necessary, we can assume that $\tilde{I} = \{j : f_j(\bar{x}+t_n v+\frac{1}{2}{t_n}^2 y_n) > f_j(\bar{x})\}$ is constant for all $n \geqq \mathbb{N}$. By the twice differentiability $f_j,\ j \in \tilde{I}$, at $\bar{x}$, we have

$$\begin{aligned}
&f_j(\bar{x} + t_n v + \frac{1}{2}{t_n}^2 y_n) - f_j(\bar{x}) \\
&= \big\langle \nabla f_j(\bar{x}), t_n v + \frac{1}{2}{t_n}^2 y_n \big\rangle + \frac{1}{2}\big\langle t_n v + \frac{1}{2}{t_n}^2 y_n, \nabla^2 f_j(\bar{x})(t_n v + \frac{1}{2}{t_n}^2 y_n)\big\rangle \\
&\quad + \|t_n v + \frac{1}{2}{t_n}^2 y_n\|^2 \cdot \beta_j(\bar{x}; t_n v + \frac{1}{2}{t_n}^2 y_n),
\end{aligned}$$

where $\lim_{n\to\infty} \beta_j(\bar{x}; t_n v + \frac{1}{2}{t_n}^2 y_n) = 0$. Since $\big\langle \nabla f_j(\bar{x}), v\big\rangle = 0$, we have

$$\begin{aligned}
&\big\langle \nabla f_j(\bar{x}), y_n \big\rangle + \big\langle v + \frac{1}{2}t_n y_n, \nabla^2 f_j(\bar{x})(v + \frac{1}{2}t_n y_n)\big\rangle \\
&\quad + 2\|v + \frac{1}{2}t_n y_n\|^2 \cdot \beta_j(\bar{x}; t_n v + \frac{1}{2}{t_n}^2 y_n) > 0
\end{aligned}$$

for all $n \geqq \mathbb{N}$ and for all $j \in \tilde{I}$. Thus $\big\langle \nabla f_j(\bar{x}), y\big\rangle + \big\langle v, \nabla^2 f_j(\bar{x})v\big\rangle \geqq 0$ for all $j \in \tilde{I}$. Since $\big\langle \nabla f_j(\bar{x}), y\big\rangle + \big\langle v, \nabla^2 f_j(\bar{x})v\big\rangle \leqq 0$ for all $j \in \tilde{I}$, $\big\langle \nabla f_j(\bar{x}), y\big\rangle + \big\langle v, \nabla^2 f_j(\bar{x})v\big\rangle = 0$. For all $j \in \tilde{I}$, we have

$$\begin{aligned}
&\frac{f_i(\bar{x})-f_i(\bar{x}+t_n v+\frac{1}{2}{t_n}^2 y_n)}{f_j(\bar{x}+t_n v+\frac{1}{2}{t_n}^2 y_n)-f_j(\bar{x})} \\
&= \frac{-\big[\frac{1}{2}\big\langle \nabla f_i(\bar{x}), y_n\big\rangle + \frac{1}{2}\big\langle v+\frac{1}{2}t_n y_n, \nabla^2 f_i(\bar{x})(v+\frac{1}{2}t_n y_n)\big\rangle + \|v+\frac{1}{2}t_n y_n\|^2 \cdot \beta_i(\bar{x}; t_n v+\frac{1}{2}{t_n}^2 y_n)\big]}{\frac{1}{2}\big\langle \nabla f_j(\bar{x}), y_n\big\rangle + \frac{1}{2}\big\langle v+\frac{1}{2}t_n y_n, \nabla^2 f_j(\bar{x})(v+\frac{1}{2}t_n y_n)\big\rangle + \|v+\frac{1}{2}t_n y_n\|^2 \cdot \beta_j(\bar{x}; t_n v+\frac{1}{2}{t_n}^2 y_n)}.
\end{aligned}$$

Thus for all $j \in \tilde{I}$,

$$\lim_{n\to\infty} \frac{f_i(\bar{x}) - f_i(\bar{x} + t_n v + \frac{1}{2}{t_n}^2 y_n)}{f_j(\bar{x} + t_n v + \frac{1}{2}{t_n}^2 y_n) - f_j(\bar{x})} = \infty,$$

which contradicts the proper efficiency of $\bar{x}$. Hence for each $i \in \{1, \ldots, p\}$,

$$\begin{aligned}
&\{y | \big\langle \nabla f_i(\bar{x}), y\big\rangle + \big\langle v, \nabla^2 f_i(\bar{x})v\big\rangle < 0\} \\
&\cap \{y | \big\langle \nabla f_j(\bar{x}), y\big\rangle + \big\langle v, \nabla^2 f_j(\bar{x})v\big\rangle \leqq 0, j \neq i\} \\
&\cap K^2(S, \bar{x}, v) = \emptyset.
\end{aligned}$$

Hence we obtain the conclusion of Theorem 2.1. □

REMARK 2.1. Since $K^2(S, \bar{x}, 0) = K(S, \bar{x})$ and $0 \in K(S, \bar{x})$, letting $v = 0$, we can obtain from Theorem 2.1 that if $\bar{x} \in S$ is a properly efficient solution of (VP), then for each $i \in \{1, \ldots, p\}$,

$$\{y \mid \langle \nabla f_i(\bar{x}), y \rangle < 0\} \cap \{y \mid \langle \nabla f_j(\bar{x}), y \rangle \leqq 0, \ j \neq i\} \cap K(S, \bar{x}) = \emptyset,$$

equivalently,

$$\{y \mid (\langle \nabla f_1(\bar{x}), y \rangle, \ldots, \langle \nabla f_p(\bar{x}), y \rangle) \in -\mathbb{R}^p_+ \setminus \{0\}\} \cap K(S, \bar{x}) = \emptyset.$$

REMARK 2.2. Suppose that the constraint set S of (VP) is closed and convex. If $\bar{x} \in S$ is a properly efficient solution of (VP), then $\bar{x} \in S$ is a solution of the following vector variational inequality:

$$\begin{aligned} &(\langle \nabla f_1(\bar{x}), s - \bar{x} \rangle, \ldots, \langle \nabla f_p(\bar{x}), s - \bar{x} \rangle) \notin -\mathbb{R}^p_+ \setminus \{0\} \\ &\text{for any } s \in S. \end{aligned}$$

The vector variational inequality can be found in [8].

We give an example illustrating Theorem 2.1.

EXAMPLE 2.1. Let $f_1(x_1, x_2, x_3) = x_1$, $f_2(x_1, x_2, x_3) = x_2$ and $f(x_1, x_2, x_3) = (x_1, x_2)$. Let $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 \geqq -x_1\}$.

Consider a vector optimization problem (VP):

$$\begin{array}{ll} \text{Minimize} & f(x_1, x_2, x_3) \\ \text{subject to} & (x_1, x_2, x_3) \in S. \end{array}$$

Let $\bar{x} = (0, 0, 0)$. Then $\bar{x}$ is a properly efficient solution of (VP) and $K(S, (0, 0, 0)) = S$. Then we have

$$\begin{aligned} A &:= \{(v_1, v_2, v_3) \mid \langle \nabla f_i(0, 0, 0), (v_1, v_2, v_3) \rangle = 0, \ i = 1, 2\} \\ &= \{(v_1, v_2, v_3) \mid (1, 0, 0)^t (v_1, v_2, v_3) = 0, \ (0, 1, 0)^t (v_1, v_2, v_3) = 0\} \\ &= \{(0, 0, v_3) \mid v_3 \in \mathbb{R}\}. \end{aligned}$$

Hence $A \cap K(S, (0, 0, 0)) = \{(0, 0, v_3) \mid v_3 \in \mathbb{R}\}$.

Also, we have

$$\begin{aligned}
&(y_1, y_2, y_3) \in K^2(S, (0,0,0), (0,0,v_3)) \\
\Longleftrightarrow\ & \exists t_n \to 0^+,\ (y_1^n, y_2^n, y_3^n) \to (y_1, y_2, y_3) \\
&\qquad \text{s.t. } (0,0,0) + t_n(0,0,v_3) + \frac{1}{2}t_n^2(y_1^n, y_2^n, y_3^n) \in S \\
\Longleftrightarrow\ & \exists t_n \to 0^+,\ (y_1^n, y_2^n, y_3^n) \to (y_1, y_2, y_3) \\
&\qquad \text{s.t. } (\frac{1}{2}t_n^2 y_1^n,\ \frac{1}{2}t_n^2 y_2^n,\ t_n v_3 + \frac{1}{2}t_n^2 y_3^n) \in S \\
\Longleftrightarrow\ & \exists t_n \to 0^+,\ (y_1^n, y_2^n, y_3^n) \to (y_1, y_2, y_3) \\
&\qquad \text{s.t. } \frac{1}{2}t_n^2 y_2^n \geqq -\frac{1}{2}t_n^2 y_1^n \\
\Longleftrightarrow\ & y_2 \geqq -y_1.
\end{aligned}$$

Hence $K^2(S, (0,0,0), (0,0,v_3)) = \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_2 \geqq -y_1\}$. Let $v = (0,0,v_3)$, where $v_3 \in \mathbb{R}$. Then we have

$$\begin{aligned}
&\{y \mid \langle \nabla f_1(\bar{x}), y\rangle + \langle v, \nabla^2 f_1(\bar{x})v\rangle < 0\} \\
&\cap \{y \mid \langle \nabla f_2(\bar{x}), y\rangle + \langle v, \nabla^2 f_2(\bar{x})v\rangle \leqq 0\} \\
&= \{y \mid (1,0,0)^t(y_1, y_2, y_3) < 0\} \cap \{y \mid (0,1,0)^t(y_1, y_2, y_3) \leqq 0\} \\
&= \{y \mid y_1 < 0,\ y_2 \leqq 0\}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\{y \mid \langle \nabla f_1(\bar{x}), y\rangle + \langle v, \nabla^2 f_1(\bar{x})v\rangle < 0\} \\
&\cap \{y \mid \langle \nabla f_2(\bar{x}), y\rangle + \langle v, \nabla^2 f_2(\bar{x})v\rangle \leqq 0\} \\
&\cap K^2(S, (0,0,0), v) = \emptyset.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\{y \mid \langle \nabla f_2(\bar{x}), y\rangle + \langle v, \nabla^2 f_2(\bar{x})v\rangle < 0\} \\
&\cap \{y \mid \langle \nabla f_1(\bar{x}), y\rangle + \langle v, \nabla^2 f_1(\bar{x})v\rangle \leqq 0\} \\
&\cap K^2(S, (0,0,0), v) = \emptyset.
\end{aligned}$$

Thus the conclusion of Theorem 2.1 holds. □

The following example shows that Theorem 2.1 can not be extended to the efficient solution of (VP).

EXAMPLE 2.2. Let $f_1(x_1, x_2, x_3) = x_1$, $f_2(x_1, x_2, x_3) = x_2$ and $f(x_1, x_2, x_3) = (x_1, x_2)$. Let $S = \{(x_1, x_2, x_3) |\ x_2 = x_1^2,\ x_3 \in \mathbb{R}\}$. Consider a vector optimization problem (VP):

$$\begin{array}{ll} \text{Minimize} & f(x_1, x_2, x_3) \\ \text{subject to} & (x_1, x_2, x_3) \in S. \end{array}$$

Then $\bar{x} = (0, 0, 0)$ is an efficient solution of (VP), but not a properly efficient solution of (VP). Also, $K(S, \bar{x}) = \{(x_1, 0, x_3) | x_1, x_3 \in \mathbb{R}\}$. Moreover we have

$$\begin{aligned} &\{v \in \mathbb{R}^3 |\ \langle \nabla f_1(\bar{x}), (v_1, v_2, v_3) \rangle = 0,\ \langle \nabla f_2(\bar{x}), (v_1, v_2, v_3) \rangle = 0\} \\ &= \{v \in \mathbb{R}^3 \mid v_1 = 0,\ v_2 = 0,\ v_3 \in \mathbb{R}\} \\ &= \{(0, 0, v_3) \in \mathbb{R}^3 \mid v_3 \in \mathbb{R}\}. \end{aligned}$$

Thus

$$v := (0, 0, 1) \in \{v \in \mathbb{R}^3 |\ \langle \nabla f_i(\bar{x}), (v_1, v_2, v_3) \rangle = 0,\ i = 1, 2\} \cap K(S, \bar{x}).$$

Also, we have

$$\begin{aligned} &(y_1, y_2, y_3) \in K^2(S, (0, 0, 0), (0, 0, 1)) \\ \Longleftrightarrow\ &\exists t_n \to 0^+,\ (y_1^n, y_2^n, y_3^n) \to (y_1, y_2, y_3) \text{ s.t.} \\ &\qquad\qquad (0, 0, 0) + t_n(0, 0, 1) + \frac{1}{2} t_n^2 (y_1^n, y_2^n, y_3^n) \in S \\ \Longleftrightarrow\ &\exists t_n \to 0^+,\ (y_1^n, y_2^n, y_3^n) \to (y_1, y_2, y_3) \text{ s.t.} \\ &\qquad\qquad (\frac{1}{2} t_n^2 y_1^n,\ \frac{1}{2} t_n^2 y_2^n,\ t_n + \frac{1}{2} t_n^2 y_3^n) \in S \\ \Longleftrightarrow\ &\exists t_n \to 0^+,\ (y_1^n, y_2^n, y_3^n) \to (y_1, y_2, y_3) \text{ s.t. } \frac{1}{2} t_n^2 y_2^n = \frac{1}{4} t_n^4 (y_1^n)^2 \\ \Longleftrightarrow\ &y_2 = 0. \end{aligned}$$

Hence we have

$$\begin{aligned} &\{y \mid \langle \nabla f_1(\bar{x}), y \rangle + \langle v, \nabla^2 f_1(\bar{x}) v \rangle < 0\} \\ &\cap \{y |\ \langle \nabla f_2(\bar{x}), y \rangle + \langle v, \nabla^2 f_2(\bar{x}) v \rangle \leqq 0\} \cap K^2(S, (0, 0, 0), (0, 0, 1)) \\ &= \{(y_1, y_2, y_3) |\ y_1 < 0,\ y_2 \leqq 0,\ y_3 \in \mathbb{R}\} \cap \{(y_1, 0, y_3) |\ y_1 \in \mathbb{R}, y_3 \in \mathbb{R}\} \\ &\neq \emptyset. \end{aligned}$$

So, the conclusion of Theorem 2.1 does not hold. □

We can easily obtain the following second order necessary optimality condition for weakly efficient solutions of (VP):

THEOREM 2.2. *Suppose that $f_i, i = 1, \ldots, p$, are twice differentiable. Let $\bar{x} \in S$ be a weakly efficient solution of* (VP). *If $\langle \nabla f_k(\bar{x}), v \rangle = 0, k = 1, \ldots, p$, and $v \in K(S, \bar{x})$, then*

$$\{y \mid \langle \nabla f_i(\bar{x}), y \rangle + \langle v, \nabla^2 f_i(\bar{x}) v \rangle < 0, i = 1, \ldots, p\} \cap K^2(S, \bar{x}, v) = \emptyset.$$

3. Nonsmooth versions of Theorems 2.1 and 2.2

In this section, using the arguments in the proofs of Ward and Lee [11, 12], we obtain the nonsmooth versions of Theorems 2.1 and 2.2.

THEOREM 3.1 ([19], (Intersection Theorem)). *Let S_i, $i = 0, 1, \ldots, p$ be a nonempty closed subset of $\mathbb{R}^n$ and $x \in \bigcap_{i=0}^{p} S_i$. Suppose that*

$$\sum_{i=0}^{p} y_i = 0, \ \ y_i \in N(S_i, x), \ i = 0, 1, \ldots, p, \ \text{imply } y_i = 0, \ \ i = 0, 1, \ldots, p.$$

Then we have

$$K^2(S_0, x, v) \cap \bigcap_{i=1}^{p} T^2(S_i, x, v) \subset K^2(\bigcap_{i=0}^{p} S_i, x, v).$$

Using the above Theorem 3.1, we can obtain the nonsmooth version of Theorem 2.1 as follows.

THEOREM 3.2. *Let the constraint set S of* (VP) *be a nonempty closed subset of $\mathbb{R}^n$ and $\bar{x} \in S$, and suppose that the functions f_i, $i = 1, \ldots, p$, of* (VP) *are lower semicontinuous. Assume that*

$$\text{(3.1)} \qquad \sum_{i=0}^{p} z_i = 0, \ \ z_0 \in N(S, \bar{x}) \ \text{and} \ z_i \in \partial^{\infty} f_i(\bar{x}), \ \ i = 1, \ldots, p$$

imply $z_i = 0$, $i = 0, 1, \ldots, p$.

Let $\bar{x} \in S$ be a properly efficient solution of (VP). *If $f_i^T(\bar{x}; v) = 0$, $i = 1, \ldots, p$, and $v \in K(S, \bar{x})$, then for each $i \in \{1, \ldots, p\}$,*

(3.2)

$$\{y \mid d^2 f_i^T(\bar{x}; v, y) < 0, \ \ d^2 f_j^T(\bar{x}; v, y) \leqq 0, \ \ j \neq i\} \cap K^2(S, \bar{x}, v) = \emptyset.$$

Proof. Let v be such that $f_i^T(\bar{x};v) = 0,\ \ i = 1,\dots,p$, and $v \in K(S,\bar{x})$. Suppose that (3.2) is false. Reordering the f_i, if necessary, we say that there exists $y \in K^2(S,\bar{x},v)$ such that

$$d^2 f_1^T(\bar{x};v,y) < 0 \ \text{ and } \ d^2 f_i^T(\bar{x};v,y) \leqq 0,\ \ i = 2,\dots,p.$$

Then we can choose $r < 0$ such that $d^2 f_1^T(\bar{x};v,y) < r$. Define

$$S_0 := S \times \mathbb{R}^p$$

and

$$S_i := \{(x, r_1,\dots,r_p) \in \mathbb{R}^{n+p} \ \mid \ f_i(x) \leqq r_i\},\ \ i = 1,\dots,p.$$

Since f_i is lower semicontinuous, S_i is closed. Since $y \in K^2(S,\bar{x},v)$, then $(y,r,0,\dots,0) \in K^2(S_0,(\bar{x},f(\bar{x})),(v,0,\dots,0))$.

Indeed,

$$\begin{aligned}
& y \in K^2(S,\bar{x},v) \\
\iff & \exists t_n \to 0^+,\ \ y_n \to y \ \text{ s.t. } \bar{x} + t_n v + \frac{1}{2}t_n^2 y_n \in S \\
& \Rightarrow (\bar{x}, f_1(\bar{x}),\dots,f_p(\bar{x})) + t_n(v,0,\dots,0) + \frac{1}{2}t_n^2(y_n,r,0,\dots,0) \\
& = (\bar{x} + t_n v + \frac{1}{2}t_n^2 y_n, f_1(\bar{x}) + \frac{1}{2}t_n^2 r, f_2(\bar{x}),\dots,f_p(\bar{x})) \\
& \in S \times \mathbb{R}^p = S_0 \\
& \Rightarrow (y,r,0,\dots,0) \in K^2(S_0,(\bar{x},f_1(\bar{x}),\dots,f_p(\bar{x})),(v,0,\dots,0)).
\end{aligned}$$

Since $d^2 f_1^T(\bar{x};v,y) < r$, $(y,r) \in \text{epi}d^2 f_1^T(\bar{x};v,\cdot) = T^2(\text{epi}f_1,(\bar{x},f_1(\bar{x})), (v, f_1^T(\bar{x},v)))$. Hence we have $(y,r,0,\dots,0) \in T^2(S_1,(\bar{x},f(\bar{x})),(v,0,\dots,0))$. Since $d^2 f_i^T(\bar{x};v,y) \leq 0,\ \ i = 2,\dots,p$, then $(y,0) \in T^2(\text{epi}f_i,(\bar{x}, f_i(\bar{x})),(v,f_i^T(\bar{x},v))$. So we have $(y,r,0,\dots,0) \in T^2(S_i,(\bar{x},f(\bar{x})),(v,0,\dots,0)),\ \ i = 2,\dots,p$. Therefore, we have

$$\begin{aligned}
& (y,r,0,\dots,0) \\
& \in K^2(S_0,(\bar{x},f(\bar{x})),(v,0,\dots,0)) \cap \bigcap_{i=1}^{p} T^2(S_i,(\bar{x},f(\bar{x})),(v,0,\dots,0)).
\end{aligned}$$

Let $d_i := (x_0^{*i}, d_1^i,\dots,d_p^i) \in N(S_i,(\bar{x},f(\bar{x}))),\ i = 0,1,\dots,p$ with $\sum_{i=0}^p d_i = 0$. We will prove that $d_i = 0,\ i = 0,1,\dots,p$. We notice that

$$\begin{aligned}
N(S_0,(\bar{x},f(\bar{x}))) &= N(S,\bar{x}) \times N(\mathbb{R}^p, f(\bar{x})) \\
&= N(S,\bar{x}) \times \{0\}.
\end{aligned}$$

Since $d_0 \in N(S_0, (\bar{x}, f(\bar{x})))$, $x_0^{*0} \in N(S, \bar{x})$ and $d_i^0 = 0$, $i = 1, \dots, p$. We can check that for $i = 1, \dots, p$,

$$N(S_i, (\bar{x}, f(\bar{x}))) = \{(x^*, y_1^*, \dots, y_p^*) \in \mathbb{R}^{n+p} \mid y_j^* = 0, \ j \neq i, \\ (x^*, y_i^*) \in N(\text{epi} f_i, (\bar{x}, f_i(\bar{x})))\}.$$

Since $d_i \in N(S_i, (\bar{x}, f(\bar{x})))$, $i = 1, \dots, p$, then $d_j^i = 0$, $j \neq i$ and $(x_0^{*i}, d_i^i) \in N(\text{epi} f_i, (\bar{x}, f_i(\bar{x})))$. Since $\sum_{i=0}^p d_i = 0$, then $\sum_{i=0}^p x_0^{*i} = 0$ and $d_i^i = 0$, $i = 1, \dots, p$. Hence we have

$$\sum_{i=0}^p x_0^{*i} = 0, \ \ x_0^{*0} \in N(S, \bar{x}) \ \text{ and } \ x_0^{*i} \in \partial^\infty f_i(\bar{x}), \ \ i = 1, \dots, p.$$

So, from assumption (3.1), $x_0^{*i} = 0$, $i = 0, 1, \dots, p$. Consequently, $d_i = 0$, $i = 0, 1, \dots, p$. Thus from Theorem 3.1, we have

$$(y, r, 0, \dots, 0) \in K^2(\bigcap_{i=0}^p S_i, (\bar{x}, f(\bar{x})), (v, 0, \dots, 0)).$$

So, there exist $t_n \to 0^+, (y_n, r_n^1 \dots, r_n^p) \to (y, r, 0, \dots, 0)$ such that

$$(\bar{x}, f_1(\bar{x}), \dots, f_p(\bar{x})) + t_n(v, 0, \dots, 0) + \frac{1}{2} t_n^2 (y_n, r_n^1, \dots, r_n^p) \in \bigcap_{i=0}^p S_i$$

that is, $\bar{x} + t_n v + \frac{1}{2} t_n^2 y_n \in S$ and

$$f_i(\bar{x} + t_n v + \frac{1}{2} t_n^2 y_n) \leqq f_i(\bar{x}) + \frac{1}{2} t_n^2 r_n^i, \ \ i = 1, \dots, p.$$

So, we have

$$(f_1(\bar{x}), \dots, f_p(\bar{x})) + \frac{1}{2} t_n^2 (r_n^1, \dots, r_n^p) \in f(S) + \mathbb{R}_+^p.$$

Since $(r_n^1, \dots, r_n^p) \to (r, 0, \dots, 0)$, then we have

$$(r, 0, \dots, 0) \in K(f(S) + \mathbb{R}_+^p, f(\bar{x})).$$

Since $r < 0$, we have $K(f(S) + \mathbb{R}_+^p, f(\bar{x})) \bigcap (-\mathbb{R}_+^p) \neq \{0\}$. Notice that $\bar{x} \in S$ is properly efficient for (VP) if and only if $K(f(S) +$

$\mathbb{R}^p_+, f(\bar{x}))\bigcap(-\mathbb{R}^p_+) = \{0\}$ (see, [6, 25]). So, $\bar{x} \in S$ is not a properly efficient solution of (VP). □

By Remarks 1.1 and 1.5, and Theorem 3.2, we can easily obtain the following corollary.

COROLLARY 3.1. *Let the constraint set S of* (VP) *be a nonempty closed subset of $\mathbb{R}^n$, and suppose that the functions $f_i, i = 1, \ldots, p$, of* (VP) *are locally Lipschitzian. Let $\bar{x} \in S$ be a properly efficient solution of* (VP). *If $f_i^+(\bar{x}; v) = 0,\ i = 1, \ldots, p$ and $v \in K(S, \bar{x})$, then for each $i \in \{1, \ldots, p\}$,*

$$\{y \mid d^2 f_i^+(\bar{x}; v, y) < 0,\ d^2 f_j^+(\bar{x}; v, y) \leqq 0,\ j \neq i\} \cap K^2(S, \bar{x}, v) = \emptyset.$$

By Remarks 1.2 and 1.4, and Theorem 3.2, we can obtain the following first order necessary optimality theorem for a properly efficient solution of (VP).

COROLLARY 3.2. *Let the constraint set S of* (VP) *be a nonempty closed subset of $\mathbb{R}^n$ and $\bar{x} \in S$, and suppose that the functions $f_i,\ i = 1, \ldots, p$, of* (VP) *are lower semicontinuous. Assume that*

$$\sum_{i=0}^{p} z_i = 0,\ \ z_0 \in N(S, \bar{x}) \ \text{ and } \ z_i \in \partial^\infty f_i(\bar{x}),\ \ i = 1, \ldots, p$$

imply $z_i = 0,\ i = 0, 1, \ldots, p$, and $v = 0$.

Let $\bar{x} \in S$ be a properly efficient solution of (VP). *If $f_i^T(\bar{x}; 0) = 0, i = 1, \ldots, p$, then for each $i \in \{1, \ldots, p\}$,*

$$\{y|\ f_i^T(\bar{x}; y) < 0,\ f_j^T(\bar{x}; y) \leqq 0,\ j \neq i\} \cap K(S, \bar{x}) = \emptyset.$$

Following the proofs of Ward and Lee [11, 12], we obtain the following second order necessary optimality theorem for weakly efficient solutions of the vector optimization problem (VP).

THEOREM 3.3. *Let the constraint set S of* (VP) *be a nonempty closed subset of $\mathbb{R}^n$ and $\bar{x} \in S$, and suppose that the functions $f_i,\ i = 1, \ldots, p$, of* (VP) *are lower semicontinuous. Assume that*

$$\sum_{i=0}^{p} z_i = 0,\ \ z_0 \in N(S, \bar{x}) \ \text{ and } \ z_i \in \partial^\infty f_i(\bar{x}),\ \ i = 1, \ldots, p \tag{3.3}$$

imply $z_i = 0, \ \ i = 0, 1, \ldots, p$.

Let $\bar{x} \in S$ *be a weakly efficient solution of* (VP). *If* $f_i^T(\bar{x}; v) = 0, \ \ i = 1, \ldots, p$ *and* $v \in K(S, \bar{x})$,

$$\{y| \ d^2 f_i^T(\bar{x}; v, y) < 0, \ \ i = 1, \ldots, p\} \cap K^2(S, \bar{x}, v) = \emptyset. \tag{3.4}$$

Proof. Let v be such that $f_i^T(\bar{x}; v) = 0, \ \ i = 1, \ldots, p$, and $v \in K(S, \bar{x})$. Suppose that (3.4) is false. Then there exists $y \in K^2(S, \bar{x}, v)$ such that

$$d^2 f_i^T(\bar{x}; v, y) < 0, \ \ i = 1, \ldots, p.$$

Then we can choose $r_i < 0$ such that $d^2 f_i^T(\bar{x}; v, y) < r_i, \ \ i = 1, \ldots, p$. Define

$$S_0 := S \times \mathbb{R}^p$$

and

$$S_i := \{(x, r_1, \ldots, r_p) \in \mathbb{R}^{n+p} \mid f_i(x) \leqq r_i\}, \ \ i = 1, \ldots, p.$$

Since f_i is lower semicontinuous, S_i is closed. Since $y \in K^2(S, \bar{x}, v)$, then $(y, r_1, \ldots, r_p) \in K^2(S_0, (\bar{x}, f(\bar{x})), (v, 0, \ldots, 0))$. Since $d^2 f_i^T(\bar{x}; v, y) < r_i$, $(y, r_i) \in \text{epi} d^2 f_i^T(\bar{x}; v, \cdot) = T^2(\text{epi} f_i, (\bar{x}, f_i(\bar{x})), (v, f_i^T(\bar{x}, v)))$. Hence we have $(y, r_1, \ldots, r_p) \in T^2(S_i, (\bar{x}, f(\bar{x})), (v, 0, \ldots, 0))$. Therefore, we have

$$\begin{aligned}&(y, r_1, \ldots, r_p) \\ &\in K^2(S_0, (\bar{x}, f(\bar{x})), (v, 0, \ldots, 0)) \cap \bigcap_{i=1}^{p} T^2(S_i, (\bar{x}, f(\bar{x})), (v, 0, \ldots, 0)).\end{aligned}$$

Let $d_i := (x_0^{*i}, d_1^i, \ldots, d_p^i) \in N(S_i, (\bar{x}, f(\bar{x}))), \ i = 0, 1, \ldots, p$ with $\sum_{i=0}^{p} d_i = 0$. By the same argument in the proof of Theorem 3.2, we can prove that $d_i = 0, \ \ i = 0, 1, \ldots, p$. Thus from Theorem 3.1, we have

$$(y, r_1, \ldots, r_p) \in K^2(\bigcap_{i=0}^{p} S_i, (\bar{x}, f(\bar{x})), (v, 0, \ldots, 0)).$$

So, there exist $t_n \to 0^+, (y_n, r_n^1, \ldots, r_n^p) \to (y, r_1, \ldots, r_p), \ \ r_n^i < 0$ such that

$$(\bar{x}, f_1(\bar{x}), \ldots, f_p(\bar{x})) + t_n(v, 0, \ldots, 0) + \frac{1}{2} t_n^2 (y_n, r_n^1, \ldots, r_n^p) \in \bigcap_{i=0}^{p} S_i,$$

that is, $\bar{x} + t_n v + \frac{1}{2}t_n^2 y_n \in S$ and

$$f_i(\bar{x} + t_n v + \frac{1}{2}t_n^2 y_n) \leqq f_i(\bar{x}) + \frac{1}{2}t_n^2 r_n^i, \quad i = 1, \dots, p.$$

Since $r_n^i < 0, \ \ i = 1, \dots, p$, then $f_i(\bar{x}+t_n v+\frac{1}{2}t_n^2 y_n) < f_i(\bar{x})$. Thus $\bar{x} \in S$ is not weakly efficient for (VP). □

By Remarks 1.1 and 1.5, and Theorem 3.3, we can easily obtain the following corollary.

COROLLARY 3.3. *Let the constraint set S of* (VP) *be a nonempty closed subset of $\mathbb{R}^n$, and suppose that the functions $f_i, \ \ i = 1, \dots, p$, of* (VP) *are locally Lipschitzian. Let $\bar{x} \in S$ be a weakly efficient solution of* (VP). *If $f_i^+(\bar{x}; v) = 0, \ \ i = 1, \dots, p$ and $v \in K(S, \bar{x})$, then*

$$\{y \mid d^2 f_i^+(\bar{x}; v, y) < 0, \ \ i = 1, \dots, p\} \cap K^2(S, \bar{x}, v) = \emptyset.$$

By Remarks 1.2 and 1.4, and Theorem 3.3, we can obtain the following first order necessary optimality theorem for a weakly efficient solution of (VP).

COROLLARY 3.4 ([12]). *Let the constraint set S of* (VP) *be a nonempty closed subset of $\mathbb{R}^n$ and $\bar{x} \in S$, and suppose that the functions $f_i, \ \ i = 1, \dots, p$, of* (VP) *are lower semicontinuous. Assume that*

$$\sum_{i=0}^{p} z_i = 0, \quad z_0 \in N(S, \bar{x}) \ \text{ and } \ z_i \in \partial^\infty f_i(\bar{x}), \quad i = 1, \dots, p$$

imply $z_i = 0, \ \ i = 0, 1, \dots, p$.

Let $\bar{x} \in S$ be a weakly efficient solution of (VP). *If $f_i^T(\bar{x}; 0) = 0, i = 1, \dots, p$, then*

$$\{y \mid f_i^T(\bar{x}; y) < 0, \ \ i = 1, \dots, p\} \cap K(S, \bar{x}) = \emptyset.$$

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