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ON CLASS ALGEBRAS

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ABSTRACT. Let $F^\alpha G$ be a twisted group algebra. A subalgebra of $F^\alpha G$ generated by all class sums of partition $\mathcal{P}$ of G is called a projective class algebra in $F^\alpha G$ associated with partition $\mathcal{P}$. In this paper we study various partitions of G determined by actions of certain operator groups on G and construct projective class algebras depending on the actions. With regard to projective class algebras, we investigate structures of associated skew group algebras and fixed group algebras.

1. Introduction

Let G be a finite group with identity 1, F be a field of characteristic $p \geq 0$ and $F^* = F \setminus \{0\}$ be the multiplicative group of F with trivial G-action. For a 2-cocycle α in $Z^2(G, F^*)$, let $F^\alpha G$ be the twisted group algebra of G over F with F-basis $\{u_g | g \in G\}$ such that $u_1 = 1 = 1_{F^\alpha G}$ and $u_g u_x = \alpha(g,x) u_{gx}$ for all $g, x \in G$.

Let $\mathcal{P} = \{\mathcal{E}_g | g \in G\}$ be a partition of G consisting of certain classes $\mathcal{E}_g$, and let $o_g^+ = \sum_{x \in \mathcal{E}_g} u_x$ be the class sum in $F^\alpha G$ containing g. An algebra Λ over F generated by all class sums is called the *projective class algebra* associated with partition $\mathcal{P}$. The projective class algebra is a subalgebra of the twisted group algebra $F^\alpha G$, and we may write $\Lambda = \oplus_g F o_g^+ < F^\alpha G$, where the sum is taken over all distinct classes $\mathcal{E}_g$ in $\mathcal{P}$.

Clearly $F^\alpha G$ itself is a projective class algebra in $F^\alpha G$. And the center algebra $Z(F^\alpha G)$ is a projective class algebra in $F^\alpha G$ associated with the partition $\mathcal{P}$ consisting of α-regular classes of G. If $\alpha = 1$, the algebra generated by $\sum_{x \in \mathcal{E}_g} x$ in the group algebra FG is *a class algebra* in FG associated with partition $\mathcal{P} = \{\mathcal{E}_g | g \in G\}$. Moreover, for all

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$\mathcal{E}_g \in \mathcal{P}$ if $\mathcal{E}_g^{-1} = \{x^{-1} | x \in \mathcal{E}_g\} = \mathcal{E}_{g^{-1}}$ belongs to $\mathcal{P}$, $\mathcal{E}_1 = \{1\}$ and if $\Lambda = \oplus_g F o_g^+$ is a subalgebra of FG with unit element then Λ is said to be a *Schur algebra* in FG, that is, a Schur algebra is a subalgebra of FG associated to a partition of G. Schur algebras were introduced by Schur and Wielandt [10], and studied intensively by Tamaschke [8] and Roesler [7]. Class algebras were investigated by Wigner [9], and recently in [4], [3] and [1].

The purpose of this paper is to study projective class algebra Λ in $F^\alpha G$ associated with partition $\mathcal{P}$ on G. Let Ω be an operator group acting on G by an action τ and $\mathcal{P}$ be the set of all Ω-orbits of G. Then $\mathcal{P}$ forms a partition of G, in this case, a projective class algebra in $F^\alpha G$ associated with $\mathcal{P}$ will be called a projective class algebra by τ. In section 2, we study projective class algebras Λ in $F^\alpha G$ by various actions $\tau : \Omega \to \mathrm{Aut}(G)$ and investigate structures of the fixed subalgebra Λ^τ by τ and the associated skew group algebra $\Lambda *_\tau \Omega$. In section 3, by making use of some linear transformations τ on G and on Galois group $\mathrm{Gal}(E/F)$, where E is an algebraic closure of F, we will determine projective class algebras $F^\alpha G *_\tau \mathrm{Gal}(E/F)$ and $F^\beta(G \times \mathrm{Gal}(E/F))$ for some 2-cocycle $\beta \in Z^2(G \times \mathrm{Gal}(E/F), F^*)$.

1. Class algebra associated with operator group

In this paper we always assume that G is a finite group, F is a field and $F^\alpha G$ is a twisted group algebra of G over F with a 2-cocycle $\alpha \in Z^2(G, F^*)$ such that $u_g u_x = \alpha(g,x) u_{gx}$ for $g, x \in G$. And we denote by $\mathrm{Aut}(G)$ the automorphism group of G. When Ω is any group acting on a group X under $\tau : \Omega \to \mathrm{Aut}(X)$, the semidirect product group $X \times_\tau \Omega$ afforded by τ satisfies the multiplication rule $(x_1, \omega_1)(x_2, \omega_2) = (x_1 \tau(\omega_1) x_2, \omega_1 \omega_2)$ for $x_i \in X$, $\omega_i \in \Omega$ $(i = 1, 2)$. We denote by $\mathrm{St}_\tau(x)$ the stabilizer of $x \in X$, i.e., $\mathrm{St}_\tau(x) = \{\omega \in \Omega \mid \tau(\omega)(x) = x\}$.

Let $\tau : \Omega \to \mathrm{Aut}(G)$ be an action of an operator group Ω on G, and let $G^\tau = \{g \in G | \ \tau(\omega) g = g \text{ for all } \omega \in \Omega\}$ denote the fixed subgroup of G by τ. For any $\omega \in \Omega$, assume that each $\tau(\omega)$ extends to an algebra isomorphism $\hat{\tau}(\omega)$ of $F^\alpha G$ by $\hat{\tau}(\omega)(\sum_{g \in G} r_g u_g) = \sum_{g \in G} r_g u_{\tau(\omega) g}$, where $r_g \in F^*$. Then for the $\hat{\tau} : \Omega \to \mathrm{Aut}(F^\alpha G)$, we denote by $(F^\alpha G)^{\hat{\tau}}$ the fixed subalgebra $\{a \in F^\alpha G | \ \tau(\omega) a = a \text{ for all } \omega \in \Omega\}$ of $F^\alpha G$ by $\hat{\tau}$.

THEOREM 1. *Let $\tau : \Omega \to Aut(G)$ be an action of an operator group Ω, and assume $\tau(\omega)$ extends to an algebra isomorphism $\hat{\tau}(\omega)$ of $F^\alpha G$*

for any $\omega \in \Omega$. Then $\alpha(g,x) = \alpha(\tau(\omega)g, \tau(\omega)x)$ for any $g, x \in G$, and $(F^\alpha G)^{\hat{\tau}} = F^\alpha(G^\tau)$, where the same symbol α is used for the restriction to G^τ. Thus the algebra $(F^\alpha G)^{\hat{\tau}}$ is a projective class algebra in $F^\alpha G$ by τ.

Proof. Let $g, x \in G$ and $\omega \in \Omega$. Then $\tau(\omega)$ satisfies

$$\hat{\tau}(\omega)u_g\hat{\tau}(\omega)u_x = u_{\tau(\omega)g}u_{\tau(\omega)x} = \alpha\Big(\tau(\omega)g, \tau(\omega)x\Big)u_{\tau(\omega)g\tau(\omega)x}$$

while

$$\hat{\tau}(\omega)(u_g u_x) = \alpha(g,x)u_{\tau(\omega)(gx)} = \alpha(g,x)u_{\tau(\omega)g\tau(\omega)x},$$

hence we have $\alpha(g,x) = \alpha\Big(\tau(\omega)g, \tau(\omega)x\Big)$.

Choose any $u_g \in (F^\alpha G)^{\hat{\tau}}$. Then $u_g = \hat{\tau}(\omega)u_g = u_{\tau(\omega)g}$ for all $\omega \in \Omega$, and $1 = u_{\tau(\omega)g}{u_g}^{-1} = \alpha^{-1}(g, g^{-1})\alpha(\tau(\omega)g, g^{-1})u_{\tau(\omega)g\cdot g^{-1}}$. Thus

$$g = \tau(\omega)g \quad \text{and} \quad \alpha(g, g^{-1}) = \alpha(\tau(\omega)g, g^{-1}).$$

But the former yields the latter, hence we have $g \in G^\tau$ and $u_g \in F^\alpha(G^\tau)$. Conversely, if $u_g \in F^\alpha(G^\tau)$ then $g \in G^\tau$ and $\hat{\tau}(\omega)u_g = u_{\tau(\omega)g} = u_g$ for all $\omega \in \Omega$, hence $u_g \in (F^\alpha G)^{\hat{\tau}}$. This proves $(F^\alpha G)^{\hat{\tau}} = F^\alpha(G^\tau)$.

Let $\mathcal{P} = \{\mathcal{O}_\tau(g) | g \in G\}$ be the partition of G consisting of orbits $\mathcal{O}_\tau(g) = \{\tau(\omega)g | \omega \in \Omega\}$ and let $o_g^+ = \sum_{x \in \mathcal{O}_\tau(g)} u_x$ be the class sum. Consider the projective class algebra $\Lambda = \oplus_g F o_g^+$ associated with $\mathcal{P}$, where the sum is taken over distinct classes in G. We now will show that $\Lambda = (F^\alpha G)^{\hat{\tau}}$. Obviously a generator o_g^+ of Λ is contained in $F^\alpha G$. And

$$\hat{\tau}(\omega)(o_g^+) = \hat{\tau}(\omega)\Big(\sum_{x \in \mathcal{O}_\tau(g)} u_x\Big) = \sum_{x \in \mathcal{O}_\tau(g)} u_{\tau(\omega)x} = \sum_{y \in \mathcal{O}_\tau(g)} u_y = o_g^+$$

for any $\omega \in \Omega$, where the third equality follows from that if we let $\tau(\omega)x = y$ then since $x \in \mathcal{O}_\tau(g)$ we can write $x = \tau(\omega')g$ for some $\omega' \in \Omega$ hence $y = \tau(\omega)x = \tau(\omega\omega')g$ and $y \in \mathcal{O}_\tau(g)$. Thus $o_g^+ \in (F^\alpha G)^{\hat{\tau}}$, and Λ is contained in $(F^\alpha G)^{\hat{\tau}}$.

On the other hand if a is any element in $(F^\alpha G)^{\hat{\tau}}$ then it forms $a = \sum_{k_i \in F, g \in G} k_i u_g$ and

$$a = \hat{\tau}(\omega)\sum k_i u_g = \sum k_i u_{\tau(\omega)g} = \sum_{k_i \in F, x \in \mathcal{O}_\tau(g)} k_i u_x = \sum_{k_i \in F} k_i o_g^+ \in \Lambda$$

for all $\omega \in \Omega$. Therefore $(F^\alpha G)^{\hat{\tau}} = \Lambda$; this completes the proof. □

If $\alpha = 1$ then each $\tau(\omega)$ always extends to an isomorphism of FG. The fixed subalgebra $(FG)^\tau$ was studied in [3], and clearly $(FG)^\tau = F(G^\tau)$.

Let Λ be any F-algebra and $\chi : \Omega \to \mathrm{Aut}_F(\Lambda)$ be a homomorphism of groups. Let $\Lambda *_\chi \Omega$ be the associated skew group algebra generated by $\{b_\omega \mid \omega \in \Omega\}$ with

$$(1) \qquad \lambda_1 b_\omega \cdot \lambda_2 b_\mu = \lambda_1 \chi(\omega)\lambda_2 b_{\omega\mu} \quad \text{for } \omega, \mu \in \Omega,\ \lambda_i \in \Lambda,\ i = 1, 2.$$

If $\Lambda = FG$ then $\chi : \Omega \to \mathrm{Aut}_F(FG)$ can be regarded as the extended map from $\chi : \Omega \to G$, and it is easy to see $FG *_\chi \Omega \cong F(G \times_\chi \Omega)$, where $G \times_\chi \Omega$ is the semidirect product group afforded by χ, hence $FG *_\chi \Omega$ is also a class algebra in $F(G \times_\chi \Omega)$ [3, (2.12)]. Similarly when $\Lambda = F^\alpha G$, we will prove that $F^\alpha G *_\chi \Omega$ is isomorphic to a certain twisted group algebra of $G \times_\chi \Omega$ over F by finding suitable 2-cocycle on $G \times_\chi \Omega$.

THEOREM 2. *Let $\hat{\tau} : \Omega \to Aut_F(F^\alpha G)$ be a homomorphism extended from the action $\tau : \Omega \to Aut(G)$. Let Λ be a projective class algebra in $F^\alpha G$ by τ. Then there is a 2-cocycle β on the semidirect product group $G \times_\tau \Omega$ such that the associated skew group algebra $\Lambda *_{\hat{\tau}} \Omega$ is a projective class algebra in $F^\beta(G \times_\tau \Omega)$.*

Proof. For any $g, x \in G$ and $\omega, \mu \in \Omega$, the operation on $G \times_\tau \Omega$ is defined by $(g,\omega)(x,\mu) = \Big(g\tau(\omega)x, \omega\mu\Big)$, so $(g,\omega)^{-1} = \Big(\tau(\omega^{-1})g^{-1}, \omega^{-1}\Big)$ and $1_{G\times_\tau\Omega} = (1_G, 1_\Omega)$. Define a map

$$\beta : (G \times_\tau \Omega) \times (G \times_\tau \Omega) \to F^* \ \text{ by } \beta\Big((g,\omega),(x,\mu)\Big) = \alpha\Big(g, \tau(\omega)x\Big).$$

Then due to Theorem 1, it can be seen that β is a 2-cocycle over $G \times_\tau \Omega$:

$$\begin{aligned}
& \beta\Big((g,\omega),(x,\mu)(y,\nu)\Big) \cdot \beta\Big((x,\mu),(y,\nu)\Big) \\
= \ & \alpha\Big(g, \tau(\omega)(x\tau(\mu)y)\Big) \cdot \alpha\Big(x, \tau(\mu)y\Big) \\
= \ & \alpha\Big(g, \tau(\omega)x\tau(\omega\mu)y\Big)\alpha\Big(x,\tau(\mu)y\Big)\alpha\Big(\tau(\omega)x, \tau(\omega\mu)y\Big) \\
& \cdot \alpha\Big(\tau(\omega)x, \tau(\omega\mu)y\Big)^{-1} \\
= \ & \alpha\Big(g\tau(\omega)x, \tau(\omega\mu)y\Big) \cdot \alpha\Big(g, \tau(\omega)x\Big) \cdot \alpha\Big(x, \tau(\mu)y\Big) \cdot \alpha\Big(x, \tau(\mu)y)\Big)^{-1} \\
= \ & \beta\Big((g,\omega)(x,\mu),(y,\nu)\Big) \cdot \beta\Big((g,\omega),(x,\mu)\Big) \quad \text{for all } \ y \in G, \nu \in \Omega.
\end{aligned}$$

If we define a relation $\sim$ on $G \times_\tau \Omega$ by $(g,\omega) \sim (x,\mu)$ whenever $x \in \mathcal{E}_g$ and $\omega = \mu \in \Omega$, then $\sim$ is an equivalent relation on $G \times_\tau \Omega$. Let $\mathcal{C}_{(g,\omega)}$ be the equivalence class containing (g,ω) and let $\mathcal{Q} = \{\mathcal{C}_{(g,\omega)} \mid (g,\omega) \in G \times_\tau \Omega\}$.

Consider the twisted group algebra $F^{\beta}(G \times_{\tau} \Omega)$ of $G \times_{\tau} \Omega$ having F-basis $\{z_{(g,\omega)} | g \in G, \omega \in \Omega\}$. Then it satisfies the multiplication rule that

$$z_{(g,\omega)} z_{(x,\mu)} = \beta\Big((g,\omega),(x,\mu)\Big) z_{(g,\omega)(x,\mu)} = \alpha\Big(g, \tau(\omega)x\Big) z_{(g\tau(\omega)x,\omega\mu)}.$$

Let $s^{+}_{(g,\omega)}$ be the class sum (with respect to $\mathcal{Q}$) in $F^{\beta}(G \times_{\tau} \Omega)$. Then the algebra

$$\Gamma = \oplus_{(g,\omega)\in G\times_{\tau}\Omega} F s^{+}_{(g,\omega)} \quad \text{with} \quad s^{+}_{(g,\omega)} = \sum_{(x,\mu)\in\mathcal{C}_{(g,\omega)}} z_{(x,\mu)}$$

is a projective class algebra in $F^{\beta}(G \times_{\tau} \Omega)$ associated with $\mathcal{Q}$. It suffices to show that the associated skew group ring $\Lambda *_{\hat{\tau}} \Omega$ is isomorphic to Γ.

Since Λ is a projective class algebra in $F^{\alpha}G$ by τ, we can write $\Lambda = \oplus_g F o^{+}_g$, where $\mathcal{P} = \{\mathcal{E}_g |\ g \in G\}$ is the partition of G afforded by τ and $o^{+}_g = \sum_{x\in\mathcal{E}_g} u_x$ is the class sum in $F^{\alpha}G$. Thus the skew group algebra $\Lambda *_{\hat{\tau}} \Omega = (\oplus_g F o^{+}_g) *_{\hat{\tau}} \Omega$ is generated by $\{b_{\omega} |\ \omega \in \Omega\}$ with multiplication

$$o^{+}_g b_{\omega} \cdot o^{+}_x b_{\mu} = o^{+}_g \hat{\tau}(\omega) o^{+}_x \cdot b_{\omega\mu} \quad \text{for } \omega, \mu \in \Omega \ \text{ and } o^{+}_g, o^{+}_x \in \Lambda.$$

Define a map

$$\theta \ : \ \Lambda *_{\hat{\tau}} \Omega = (\oplus_g F o^{+}_g) *_{\hat{\tau}} \Omega \ \longrightarrow \ \Gamma = \oplus_{(g,\omega)} F s^{+}_{(g,\omega)}$$

by $\theta(o^{+}_g b_{\omega}) = s^{+}_{(g,\omega)}$. Clearly θ is a bijection and satisfies

$$\sum_{(x,\mu)\in\mathcal{C}_{(g,\omega)}} z_{(x,\mu)} = s^{+}_{(g,\omega)} = \theta(o^{+}_g b_{\omega}) = \theta(\sum_{x\in\mathcal{E}_g} u_x b_{\omega})$$

for any $g, y \in G$ and $\omega, \mu \in \Omega$, because $\mathcal{C}_{(g,\omega)} = \mathcal{E}_g \times \{\omega\}$. Hence we have

$$\begin{aligned}
\theta\Big(o^{+}_g b_{\omega}\Big) \cdot \theta\Big(o^{+}_y b_{\mu}\Big) &= s^{+}_{(g,\omega)} s^{+}_{(y,\mu)} = \sum_{(x,\nu)\in\mathcal{C}_{(g,\omega)}} z_{(x,\nu)} \sum_{(k,\eta)\in\mathcal{C}_{(y,\mu)}} z_{(k,\eta)} \\
&= \sum_{(x,\nu)\in\mathcal{C}_{(g,\omega)}} \sum_{(k,\eta)\in\mathcal{C}_{(y,\mu)}} \alpha\Big(x, \tau(\omega)k\Big) z_{(x\tau(\omega)k,\omega\mu)} \\
&= \theta\Big(\sum_{x\in\mathcal{E}_g} \sum_{k\in\mathcal{E}_y} \alpha\Big(x, \tau(\omega)k\Big) u_{x\tau(\omega)k} b_{\omega\mu}\Big) = \theta\Big(\sum_{x\in\mathcal{E}_g} u_x \sum_{k\in\mathcal{E}_y} u_{\tau(\omega)k} \cdot b_{\omega\mu}\Big) \\
&= \theta\Big(\sum_{x\in\mathcal{E}_g} u_x \cdot \hat{\tau}(\omega) \sum_{k\in\mathcal{E}_y} u_k \cdot b_{\omega\mu}\Big) = \theta\Big(o^{+}_g \tau(\omega) o^{+}_y \cdot b_{\omega\mu}\Big) \\
&= \theta\Big(o^{+}_g b_{\omega} \cdot o^{+}_y b_{\mu}\Big),
\end{aligned}$$

thus $\Lambda *_{\hat{\tau}} \Omega \cong \Gamma$ as is desired. □

In particular if $\Lambda = F^\alpha G$ then the map $\theta : F^\alpha G *_\tau \Omega \to F^\beta(G \times_{\hat{\tau}} \Omega)$ defined by $\theta(u_g b_\omega) = z_{(g,\omega)}$ is a bijective homomorphism because

$$\begin{aligned} & \theta\Big(u_g b_\omega \cdot u_x b_\mu\Big) = \alpha\Big(g, \tau(\omega)x\Big)\theta\Big(u_{g\tau(\omega)x} b_{\omega\mu}\Big) \\ = & \ \alpha\Big(g, \tau(\omega)x\Big) \cdot z_{(g\tau(\omega)x, \omega\mu)} = \beta\Big((g,\omega),(x,\mu)\Big) \cdot z_{(g,\omega)(x,\mu)} \\ = & \ z_{(g,\omega)} z_{(x,\mu)} = \theta\Big(u_g b_\omega\Big)\theta\Big(u_x b_\mu\Big). \end{aligned}$$

Hence the next corollary follows immediately.

COROLLARY 3. *Let the context be same as in Theorem 2 and $F^\alpha G$ be a twisted group algebra. Then there is a 2-cocycle β on $G \times_\tau \Omega$ satisfying $F^\alpha G *_{\hat{\tau}} \Omega \cong F^\beta(G \times_\tau \Omega)$, hence $F^\alpha G *_{\hat{\tau}} \Omega$ is a projective class algebra.*

Obviously the dimension of $F^\beta(G \times_\tau \Omega)$ is equal to $|\Omega|$ times the dimension of $F^\alpha G$, for $\dim F^\beta(G \times_\tau \Omega) = |G \times_\tau \Omega| = |G||\Omega| = |\Omega| \dim F^\alpha G$. In next theorem we will observe dimensions of projective class algebras in $F^\beta(G \times_\tau \Omega)$ and $F^\alpha G$.

THEOREM 4. *Assume that a finite group Ω acts on G by $\tau : \Omega \to Aut(G)$. Then there is a homomorphism $\theta : \Omega \to Aut(G \times_\tau \Omega)$ such that $St_\theta(g,\omega)$ equals $St_\tau(g)$ for all $(g,\omega) \in G \times_\tau \Omega$. And the dimension of projective class algebra by θ in a twisted group algebra of $G \times_\tau \Omega$ is equal to $|\Omega|$ times the dimension of projective class algebra by τ in a twisted group algebra of G.*

Proof. Let (g,ω) be any element in $G \times_\tau \Omega$. If we define a map

$$\theta \ : \ \Omega \to \mathrm{Aut}(G \times_\tau \Omega) \ \text{ by } \ \theta(\nu)(g,\omega) = (\tau(\nu)g, \omega)$$

for any $\nu \in \Omega$, then θ is a homomorphism because

$$\theta(\nu_1)\theta(\nu_2)(g,\omega) = \Big(\tau(\nu_1)\tau(\nu_2)g, \omega\Big) = \Big(\tau(\nu_1\nu_2)g, \omega\Big) = \theta(\nu_1\nu_2)(g,\omega)$$

for all $\nu_i \in \Omega$. Thus it is easy to see $\mathrm{St}_\theta(g,\omega) = \mathrm{St}_\tau(g) < \Omega$, because $\nu \in \mathrm{St}_\theta(g,\omega)$ if and only if $(g,\omega) = \theta(\nu)(g,\omega) = (\tau(\nu)g,\omega)$, or equivalently, $g = \tau(\nu)g$, if and only if $\nu \in \mathrm{St}_\tau(g)$.

We now define a relation $\sim$ on $G \times_\tau \Omega$ by, for (g,ω) and (x,μ) in $G \times_\tau \Omega$, $(g,\omega) \sim (x,\mu)$ if there is $\nu \in \Omega$ such that $x = \tau(\nu)g$ and $\mu = \omega \in \Omega$. Clearly $\sim$ form equivalence classes $\mathcal{E}$, and we denote by $s^+_{(g,\omega)}$ the class sum of $\mathcal{E}$ in $F^\beta(G \times_\tau \Omega)$ for some 2-cocycle $\beta \in Z^2(G \times_\tau \Omega, F^*)$. Then $A = \oplus_{(g,\omega)} F s^+_{(g,\omega)}$ (the sum is taken over distinct classes $\mathcal{E}$ in $G \times_\tau \Omega$) is a projective class algebra by θ in $F^\beta(G \times_\tau \Omega)$.

Since the equivalence class $\mathcal{E}$ containing (g, ω) corresponds to the orbit $\mathcal{O}_\theta(g, \omega)$ under θ, for $\mathcal{O}_\theta(g,\omega) = \{\theta(\nu)(g,\omega) | \nu \in \Omega\} = \{(\tau(\nu)g, \omega) | \nu \in \Omega\} = \{(x, \mu) | (g, \omega) \sim (x, \mu)\}$, the dimension of A is equal to the number of distinct orbits under θ.

We recall a fact that if Y is any group acting on a finite set X by an operation $\tau : Y \to \mathrm{Perm}(X)$ then the number of Y-orbits in X equals $\sum_{x \in X} \frac{1}{|Y : \mathrm{St}_\tau(x)|}$. This is also equal to $\frac{1}{|Y|} \cdot \sum_{x \in X} |\mathrm{St}_\tau(x)|$ if Y is finite.

Therefore the dimension of the projective class algebra A by θ in $F^\beta(G \times_\tau \Omega)$ is

$$\frac{1}{|\Omega|} \sum_{(g,\omega) \in G \times_\tau \Omega} |\mathrm{St}_\theta(g, \omega)| = \frac{1}{|\Omega|} \sum_{(g,\omega) \in G \times_\tau \Omega} |\mathrm{St}_\tau(g)| = |\Omega| \frac{1}{|\Omega|} \sum_{g \in G} |\mathrm{St}_\tau(g)|,$$

which is indeed equal to $|\Omega|$ times the number of orbits under τ, i.e., $|\Omega|$ times the dimension of projective class algebra by τ. □

2. Class algebras associated with linear transformation

We let F denote a field of any characteristic $p \geq 0$ and E denote an algebraic closure of F with Galois group $\mathcal{G} = \mathrm{Gal}(E/F)$. Let $\alpha \in Z^2(G, F^*)$. Then $E \otimes F^\alpha G = E^\alpha G$ is a twisted group algebra of G over E with the same basis $\{u_g |\ g \in G\}$ of $F^\alpha G$. In this section, we will discuss E-linear transformations on G and $\mathcal{G}$, and study projective class algebras of $E^\alpha G$ associated with the actions.

Let n be a positive integer divisible by $\exp(G)$. For each $\sigma \in \mathcal{G} = \mathrm{Gal}(E/F)$, let $m(\sigma)$ be a positive integer satisfying

$$m(\sigma) \equiv 1 \pmod{n_p} \quad \text{and} \quad \varepsilon_{n_{p'}}^\sigma = \varepsilon_{n_{p'}}^{m(\sigma)},$$

where ε_j $(j > 0)$ denotes a primitive j-th root of unity in E, and n_p and $n_{p'}$ are p- and p'-parts of n such that $n = n_p n_{p'}$. If $p = 0$ or $p \neq 0$ does not divide $|G|$ then $n_p = 1$ and $n_{p'} = n$. For each $g \in G$, choose $v(g) \in E$ such that

$$(2) \qquad v(g)^n = t(g) \quad \text{where} \quad t(g) = \prod_{i=1}^{n-1} \alpha(g^i, g) \in F.$$

There are monomial transformations $K_{F^\alpha G}$ and $S_{F^\alpha G}$ of $E^\alpha G$ (refer to [6]):

$$K_{F^\alpha G} : G \to \mathrm{Aut}(E^\alpha G) \text{ by } u_g K_{F^\alpha G}(x) = u_x^{-1} u_g u_x,$$
$$S_{F^\alpha G} : \mathcal{G} \to \mathrm{Aut}(E^\alpha G) \text{ by } u_g S_{F^\alpha G}(\sigma) = v(g)^{\sigma^{-1}} v(g)^{-m(\sigma^{-1})} u_g^{m(\sigma^{-1})}.$$

And also there are permutations k_G and s_G of G such that:

$$k_G : G \to \mathrm{Aut}(G) \text{ by } g k_G(x) = g^x;$$
$$s_G : \mathcal{G} \to \mathrm{Aut} \text{ by } g s_G(\sigma) = g^{m(\sigma^{-1})},$$

here we write $g^x = x^{-1}gx$. Furthermore, over the abstract direct product group $\mathcal{G} \times G$ with multiplication $(\sigma, x)(\tau, y) = (\sigma\tau, xy)$, it is defined by $d_G = s_G \times k_G$ and $D_{F^\alpha G} = S_{F^\alpha G} \times K_{F^\alpha G}$, hence

$$g d_G(\sigma, x) = (g^{m(\sigma^{-1})})^x,$$
$$u_g D_{F^\alpha G}(\sigma, x) = v(g)^{\sigma^{-1}} v(g)^{-m(\sigma^{-1})} u_x^{-1} u_g^{m(\sigma^{-1})} u_x.$$

LEMMA 5. *The $K_{F^\alpha G}$ is a (right) G-action on $E^\alpha G$, while k_G is a G-action on G. And d_G [resp. s_G] is a $\mathcal{G} \times G$ [resp. $\mathcal{G}$]-action on G if G is abelian.*

Proof. Let g, x and y be elements in G. Then the next two identities

$$u_g K_{F^\alpha G}(x) \cdot u_y K_{F^\alpha G}(x) = u_x^{-1} u_g u_x u_x^{-1} u_y u_x = \alpha(g, y) u_x^{-1} u_{gy} u_x$$
$$= \alpha(g, y) u_{gy} K_{F^\alpha G}(x) = u_g u_y K_{F^\alpha G}(x),$$
$$u_g (K_{F^\alpha G}(x) K_{F^\alpha G}(y)) = u_x^{-1} u_g u_x K_{F^\alpha G}(y) = u_y^{-1} u_x^{-1} u_g u_x u_y$$
$$= (u_x u_y)^{-1} u_g u_x u_y = \alpha^{-1}(x, y) u_{xy}^{-1} u_g \alpha(x, y) u_x y = u_g K_{F^\alpha G}(xy)$$

imply that $K_{F^\alpha G} : G \to \mathrm{Aut}(E^\alpha G)$ is a homomorphism. Also for any $\sigma, \tau \in \mathcal{G}$, since $m(\sigma\tau) \equiv m(\sigma)m(\tau) \equiv m(\tau\sigma) \pmod{n}$, we have

$$g(d_G(\sigma, x) d_G(\tau, y)) = (g^{m(\sigma^{-1})})^x d_G(\tau, y) = (g^{m(\tau^{-1}\sigma^{-1})})^{xy} = g d_G(\sigma\tau, xy)$$

and

$$g d_G(\sigma, x) \cdot y d_G(\sigma, x) = (g^{m(\sigma^{-1})} y^{m(\sigma^{-1})})^x = ((gy)^{m(\sigma^{-1})})^x = gy d_G(\sigma, x),$$

hence d_G is a homomorphism. The rest part of the proof is clear. □

An element $g \in G$ is said to be α-regular if $\alpha(g, x) = \alpha(x, g)$ for all x in the centralizer $C_G(g)$ of g. Hence g is α-regular if and only if $u_g K_{F^\alpha G}(x) = u_g$ for all $x \in G$ with $gx = xg$. A conjugacy class containing an α-regular element is called an α-regular class. A 2-cocycle $\alpha \in Z^2(G, F^*)$ is called normal if α satisfies $\alpha(g, x) = \alpha(x, x^{-1}gx)$ for any α-regular $g \in G$ and any $x \in G$, or equivalently, $u_x^{-1} u_g u_x = u_{x^{-1}gx}$

in terms of basis of algebra $F^\alpha G$. If, moreover $u_x^{-1} = u_{x^{-1}}$ for all x then α is called standard. Thus α is standard if $\alpha(x, x^{-1}) = 1$ and $\alpha(x^{-1}g, x)\alpha(x^{-1}, g) = 1$ for any α-regular element $g \in G$ and any $x \in G$ (refer to [5, p.97]).

We will study structures of the associated skew group algebras by actions $K_{F^\alpha G}$ and s_G in next theorems. From now on, we denote the group generated by all α-regular elements in G by G_0.

THEOREM 6. *The algebra $E^\alpha G_0 *_{K_{F^\alpha G}} G$ is isomorphic to the twisted group algebra $E^\beta(G_0 \times_{k_G} G)$ for some $\beta \in Z^2(G \times_{k_G} G, E^*)$. In particular if α is normal then $F^\alpha G_0 *_{K_{F^\alpha G}} G \cong F^\beta(G_0 \times_{k_G} G)$ for $\beta \in Z^2(G \times_{k_G} G, F^*)$.*

Proof. The multiplication on $G \times_{k_G} G$ satisfies

$$(g_1, x_1)(g_2, x_2) = (g_1 g_2 k_G(x_1), x_1 x_2) = (g_1 g_2^{x_1}, x_1 x_2)$$

for $g_i, x_i \in G$ $(i = 1, 2)$. And from $\alpha \in Z^2(G, F^*)$, we have a 2-cocycle β on $G \times_{k_G} G$ satisfying $\beta\Big((g_1, x_1), (g_2, x_2)\Big) = \alpha\Big(g_1, g_2 k_G(x_1)\Big) = \alpha(g_1, g_2^{x_1})$ as in Theorem 2. Thus if we let $z_{(g,x)}$ be an F-basis of $F^\beta(G_0 \times_{k_G} G)$, where $g \in G_0$, $x \in G$, then

$$\begin{aligned} z_{(g_1,x_1)} z_{(g_2,x_2)} &= \beta\Big((g_1, x_1), (g_2, x_2)\Big) z_{(g_1,x_1)(g_2,x_2)} \\ &= \alpha(g_1, g_2^{x_1}) z_{(g_1 g_2^{x_1}, x_1 x_2)}. \end{aligned} \tag{3}$$

On the other hand, let $\{b_x | \ x \in G\}$ be a basis of $F^\alpha G_0 *_{K_{F^\alpha G}} G$ afforded by the homomorphism $K_{F^\alpha G} : G \to \mathrm{Aut}(E^\alpha G)$. Then it satisfies

$$u_{g_1} b_{x_1} \cdot u_{g_2} b_{x_2} = u_{g_1} \cdot u_{g_2} K_{F^\alpha G}(x_1) b_{x_1 x_2} = u_{g_1} u_{x_1}^{-1} u_{g_2} u_{x_1} b_{x_1 x_2}$$

(see (1)) where $\{u_g | g \in G_0\}$ is the F-basis of $F^\alpha G$. For any $g \in G_0$ and $x \in G$, we define a map

$$\bar{\theta} : F^\alpha G_0 *_{K_{F^\alpha G}} G \to F^\beta(G_0 \times_{k_G} G) \quad \text{by} \quad \bar{\theta}(u_g b_x) = z_{(g,x)}. \tag{4}$$

When α is normal then since $u_{x_i}^{-1} u_{g_j} u_{x_i} = u_{x_i^{-1} g_j x_i}$ for $g_j \in G_0$, we have

$$\begin{aligned} \bar{\theta}(u_{g_1} b_{x_1} \cdot u_{g_2} b_{x_2}) &= \bar{\theta}(u_{g_1} u_{x_1}^{-1} u_{g_2} u_{x_1} b_{x_1 x_2}) = \bar{\theta}(u_{g_1} u_{x_1^{-1} g_2 x_1} b_{x_1 x_2}) \\ &= \alpha(g_1, g_2^{x_1}) z_{(g_1 g_2^{x_1}, x_1 x_2)} = z_{(g_1,x_1)} z_{(g_2,x_2)} = \bar{\theta}(u_{g_1} b_{x_1}) \bar{\theta}(u_{g_2} b_{x_2}) \end{aligned}$$

for any $u_{g_i} b_{x_i} \in F^\alpha G_0 *_{K_{F^\alpha G}} G$ $(i = 1, 2)$. Hence $\bar{\theta}$ gives the isomorphism $F^\alpha G_0 *_{K_{F^\alpha G}} G \cong F^\beta(G_0 \times_{k_G} G)$.

When α is not normal, there is a normal cocycle $\gamma \in Z^2(G, F^*)$ which is cohomologous to α (refer to [5, (2.6.2)]). Moreover since E is an algebraic closure, we may assume $\gamma \in Z^2(G, E^*)$ is standard (refer to [5, (2.6.4)]), and $\alpha(x,y) = f(x)f(y)f^{-1}(xy)\gamma(x,y)$ for a map $f : G \to E^*$ with $f(1) = 1$ for $x, y \in G$. Since every α-regular element is γ-regular, we shall use the same notation G_0 for both sets of α-regular elements and of γ-regular elements.

Define a 2-cocycle $\beta \in Z^2(G, E^*)$ with regard to γ as before, that is, with $g_i \in G_0$, $x_i \in G$, $\beta\Big((g_1, x_1), (g_2, x_2)\Big) = \gamma(g_1, g_2^{x_1})$ (see (3)), and using the same notations $\{z_{(g,x)}\}$ for E-basis of $E^\beta(G \times_{k_G} G)$, it satisfies $z_{(g_1,x_1)}z_{(g_2,x_2)} = \gamma(g_1, g_2^{x_1})z_{(g_1g_2^{x_1}, x_1x_2)}$. Thus the associated skew group algebra $E^\gamma G_0 *_{K_{F^\gamma G}} G$ by $K_{F^\gamma G} : G \to \mathrm{Aut}(E^\gamma G)$ is isomorphic to

$$E^\gamma G_0 *_{K_{F^\gamma G}} G \cong E^\beta(G_0 \times_{k_G} G)$$

(see (4)). We now suffices to show that $E^\alpha G_0 *_{K_{F^\alpha G}} G \cong F^\gamma G_0 *_{K_{F^\gamma G}} G$.

With E-basis $\{u_x | x \in G\}$ and $\{v_x | x \in G\}$ of $E^\alpha G$ and $E^\gamma G$ respectively, there is an E-algebra isomorphism $\theta : E^\alpha G \to E^\gamma G$ such that $\theta(u_x) = f(x)v_x$ for all $x \in G$.

Now let $\{c_x |\ x \in G\}$ be a basis of $E^\gamma G_0 *_{K_{F^\gamma G}} G$. Then c_x satisfies

$$v_{g_1}c_{x_1} \cdot v_{g_2}c_{x_2} = v_{g_1}v_{x_1}^{-1}v_{g_2}v_{x_1}c_{x_1x_2} = v_{g_1}v_{g_2^{x_1}}c_{x_1x_2} = \gamma(g_1, g_2^{x_1})v_{g_2^{x_1}}c_{x_1x_2},$$

because γ is standard. We will define a map $\bar\theta$ induced from θ by

$$\bar\theta : E^\alpha G_0 *_{K_{F^\alpha G}} G \to E^\gamma G_0 *_{K_{F^\gamma G}} G\ ,\quad \bar\theta(u_g b_x) = \theta(u_g)c_x = f(g)v_g c_x$$

for $g \in G_0$ and $x \in G$. We first note that, for $g_i \in G_0$ and any $x_i \in G$,

$$\begin{aligned}
& u_{g_1}b_{x_1}u_{g_2}b_{x_2} \\
= \;& \alpha^{-1}(x_1, x_1^{-1})\alpha(x_1^{-1}, g_2)\alpha(x_1^{-1}g_2, x_1)\alpha(g_1, g_2^{x_1})u_{g_1g_2^{x_1}}b_{x_1x_2} \\
= \;& f^{-1}(x_1)f^{-1}(x_1^{-1})f(1)\gamma^{-1}(x_1, x_1^{-1})f(x_1^{-1})f(g_2)f^{-1}(x_1^{-1}g_2) \\
& \cdot \gamma(x_1^{-1}, g_2)f(x_1^{-1}g_2)f(x_1)f^{-1}(g_2^{x_1})\gamma(x_1^{-1}g_2, x_1) \\
& \cdot f(g_1)f(g_2^{x_1})f^{-1}(g_1g_2^{x_1})\gamma(g_1, g_2^{x_1})u_{g_1g_2^{x_1}}b_{x_1x_2} \\
= \;& f(g_1)f(g_2)f^{-1}(g_1g_2^{x_1})\gamma^{-1}(x_1, x_1^{-1})\gamma(x_1^{-1}, g_2)\gamma(x_1^{-1}g_2, x_1) \\
& \cdot \gamma(g_1, g_2^{x_1})u_{g_1g_2^{x_1}}b_{x_1x_2} \\
= \;& f(g_1)f(g_2)f^{-1}(g_1g_2^{x_1})\gamma(g_1, g_2^{x_1})u_{g_1g_2^{x_1}}b_{x_1x_2},
\end{aligned}$$

where the last equality is due to γ standard. Then it follows that

$$\begin{aligned}
& \bar\theta(u_{g_1}b_{x_1}u_{g_2}b_{x_2}) = f(g_1)f(g_2)f^{-1}(g_1g_2^{x_1})\gamma(g_1, g_2^{x_1})\bar\theta(u_{g_1g_2^{x_1}}b_{x_1x_2}) \\
& = f(g_1)f(g_2)\gamma(g_1, g_2^{x_1})v_{g_1g_2^{x_1}}c_{x_1x_2} = f(g_1)f(g_2)v_{g_1}c_{x_1}v_{g_2}c_{x_2} \\
& = f(g_1)v_{g_1}c_{x_1}f(g_2)v_{g_2}c_{x_2} = \bar\theta(u_{g_1}b_{x_1})\bar\theta(u_{g_2}b_{x_2}),
\end{aligned}$$

therefore we have $E^{\alpha}G_0 *_{K_{F^{\alpha}G}} G \cong E^{\gamma}G_0 *_{K_{F^{\gamma}G}} G \cong E^{\beta}(G_0 \times_{k_G} G)$.

THEOREM 7. *If G is abelian then $F^{\alpha}G *_{s_G} \mathcal{G}$ is isomorphic to $F^{\beta}(G \times_{s_G} \mathcal{G})$ for some $\beta \in Z^2(G \times_{s_G} \mathcal{G}, F^*)$.*

Proof. For the algebra $F^{\alpha}G *_{s_G} \mathcal{G}$, we may consider that $s_G(\sigma)$ extends to an algebra isomorphism of $F^{\alpha}G$ for each $\sigma \in \mathcal{G}$. Hence for all $u_g \in F^{\alpha}G$ and $\sigma \in \mathcal{G}$, we have $u_g s_G(\sigma) = u_{g^{m(\sigma^{-1})}}$ thus

$$\begin{aligned} u_g s_G(\sigma) \cdot u_x s_G(\sigma) &= (u_g u_x) s_G(\sigma) = \alpha(g,x) u_{(gx)^{m(\sigma^{-1})}} \\ &= \alpha(g,x)\alpha^{-1}\Big(g^{m(\sigma^{-1})}, x^{m(\sigma^{-1})}\Big) u_g s_G(\sigma) \cdot u_x s_G(\sigma), \end{aligned}$$

which shows $\alpha(g,x) = \alpha\Big(g^{m(\sigma^{-1})}, x^{m(\sigma^{-1})}\Big) = \alpha\Big(g s_G(\sigma), x s_G(\sigma)\Big)$ (see Theorem 2). Let $\{b_\sigma | \sigma \in \mathcal{G}\}$ be a basis of $F^{\alpha}G *_{s_G} \mathcal{G}$ with multiplication

$$u_g b_\sigma u_x b_\tau = u_g u_x s_G(\sigma) b_{\sigma\tau} = \alpha\Big(g, x^{m(\sigma^{-1})}\Big) u_{g x^{m(\sigma^{-1})}} b_{\sigma\tau}$$

for any $g, x \in G,\ \sigma, \tau \in \mathcal{G}$.

Since multiplication on $G \times_{s_G} \mathcal{G}$ satisfies $(g,\sigma)(x,\tau) = \Big(g \cdot x s_G(\sigma), \sigma\tau\Big)$ $= \Big(g x^{m(\sigma^{-1})}, \sigma\tau\Big)$, if we define a map

$$\beta : (G \times_{s_G} \mathcal{G}) \times (G \times_{s_G} \mathcal{G}) \to F^* \text{ by } \beta\Big((g,\sigma),(x,\tau)\Big) = \alpha\Big(g, x^{m(\sigma^{-1})}\Big),$$

then it is routine to see that β is a 2-cocycle in $Z^2(G \times_{s_G} \mathcal{G}, F^*)$. Let $z_{(g,\sigma)}$ be the basis of $F^{\beta}(G \times_{s_G} \mathcal{G})$. Then it satisfies

$$z_{(g,\sigma)} z_{(x,\tau)} = \beta\Big((g,\sigma),(x,\tau)\Big) z_{(g,\sigma)(x,\tau)} = \alpha\Big(g, x^{m(\sigma^{-1})}\Big) z_{(g x^{m(\sigma^{-1})}, \sigma\tau)},$$

thus the map $\theta : F^{\alpha}G *_{s_G} \mathcal{G} \to F^{\beta}(G \times_{s_G} \mathcal{G})$ defined by $\theta(u_g b_\sigma) = z_{(g,\sigma)}$ is an isomorphism because

$$\begin{aligned} \theta\Big(u_g b_\sigma \cdot u_x b_\tau\Big) &= \alpha\Big(g, x^{m(\sigma^{-1})}\Big) \cdot z_{(g x^{m(\sigma^{-1})}, \sigma\tau)} \\ &= z_{(g,\sigma)} z_{(x,\tau)} = \theta\Big(u_g b_\sigma\Big) \chi\Big(u_x b_\tau\Big), \end{aligned}$$

this completes the proof. □

Since any cocycle α is cohomologous to a normal cocycle and cohomologous cocycles yield an isomorphism of twisted group algebras, we may assume α is normal. We now will investigate structures of fixed twisted group algebras as projective class algebras.

THEOREM 8. (a) *The fixed algebra* $(F^\alpha G_0)^{K_{F^\alpha G}}$ *equals* $(F^\alpha G_0)^{k_G}$.

(b) *The fixed algebra* $(E^\alpha G)^{S_{F^\alpha G}}$ *is generated by all elements* $u_g \in E^\alpha G$ *that satisfy* $g^{m(\sigma)} = g$ *for all* $\sigma \in \mathcal{G} = Gal(E/F)$.

Proof. Without loss of generality we may assume $\alpha \in Z^2(G, F^*)$ is normal. From the fixed algebra $(F^\alpha G_0)^{k_G}$, $k_G(x)$ is regarded as an extended algebra homomorphism of $F^\alpha G_0$ for each $x \in G$, hence $u_g k_G(x) = u_{gk_G(x)} = u_{g^x}$ for any α-regular element g in G_0. Moreover since $u_g K_{F^\alpha G}(x) = u_x^{-1} u_g u_x = u_g k_G(x)$, this shows that $K_{F^\alpha G}$ is the linearly extended mapping of k_G to $\mathrm{Aut}(F^\alpha G_0)$. Thus we have $(F^\alpha G_0)^{K_{F^\alpha G}} = (F^\alpha G_0)^{k_G}$.

Let $\mathcal{E}_g$ be the α-regular class containing $g \in G$. Since the class sums $o_g^+ = \sum_{y \in \mathcal{E}_g} u_y$ constitute the center $Z(F^\alpha G)$ ([5, (2.6.3)]), the equality

$$o_g^+ K_{F^\alpha G} u_x = u_x^{-1} \sum_{y \in \mathcal{E}_g} u_y u_x = \sum_{y \in \mathcal{E}_g} u_{x^{-1} y x} = \sum_{z \in \mathcal{E}_g} u_z = o_g^+$$

for any $u_x \in F^\alpha G$ shows that $o_g^+ \in (F^\alpha G_0)^{K_{F^\alpha G}}$ and $Z(F^\alpha G)$ is a subset of $(F^\alpha G_0)^{K_{F^\alpha G}}$. The other inclusion is clear, for $u_g \in (F^\alpha G_0)^{K_{F^\alpha G}}$ if and only if $g \in G_0$ and $u_g \in Z(F^\alpha G)$.

For the second statement, let u_g be any E-basis element in $E^\alpha G$. Then u_g is in $(E^\alpha G)^{S_{F^\alpha G}}$ if and only if $u_g = v(g)^{\sigma^{-1}} v(g)^{-m(\sigma^{-1})} u_g^{m(\sigma^{-1})}$ for all $\sigma \in \mathcal{G}$. This is equivalent to say that, u_1 equals

$$v(g)^{\sigma^{-1}} v(g)^{-m(\sigma^{-1})} \prod_{i=1}^{m(\sigma^{-1})-1} \alpha(g^i, g) \alpha^{-1}(g, g^{-1}) \alpha(g^{m(\sigma^{-1})}, g^{-1}) u_{g^{m(\sigma^{-1})-1}},$$

which drives the following two identities that, for all $\sigma \in \mathcal{G}$,

(i) $1 = v(g)^{\sigma^{-1}} v(g)^{-m(\sigma^{-1})} \prod_{i=1}^{m(\sigma^{-1})-1} \alpha(g^i, g) \alpha^{-1}(g, g^{-1}) \alpha(g^{m(\sigma^{-1})}, g^{-1})$

(ii) $g = g^{m(\sigma^{-1})}$.

However we will show that (i) follows from (ii). In fact, since $v(g)^n = t(g) = \prod_{i=1}^{n-1} \alpha(g^i, g) \in F^*$ (see (2)), we have

$$\prod_{i=1}^{m(\sigma^{-1})n} \alpha(g^i, g) = \Big(\prod_{i=1}^{n-1} \alpha(g^i, g)\Big)^{m(\sigma^{-1})} = t(g)^{m(\sigma^{-1})}.$$

On the other hand, we also can write

$$\prod_{i=1}^{m(\sigma^{-1})n} \alpha(g^i,g) = \Big(\prod_{i=1}^{m(\sigma^{-1})-1} \alpha(g^i,g)\Big)^n \cdot \prod_{i=1}^{n} \alpha(g^i,g) = \Big(\prod_{i=1}^{m(\sigma^{-1})-1} \alpha(g^i,g)\Big)^n t(g).$$

Thus the above two identities give rise to

$$\Big(\prod_{i=1}^{m(\sigma^{-1})-1} \alpha(g^i,g)\Big)^n t(g) = t(g)^{m(\sigma^{-1})}.$$

Since $t(g) \in F^*$ and $t(g)^\sigma = t(g)$ for all $\sigma \in \mathcal{G}$, $g = g^{m(\sigma^{-1})}$ in (ii) yields

$$\Big(v(g)^{\sigma^{-1}} v(g)^{-m(\sigma^{-1})} \cdot \prod_{i=1}^{m(\sigma^{-1})-1} \alpha(g^i,g)\alpha^{-1}(g,g^{-1})\alpha(g^{m(\sigma^{-1})},g^{-1})\Big)^n$$
$$= t(g)^{\sigma^{-1}} t(g)^{-m(\sigma^{-1})} \cdot \Big(\prod_{i=1}^{m(\sigma^{-1})-1} \alpha(g^i,g)\Big)^n = 1,$$

and by taking n-th root of unity as 1, (i) follows immediately. Therefore $(E^\alpha G)^{S_{F^\alpha G}}$ is generated by $u_g \in E^\alpha G$ with $g^{m(\sigma)} = g$ for all $\sigma \in \mathcal{G}$. □

To observe an explicit example for Theorem 8, we recall that a finite group G is called an F-group if all irreducible E-characters of G have values in F. Hence G is an abelian F-group if $\mathrm{Hom}(G,F^*) = \mathrm{Hom}(G,E^*)$. It was proved in [2] that G is an abelian F-group if and only if $s_G(\sigma)$ fixes each g in G for all $\sigma \in \mathcal{G}$, i.e., $g = gs_G(\sigma^{-1}) = g^{m(\sigma)}$. Thus, if G is an abelian F-group then the fixed algebra $(E^\alpha G)^{S_{F^\alpha G}}$ is generated by $u_g \in E^\alpha G$ for all $g \in G$. Hence the next corollary follows immediately.

COROLLARY 9. *If G is an abelian F-group then $(E^\alpha G)^{S_{F^\alpha G}} = E^\alpha G$.*

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