

MULTIDIMENSIONAL INTEGRATION VIA TRAPEZOIDAL AND THREE POINT GENERATORS

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ABSTRACT. Multidimensional integrals are expressed in terms of lower dimensional integrals and function evaluations. An iterative process is used where a trapezoidal and three point identities are used as *generators* for higher dimensional identities. Bounds are obtained utilising the resulting identities. It is demonstrated that earlier Ostrowski type results are obtained as particular instances of the current work.

1. Introduction

We firstly present one-dimensional identities which may be used as generators for higher dimensional results.

For $f : [a, b] \rightarrow \mathbb{R}$ we define the Ostrowski and Trapezoidal functionals by

$$(1.1) \quad \mathcal{S}(f; c, x, d) := f(x) - \mathcal{M}(f; c, d)$$

and

$$(1.2) \quad T(f; c, x, d) := \left(\frac{x-c}{d-c}\right) f(c) + \left(\frac{d-x}{d-c}\right) f(d) - \mathcal{M}(f; c, d),$$

respectively, where

$$(1.3) \quad \mathcal{M}(f; c, d) := \frac{1}{d-c} \int_c^d f(u) du, \text{ the integral mean.}$$

We note that

$$(1.4) \quad (b-a) \mathcal{S}\left(f; a, \frac{a+b}{2}, b\right) = (b-a) f\left(\frac{a+b}{2}\right) - \int_a^b f(u) du$$

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and

$$(1.5) \quad (b-a)T\left(f; a, \frac{a+b}{2}, b\right) = \frac{b-a}{2} [f(a) + f(b)] - \int_a^b f(u) du,$$

recapturing the midpoint and trapezoidal rules for the evaluation of the integrals. With this in mind, the most common task is to obtain bounds on the above functionals. This task is perhaps best accomplished from identities involving the functionals. The following identities may be easily shown to hold for f of bounded variation, by an integration by parts argument of the Riemann-Stieltjes integrals and so

$$(1.6) \quad \mathcal{S}(f; c, x, d) = \int_c^d p(x, t, c, d) df(t), \quad p(x, t, c, d) = \begin{cases} \frac{t-c}{d-c}, & t \in [c, x] \\ \frac{t-d}{d-c}, & t \in (x, d] \end{cases}$$

and

$$(1.7) \quad \mathcal{T}(f; c, x, d) = \int_c^d q(x, t, c, d) df(t), \quad q(x, t, c, d) = \frac{t-x}{d-c}, \quad x, t \in [c, d].$$

The book [15] is devoted to Ostrowski type results involving (1.1) and numerous generalisations. See also [1], [16], [19] and [21].

Further, define the three point functional $\mathfrak{T}(f; a, \alpha, x, \beta, b)$ which involves the difference between the integral mean and, a weighted combination of a function evaluated at the end points and an interior point. Namely, for $a \leq \alpha < x < \beta \leq b$,

$$(1.8) \quad \mathfrak{T}(f; a, \alpha, x, \beta, b) := \left(\frac{\alpha-a}{b-a}\right) f(a) + \left(\frac{\beta-\alpha}{b-a}\right) f(x) + \left(\frac{b-\beta}{b-a}\right) f(b) - \mathcal{M}(f; a, b).$$

Cerone and Dragomir [7] showed that for f of bounded variation, the identity

$$(1.9) \quad \mathcal{T}(f; a, \alpha, x, \beta, b) = \int_a^b r(x, t) df(t), \quad r(x, t) = \begin{cases} \frac{t-\alpha}{b-a}, & t \in [a, x] \\ \frac{t-\beta}{b-a}, & t \in (x, b] \end{cases}$$

is valid. They effectively demonstrated that the Ostrowski functional and the trapezoid functional could be recaptured as particular instances.

Specifically, from (1.8) and (1.9)

$$\mathcal{S}(f; a, x, b) = \mathfrak{T}(f; a, a, x, b, b)$$

and

$$T(f; a, x, b) = \mathfrak{T}(f; a, x, x, x, b),$$

where $\mathcal{S}(f; a, x, b)$ and $T(f; a, x, b)$ are defined by (1.1) and (1.2) and satisfy identities (1.6) and (1.7) respectively.

It should be noted at this stage that

$$\begin{aligned} (b-a) \mathfrak{T}\left(f; a, \frac{5a+b}{6}, \frac{a+b}{2}, \frac{a+5b}{6}, b\right) \\ = \frac{b-a}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \int_a^b f(x) dx \end{aligned}$$

is the Simpson functional.

Further, if $f(t)$ is assumed to be absolutely continuous for t over its respective interval, then $df(t) = f'(t) dt$ and the Riemann-Stieltjes integrals in (1.8) and (1.9) are equivalent to Riemann integrals.

Pachpatte [22] obtains, with the following notation, a trapezoidal type result for double integrals.

He lets \mathbb{R} denote the set of real numbers and $\mathbb{R}^+ = [0, \infty)$ and uses the notation $\Delta = [a, b] \times [c, d]$ for a, b, c, d in \mathbb{R}^+ . If $f(x, y)$ is a differentiable function defined on Δ , then its partial derivatives are denoted by $D_1f(x, y) = \frac{\partial}{\partial x}f(x, y)$, $D_2f(x, y) = \frac{\partial}{\partial y}f(x, y)$, $D_2D_1f(x, y) = \frac{\partial^2}{\partial y \partial x}f(x, y)$. He denotes by $F(\Delta)$ the class of continuous functions $f : \Delta \rightarrow \mathbb{R}$ for which $D_1f(x, y)$, $D_2f(x, y)$, $D_2D_1f(x, y)$ exist and are continuous on Δ .

THEOREM 1. *If $f \in F(\Delta)$, then*

$$\begin{aligned} (1.10) \quad & \left| \int_a^b \int_c^d f(s, t) dt ds - \frac{1}{2} \left[(d-c) \int_a^b [f(s, c) + f(s, d)] ds \right. \right. \\ & \left. \left. + (b-a) \int_c^d [f(a, t) + f(b, t)] dt \right] \right. \\ & \left. + \frac{1}{4} (b-a)(d-c) [f(a, c) + f(a, d) + f(b, c) + f(b, d)] \right| \\ & \leq \frac{1}{4} (b-a)(d-c) \int_a^b \int_c^d |D_2D_1f(s, t)| dt ds. \end{aligned}$$

Using a similar argument, Pachpatte [23] obtains a trapezoidal type result for a triple integral involving function evaluation on the boundary,

single integrals and double integrals. The error is again as in (1.10) obtained only for the first partial derivatives over $\Delta \in L_1[\Delta]$.

In the current work, the generalised trapezoidal and three point identities (1.6)-(1.7) and (1.8)-(1.9) for absolutely continuous functions are used as generators to produce identities involving multidimensional integrals in terms of lower dimensional integrals and function evaluations. These are used to procure bounds for $\frac{\partial^n f}{\partial t_n \dots \partial t_1} \in L_p[I^n]$, $1 \leq p \leq \infty$, where $I^n = [a_1, b_1] \times \dots \times [a_n, b_n]$. Here for $h : I^n \rightarrow \mathbb{R}$ we mean by $h \in L_p[I^n]$, the Lebesgue norms, that is,

$$(1.11) \quad \|h\|_p := \left(\int_{a_n}^{b_n} \dots \int_{a_1}^{b_1} |h(t_1, t_2, \dots, t_n)|^p dt_1 \dots dt_n \right)^{\frac{1}{p}},$$

$$1 \leq p < \infty, \text{ for } h \in L_p[I^n],$$

and

$$(1.12) \quad \|h\|_\infty := \operatorname{ess\,sup}_{t \in I^n} |h(t_1, t_2, \dots, t_n)|, \text{ for } h \in L_\infty[I^n],$$

where $t = (t_1, t_2, \dots, t_n)$ and so $t_i \in [a_i, b_i]$, $i = 1, 2, \dots, n$.

The methodology of Cerone [5] is used for the current work. That work turns out to be a particular case of the three point development in Section 3 of the current paper. The generalised trapezoidal results of Section 2 are also specialisations of the three point results of Section 2.

2. Identities from an iterative approach and their bounds

The following theorem obtained by Cerone [5] uses an iterative approach to extend the Ostrowski functional identity to multidimensions. Firstly, we will require some notation.

Let $I^n = \prod_{i=1}^n [a_i, b_i] = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$. Further, let $f : I^n \rightarrow \mathbb{R}$ and define operators $F_i(f)$ and $\lambda_i(f)$ by

$$(2.1) \quad F_i(f) := f(t_1, \dots, t_{i-1}, x_i, t_{i+1}, \dots, t_n) \text{ where } x_i \in [a_i, b_i]$$

and

$$(2.2) \quad \lambda_i(f) := \frac{1}{d_i} \int_{a_i}^{b_i} f(t_1, \dots, t_{i-1}, t_i, t_{i+1}, \dots, t_n) dt_i.$$

That is, $F_i(f)$ evaluates $f(\cdot)$ in the i th variable at $x_i \in [a_i, b_i]$ and $\lambda_i(f)$ is the integral mean of $f(\cdot)$ in the i th variable. Assuming that $f(\cdot)$ is

absolutely continuous in the i th variable $t_i \in [a_i, b_i]$, we have

$$(2.3) \quad \mathcal{L}_i(f) = \frac{1}{d_i} \int_{a_i}^{b_i} p_i(x_i, t_i) \frac{\partial f}{\partial t_i} dt_i = (F_i - \lambda_i)(f),$$

for $i = 1, 2, \dots, n$, where

$$(2.4) \quad \frac{p_i(x_i, t_i)}{d_i} = \begin{cases} \frac{t_i - a_i}{b_i - a_i}, & t_i \in [a_i, x_i] \\ \frac{t_i - b_i}{b_i - a_i}, & t_i \in (x_i, b_i], \end{cases}$$

and $d_i = b_i - a_i$.

Thus (2.3)-(2.4) is ostensibly equivalent to the Montgomery identity for $f(t_1, \dots, t_{i-1}, t_i, t_{i+1}, \dots, t_n)$ absolutely continuous for $t_i \in [a_i, b_i]$.

THEOREM 2. *Let $f : I^n \rightarrow \mathbb{R}$ be absolutely continuous in such a manner that the partial derivatives of order one with respect to every variable exist. Then*

$$(2.5) \quad \begin{aligned} E_n(f) &= f(x_1, x_2, \dots, x_n) \\ &\quad - \sum_{i=1}^n \frac{1}{d_i} \int_{a_i}^{b_i} f(x_1, x_2, \dots, x_{i-1}, t_i, x_{i+1}, \dots, x_n) dt_i \\ &\quad + \sum_{i < j}^n \frac{1}{d_i d_j} \int_{a_j}^{b_j} \int_{a_i}^{b_i} f(x_1, \dots, x_{i-1}, t_i, x_{i+1}, \dots, t_j, \dots, x_n) dt_i dt_j \\ &\quad - \dots - \frac{(-1)^n}{D_n} \int_{a_n}^{b_n} \dots \int_{a_1}^{b_1} f(t_1, \dots, t_n) dt_1 \dots dt_n \\ &:= \tau_n(a, x, b), \end{aligned}$$

where

$$(2.6) \quad E_n(f) = \frac{1}{D_n} \int_{a_n}^{b_n} \dots \int_{a_1}^{b_1} \prod_{i=1}^n p_i(x_i, t_i) \frac{\partial^n f}{\partial t_n \dots \partial t_1} dt_1 \dots dt_n,$$

$p_i(x_i, t_i)$ is given by (2.4), and

$$(2.7) \quad D_n = \prod_{i=1}^n d_i, \quad d_i = b_i - a_i.$$

THEOREM 3. *Let the conditions of Theorem 2 continue to hold. Then*

$$(2.8) \quad D_n \left| \tau_n \left(\underset{\sim}{a}, \underset{\sim}{x}, \underset{\sim}{b} \right) \right| \leq \begin{cases} \prod_{i=1}^n P_i(1) \left\| \frac{\partial^n f}{\partial t_n \dots \partial t_1} \right\|_{\infty}, & \frac{\partial^n f}{\partial t_n \dots \partial t_1} \in L_{\infty} [I^n]; \\ \left(\prod_{i=1}^n P_i(q) \right)^{\frac{1}{q}} \left\| \frac{\partial^n f}{\partial t_n \dots \partial t_1} \right\|_p, & \frac{\partial^n f}{\partial t_n \dots \partial t_1} \in L_p [I^n], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \prod_{i=1}^n \theta_i \left\| \frac{\partial^n f}{\partial t_n \dots \partial t_1} \right\|_1, & \frac{\partial^n f}{\partial t_n \dots \partial t_1} \in L_1 [I^n], \end{cases}$$

where $\tau_n \left(\underset{\sim}{a}, \underset{\sim}{x}, \underset{\sim}{b} \right)$ is as defined by (2.5),

$$(2.9) \quad (q + 1) P_i(q) = (x_i - a_i)^{q+1} + (b_i - x_i)^{q+1},$$

$$(2.10) \quad \theta_i = \frac{b_i - a_i}{2} + \left| x_i - \frac{a_i + b_i}{2} \right|.$$

REMARK 1. It was stated in Cerone [5] that the expression for $\tau_n(\underset{\sim}{a}, \underset{\sim}{x}, \underset{\sim}{b})$ may be written in a less explicit form which is perhaps more appealing. Namely,

$$(2.11) \quad \tau_n \left(\underset{\sim}{a}, \underset{\sim}{x}, \underset{\sim}{b} \right) = f(x_1, x_2, \dots, x_n) + \sum_{k=1}^{n-1} (-1)^k \sum_k \mathcal{M}_k + (-1)^n \mathcal{M}_n,$$

where \mathcal{M}_k represents the integral means in k variables with the remainder being evaluated at their respective interior point and $\sum_k \mathcal{M}_k$ is a sum over all $\binom{n}{k}$, k -dimensional integral means. Here

$$\mathcal{M}_n = \frac{1}{D_n} \int_{a_n}^{b_n} \dots \int_{a_1}^{b_1} f(t_1, \dots, t_n) dt_1 \dots dt_n$$

and

$$\begin{aligned} & \sum_1 \mathcal{M}_1 \\ = & \frac{1}{d_1} \int_{a_1}^{b_1} f(t_1, x_2, \dots, x_n) dt_1 + \frac{1}{d_2} \int_{a_2}^{b_2} f(x_1, t_2, x_3, \dots, x_n) dt_2 \\ & + \dots + \frac{1}{d_n} \int_{a_n}^{b_n} f(x_1, x_2, \dots, x_{n-1}, t_n) dt_n. \end{aligned}$$

It should be noted that (2.11) may be written as

$$(2.12) \quad \tau_n \left(\underset{\sim}{a}, \underset{\sim}{x}, \underset{\sim}{b} \right) = \sum_{k=0}^n (-1)^k \sum_k \mathcal{M}_k$$

if we define the degenerate 0–th integral mean $\mathcal{M}_0 = f(x_1, x_2, \dots, x_n)$.

The work of Cerone [3] used the Ostrowski functional (1.1) as a *seed* or *generator* for extension to higher dimensions using the Montgomery identity. We may prove, in an equivalent manner, utilising the generalised trapezoidal identity (1.7) with (1.2) as the *generator* of a higher dimensional result. We will restrict the current work to absolutely continuous functions so that the Riemann integral identity corresponding to (1.7) will be used. Let $f : I^n \rightarrow \mathbb{R}$ and define the operator

$$(2.13) \quad G_i(f) : = \frac{A_i}{d_i} f(t_1, \dots, t_{i-1}, a_i, t_{i+1}, \dots, t_n) + \frac{B_i}{d_i} f(t_1, \dots, t_{i-1}, b_i, t_{i+1}, \dots, t_n),$$

where

$$(2.14) \quad A_i = x_i - a_i, \quad B_i = b_i - x_i, \quad d_i = b_i - a_i.$$

Here $d_i G_i(f)$ represents the generalised trapezoid in the i^{th} variable giving the standard trapezoid when $x_i = \frac{a_i + b_i}{2}$.

We note that $A_i + B_i = d_i$ and if we extend the notation to

$$\tilde{A}_i(x_i) = x_i - a_i, \quad \text{and} \quad \tilde{B}_i(x_i) = b_i - x_i,$$

then we see that

$$\tilde{A}_i(x_i) = \begin{cases} 0, & x_i = a_i \\ d_i, & x_i = b_i \end{cases} \quad \text{and} \quad \tilde{B}_i(x_i) = \begin{cases} d_i, & x_i = a_i \\ 0, & x_i = b_i. \end{cases}$$

Now, for $f(\cdot)$ absolutely continuous in the i^{th} variable $t_i \in [a_i, b_i]$ we have

$$(2.15) \quad \mathfrak{M}_i(f) = \frac{1}{d_i} \int_{a_i}^{b_i} q_i(x_i, t_i) \frac{\partial f}{\partial t_i} dt_i = (G_i - \lambda_i)(f), \quad i = 1, 2, \dots, n,$$

where $G_i(f)$ and $\lambda_i(f)$ are as given by (2.13)-(2.14) and (2.2) respectively and,

$$(2.16) \quad \frac{q_i(x_i, t_i)}{d_i} = \frac{t_i - x_i}{b_i - a_i}, \quad x_i, t_i \in [a_i, b_i].$$

Let $c^{(0)} = (c_1, c_2, \dots, c_n)$, where $c_i = a_i$ or b_i in the i^{th} position for $i = 1, 2, \dots, n$. Also, let $\sigma_0(c^{(0)})$ be the set of all such vectors which consists of 2^n possibilities. Further, let

$$(2.17) \quad \chi_k = \prod_{j=1}^n \binom{k}{j} \frac{C_j}{d_j}, \quad k = 0, 1, \dots, n,$$

where $C_j = A_j$ or B_j with the exception that k of the $C_j = d_j$ and so $\frac{C_j}{d_j} = 1$.

In a similar fashion, let $c^{(k)}$ be a vector taking on the fixed values a_i or b_i in the i^{th} position except for k of the positions which are variable, t_{\bullet} . Let \mathcal{M}_k be k -dimensional integral means for $f(c^{(k)})$. Here $c^{(k)} \in \sigma_k(c^{(k)})$ the set of all such elements, of which there are $\binom{n}{k} 2^{n-k}$.

With the above notation in place, the following theorem holds.

THEOREM 4. *Let $f : I^n \rightarrow \mathbb{R}$ be absolutely continuous and be such that all partial derivatives of order one in each of the variables exist. Then*

$$(2.18) \quad R_n(f) = \sum_0 \chi_0 f(c^{(0)}) - \sum_1 \chi_1 \mathcal{M}_1 + \sum_2 \chi_2 \mathcal{M}_2 - \dots - (-1)^{n-1} \sum_{n-1} \chi_{n-1} \mathcal{M}_{n-1} + (-1)^n \mathcal{M}_n$$

$$: = \rho_n(\underset{\sim}{a}, \underset{\sim}{x}, \underset{\sim}{b}),$$

where χ_k is as defined in (2.17), \mathcal{M}_k is the k -dimensional integral mean for $f(c^{(k)})$, specifically,

$$\mathcal{M}_n = \frac{1}{D_n} \int_{a_n}^{b_n} \dots \int_{a_1}^{b_1} f(t_1, t_2, \dots, t_n) dt_1 \dots dt_n$$

and \sum_k is a sum involving each of the elements of $\sigma_k(c^{(k)})$ of which there are $\binom{n}{k} 2^{n-k}$ terms.

Further,

$$(2.19) \quad R_n(f) = \frac{1}{D_n} \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \prod_{i=1}^n q_i(x_i, t_i) \frac{\partial^n f}{\partial t_n \cdots \partial t_1} dt_1 \cdots dt_n,$$

and $q_i(x_i, t_i)$ is given by (2.16), D_n by (2.7). Here, c_i is equal to either a_i or b_i in which case $C_i = A_i$ or B_i .

Proof. Let $R_r(f)$ be defined by

$$(2.20) \quad R_r(f) := \left(\prod_{i=1}^r \mathfrak{M}_i \right) (f),$$

then from the left identity in (2.15), $R_n(f)$ is as given by (2.19). Now,

$$(2.21) \quad R_r(f) = \mathfrak{M}_r(R_{r-1}(f)), \text{ for } r = 1, 2, \dots, n,$$

where $R_0(f) = f$.

Thus, from (2.20)

$$R_1(f) = \mathfrak{M}_1(f) = (G_1 - \lambda_1)(f),$$

which is the generalised trapezoidal identity for $t_1, x_1 \in [a_1, b_1]$

$$(2.22) \quad \begin{aligned} R_1(f) &= \frac{1}{d_1} \int_{a_1}^{b_1} q_1(x_1, t_1) \frac{\partial f}{\partial t_1}(t_1, t_2, \dots, t_n) dt_1 \\ &= \frac{A_1}{d_1} f(a_1, t_2, \dots, t_n) + \frac{B_1}{d_1} f(b_1, t_2, \dots, t_n) \\ &\quad - \frac{1}{d_1} \int_{a_1}^{b_1} f(t_1, t_2, \dots, t_n) dt_1 \end{aligned}$$

contains three entities; two function evaluations and one integral. Further,

$$\begin{aligned} R_2(f) &= \mathfrak{M}_2(R_1(f)) = (G_2 - \lambda_2)(R_1(f)) \\ &= G_2(R_1(f)) - \lambda_2(R_1(f)) \\ &= \frac{A_2}{d_2} R_1(f) \Big|_{t_2=a_2} + \frac{B_2}{d_2} R_1(f) \Big|_{t_2=b_2} - \frac{1}{d_2} \int_{a_2}^{b_2} R_1(f) dt_2 \end{aligned}$$

contains nine entities. Thus,

$$\begin{aligned}
& R_2(f) \\
= & \frac{A_2}{d_2} \left\{ \frac{A_1}{d_1} f(a_1, a_2, t_3, \dots, t_n) + \frac{B_1}{d_1} f(b_1, a_2, t_3, \dots, t_n) \right. \\
& \quad \left. - \frac{1}{d_1} \int_{a_1}^{b_1} f(t_1, a_2, t_3, \dots, t_n) dt_1 \right\} \\
& + \frac{B_2}{d_2} \left\{ \frac{A_1}{d_1} f(a_1, b_2, t_3, \dots, t_n) + \frac{B_1}{d_1} f(b_1, b_2, t_3, \dots, t_n) \right. \\
& \quad \left. - \frac{1}{d_1} \int_{a_1}^{b_1} f(t_1, b_2, t_3, \dots, t_n) dt_1 \right\} \\
& - \frac{1}{d_2} \int_{a_2}^{b_2} \left\{ \frac{A_1}{d_1} f(a_1, t_2, \dots, t_n) + \frac{B_1}{d_1} f(b_1, t_2, t_3, \dots, t_n) \right. \\
(2.23) \quad & \quad \left. - \frac{1}{d_1} \int_{a_1}^{b_1} f(t_1, t_2, \dots, t_n) dt_1 \right\} dt_2 \\
= & \frac{A_2 A_1}{d_2 d_1} f(a_1, a_2, t_3, \dots, t_n) + \frac{A_2 B_1}{d_2 d_1} f(b_1, a_2, t_3, \dots, t_n) \\
& + \frac{B_2 A_1}{d_2 d_1} f(a_1, b_2, t_3, \dots, t_n) + \frac{B_2 B_1}{d_2 d_1} f(b_1, b_2, t_3, \dots, t_n) \\
& - \frac{A_2}{d_2} \cdot \frac{1}{d_1} \int_{a_1}^{b_1} f(t_1, a_2, t_3, \dots, t_n) dt_1 \\
& - \frac{B_2}{d_2} \cdot \frac{1}{d_1} \int_{a_1}^{b_1} f(t_1, b_2, t_3, \dots, t_n) dt_1 \\
& - \frac{A_1}{d_1} \cdot \frac{1}{d_2} \int_{a_2}^{b_2} f(a_1, t_2, \dots, t_n) dt_2 \\
& - \frac{B_1}{d_1} \cdot \frac{1}{d_2} \int_{a_2}^{b_2} f(b_1, t_2, t_3, \dots, t_n) dt_2 \\
& + \frac{1}{d_2 d_1} \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(t_1, t_2, \dots, t_n) dt_1 dt_2.
\end{aligned}$$

From the 3^2 entities of $R_2(f)$ there are 2^2 function evaluations, 2×2 single integrals and one double integral

$$\begin{aligned}
R_3(f) &= \mathfrak{M}_3(R_2(f)) = (G_3 - \lambda_3)(R_2(f)) = G_3(R_2(f)) - \lambda_3(R_2(f)) \\
&= \frac{A_3}{d_3} R_2(f) \Big|_{t_3=a_3} + \frac{B_3}{d_3} R_2(f) \Big|_{t_3=b_3} - \frac{1}{d_3} \int_{a_3}^{b_3} R_2(f) dt_3.
\end{aligned}$$

This will produce 3^2 entities with $\binom{3}{0} 2^3$ function evaluations, $\binom{3}{1} 2^2$ single integrals, $\binom{3}{2} 2^1$ double integrals and $\binom{3}{0} 2^0$ triple integrals. The 2 occurs since evaluation is at either the a_i or the b_i .

Continuing in this manner we obtain the result as stated where there are 3^n entities for $R_n(f)$, with $\binom{n}{0} 2^n$ function evaluations only, $\binom{n}{1} 2^{n-1}$ single integrals, $\binom{n}{2} 2^{n-2}$ double integrals, ... , $\binom{n}{n-1} 2$, $(n-1)^{\text{th}}$ integrals and one n -dimensional integral. \square

COROLLARY 1. *Let the conditions of Theorem 4 continue to hold. Then*

$$(2.24) \quad \begin{aligned} \tilde{R}_n(f) &= \frac{1}{2^n} \sum_0 f(c^{(0)}) - \frac{1}{2^{n-1}} \sum_1 \mathcal{M}_1 + \dots + \frac{(-1)^{n-1}}{2} \sum_{n-1} \mathcal{M}_{n-1} \\ &\quad + \frac{(-1)^n}{D_n} \int_{a_n}^{b_n} \dots \int_{a_1}^{b_1} f(t_1, t_2, \dots, t_n) dt_1 \dots dt_n, \end{aligned}$$

where c_i is either a_i or b_i for $i = 1, 2, \dots, n$ and

$$\tilde{R}_n(f) = \frac{1}{D_n} \int_{a_n}^{b_n} \dots \int_{a_1}^{b_1} \prod_{i=1}^n q_i(\gamma_i, t_i) \frac{\partial^n f}{\partial t_n \dots \partial t_1} dt_1 \dots dt_n,$$

$$\gamma_i = \frac{a_i + b_i}{2}.$$

Proof. The proof is obvious. On placing $x_i = \gamma_i = \frac{a_i + b_i}{2}$ we get $A_i = B_i = \frac{b_i - a_i}{2} = \frac{d_i}{2}$. \square

REMARK 2. Taking $n = 1$ in (2.2) gives

$$(2.25) \quad \begin{aligned} &\frac{1}{d_1} \int_{a_1}^{b_1} q_1(x_1, t_1) f'(t_1) dt_1 \\ &= \frac{A_1}{d_1} f(a_1) + \frac{B_1}{d_1} f(b_1) - \frac{1}{d_1} \int_{a_1}^{b_1} f(t_1) dt_1 \end{aligned}$$

and so

$$\begin{aligned} &\frac{1}{d_1} \int_{a_1}^{b_1} q_1\left(\frac{a_1 + b_1}{2}, t_1\right) f'(t_1) dt_1 \\ &= \frac{1}{2} [f(a_1) + f(b_1)] - \frac{1}{d_1} \int_{a_1}^{b_1} f(t_1) dt_1, \end{aligned}$$

the trapezoidal identity involving the first derivative.

If $n = 2$, then from (2.23) and dropping the third to n^{th} dimensions gives,

$$\begin{aligned}
 & R_2(f) \\
 = & \frac{A_2}{d_2} \cdot \frac{A_1}{d_1} f(a_1, a_2) + \frac{A_2}{d_2} \cdot \frac{B_1}{d_1} f(b_1, a_2) \\
 & + \frac{B_2}{d_2} \cdot \frac{A_1}{d_1} f(a_1, b_2) + \frac{B_2}{d_2} \cdot \frac{B_1}{d_1} f(b_1, b_2) \\
 (2.26) \quad & - \frac{A_2}{d_2} \cdot \frac{1}{d_1} \int_{a_1}^{b_1} f(t_1, a_2) dt_1 - \frac{B_2}{d_2} \cdot \frac{1}{d_1} \int_{a_1}^{b_1} f(t_1, b_2) dt_1 \\
 & - \frac{A_1}{d_1} \cdot \frac{1}{d_2} \int_{a_2}^{b_2} f(a_1, t_2) dt_2 - \frac{B_1}{d_1} \cdot \frac{1}{d_2} \int_{a_2}^{b_2} f(b_1, t_2) dt_2 \\
 & + \frac{1}{d_2 d_1} \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(t_1, t_2) dt_1 dt_2.
 \end{aligned}$$

We now obtain bounds for $\rho_n(\underline{a}, \underline{x}, \underline{b})$ as defined in (2.18) for $\frac{\partial^n f}{\partial t_n \dots \partial t_1} \in L_p[I^n]$, $1 \leq p \leq \infty$ should be with the usual Lebesgue norms.

THEOREM 5. *Let the conditions of Theorem 4 persist. Then,*

$$(2.27) \quad D_n \left| \rho_n(\underline{a}, \underline{x}, \underline{b}) \right| \leq \begin{cases} \prod_{i=1}^n Q_i(1) \left\| \frac{\partial^n f}{\partial t_n \dots \partial t_1} \right\|_{\infty}, & \frac{\partial^n f}{\partial t_n \dots \partial t_1} \in L_{\infty}[I^n]; \\ \left(\prod_{i=1}^n Q_i(q) \right)^{\frac{1}{q}} \left\| \frac{\partial^n f}{\partial t_n \dots \partial t_1} \right\|_p, & \frac{\partial^n f}{\partial t_n \dots \partial t_1} \in L_p[I^n], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1 \\ \prod_{i=1}^n \phi_i \left\| \frac{\partial^n f}{\partial t_n \dots \partial t_1} \right\|_1, & \frac{\partial^n f}{\partial t_n \dots \partial t_1} \in L_1[I^n]; \end{cases}$$

where $\rho_n(\underline{a}, \underline{x}, \underline{b})$ is as defined by (2.18),

$$(2.28) \quad (q + 1) Q_i(q) = A_i^{q+1} + B_i^{q+1},$$

$$(2.29) \quad \phi_i = \frac{d_i}{2} + |x_i - \gamma_i|.$$

Proof. From (2.18) and (2.19) we have

(2.30)

$$\begin{aligned} \left| \rho_n \left(\tilde{a}, \tilde{x}, \tilde{b} \right) \right| &= |R_n(f)| \\ &\leq \frac{1}{D_n} \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \left| \prod_{i=1}^n q_i(x_i, t_i) \frac{\partial^n f}{\partial t_n \cdots \partial t_1} \right| dt_1 \cdots dt_n. \end{aligned}$$

Now, for $\frac{\partial^n f}{\partial t_n \cdots \partial t_1} \in L_\infty[I^n]$, then

(2.31)

$$\begin{aligned} |R_n(f)| &\leq \left\| \frac{\partial^n f}{\partial t_n \cdots \partial t_1} \right\|_\infty \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \left| \prod_{i=1}^n q_i(x_i, t_i) \right| dt_1 \cdots dt_n \\ &= \left\| \frac{\partial^n f}{\partial t_n \cdots \partial t_1} \right\|_\infty \prod_{i=1}^n \int_{a_i}^{b_i} |q_i(x_i, t_i)| dt_i \\ &= \left\| \frac{\partial^n f}{\partial t_n \cdots \partial t_1} \right\|_\infty \prod_{i=1}^n \left[\int_{a_i}^{x_i} (t_i - a_i) dt_i + \int_{x_i}^{b_i} (b_i - t_i) dt_i \right] \\ &= \frac{1}{2^n} \prod_{i=1}^n [A_i^2 + B_i^2] \left\| \frac{\partial^n f}{\partial t_n \cdots \partial t_1} \right\|_\infty. \end{aligned}$$

Thus, combining (2.30) and (2.31) gives the first inequality in (2.27).

Further, using the Hölder inequality for multiple integrals, we have from (2.30) for $\frac{\partial^n f}{\partial t_n \cdots \partial t_1} \in L_p[I^n]$, $1 < p < \infty$,

(2.32)

$$D_n |R_n(f)| \leq \left\| \frac{\partial^n f}{\partial t_n \cdots \partial t_1} \right\|_p \left(\int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \left| \prod_{i=1}^n q_i(x_i, t_i) \right|^q dt_1 \cdots dt_n \right)^{\frac{1}{q}}.$$

Here, on using (2.16), we have

$$\begin{aligned} &\int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \left| \prod_{i=1}^n q_i(x_i, t_i) \right|^q dt_1 \cdots dt_n \\ &= \prod_{i=1}^n \int_{a_i}^{b_i} |p_i(x_i, t_i)|^q dt_i \end{aligned}$$

$$\begin{aligned}
&= \prod_{i=1}^n \left[\int_{a_i}^{x_i} (t_i - a_i)^q dt_i + \int_{x_i}^{b_i} (b_i - t_i)^q dt_i \right] \\
&= \frac{1}{(q+1)^n} \prod_{i=1}^n \left[A_i^{q+1} + B_i^{q+1} \right]
\end{aligned}$$

and so the second inequality in (2.27) holds on utilising (2.30) and noting (2.28).

The final inequality in (2.27) is obtained from (2.31) for $\frac{\partial^n f}{\partial t_n \cdots \partial t_1} \in L_1 [I^n]$ giving from (2.30)

$$\begin{aligned}
D_n |R_n(f)| &\leq \sup_{t \in [\underline{a}, \underline{b}]} \left| \prod_{i=1}^n q_i(x_i, t_i) \right| \left\| \frac{\partial^n f}{\partial t_n \cdots \partial t_1} \right\|_1 \\
&= \prod_{i=1}^n \sup_{t_i \in [a_i, b_i]} |q_i(x_i, t_i)| \left\| \frac{\partial^n f}{\partial t_n \cdots \partial t_1} \right\|_1 \\
&= \prod_{i=1}^n \max \{A_i, B_i\} \left\| \frac{\partial^n f}{\partial t_n \cdots \partial t_1} \right\|_1.
\end{aligned}$$

Noting that

$$\max \{X, Y\} = \frac{X+Y}{2} + \frac{|X-Y|}{2}$$

gives the final result. \square

REMARK 3. It should be noted that the bounds given on $\left| \tau_n \left(\underline{a}, \underline{x}, \underline{b} \right) \right|$ by (2.8) are exactly the same bounds as those given by (2.27) for $\left| \rho_n \left(\underline{a}, \underline{x}, \underline{b} \right) \right|$. It was shown by Cerone [6] that the bounds in terms of the Lebesgue norms on $|S(f; c, x, d)|$ and $|T(f; c, x, d)|$, as defined in (1.1) and (1.2), are the same. Since the identities for these two functionals were used as **generators** for the multidimensional extension, the equality of the two sets of bounds (2.8) and (2.27) should not come as a great surprise. Finally we notice, due to convexity of the bounds in (2.27), that the sharpest bounds occur at $x_i = \gamma_i = \frac{a_i + b_i}{2}$.

3. Three point identities and their bounds

For $f(\cdot)$ absolutely continuous, then from (1.8)-(1.9)

$$(3.1) \quad \mathfrak{F}(f; a, \alpha, x, \beta, b) = \int_a^b r(x, t) f'(t) dt,$$

with

$$(3.2) \quad r(x, t) = \begin{cases} \frac{t - \alpha}{b - a}, & t \in [a, x] \\ \frac{t - \beta}{b - a}, & t \in (x, b]. \end{cases}$$

However, it may be noticed that

$$(3.3) \quad \mathfrak{F}(f; a, \alpha, x, \beta, b) = T(f; a, \alpha, x) + T(f; x, \beta, b)$$

and

$$(3.4) \quad r(x, t) = \begin{cases} q(\alpha, t), & t \in [a, x] \\ q(\beta, t), & t \in (x, b], \end{cases}$$

where $q(x, t) = \frac{t-x}{b-a}$, $t, x \in [a, b]$.

Thus we have the identity

$$(3.5) \quad \begin{aligned} \Psi_n \left(\underset{\sim}{a}, \underset{\sim}{\alpha}, \underset{\sim}{x}, \underset{\sim}{\beta}, \underset{\sim}{b} \right) &:= \rho_n \left(\underset{\sim}{a}, \underset{\sim}{\alpha}, \underset{\sim}{x} \right) + \rho_n \left(\underset{\sim}{x}, \underset{\sim}{\beta}, \underset{\sim}{b} \right) \\ &= \frac{1}{D_n} \int_{a_n}^{x_n} \cdots \int_{a_1}^{x_1} \prod_{i=1}^n (t_i - \alpha_i) \frac{\partial^n f}{\partial t_n \cdots \partial t_1} dt_1 \cdots dt_n \\ &+ \frac{1}{D_n} \int_{x_n}^{b_n} \cdots \int_{x_1}^{b_1} \prod_{i=1}^n (t_i - \beta_i) \frac{\partial^n f}{\partial t_n \cdots \partial t_1} dt_1 \cdots dt_n \\ &= \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \prod_{i=1}^n r_i(x_i, t_i) \frac{\partial^n f}{\partial t_n \cdots \partial t_1} dt_1 \cdots dt_n, \end{aligned}$$

where

$$(3.6) \quad \frac{r_i(x_i, t_i)}{d_i} = \begin{cases} \frac{t_i - \alpha_i}{d_i}, & t_i \in [a_i, x_i] \\ \frac{t_i - \beta_i}{d_i}, & t_i \in (x_i, b_i]. \end{cases}$$

It is important to obtain an identity for the three point rule since the bounds are tighter than using the bounds of the two trapezoidal rules as this would entail using the triangle inequality. We notice that $\Psi_n \left(\underset{\sim}{a}, \underset{\sim}{\alpha}, \underset{\sim}{x}, \underset{\sim}{\beta}, \underset{\sim}{b} \right)$ in (3.5) is not expressed explicitly. This may be accomplished by returning to (2.18) or else we may use the *generator* methodology utilised to obtain the results in Section 2.

Let $f : I^n \rightarrow \mathbb{R}$ and define the operator

$$(3.7) \quad H_i(f) := \frac{\nu_i^{(a)}}{d_i} f(t_1, \dots, t_{i-1}, a_i, t_{i+1}, \dots, t_n) \\ + \frac{\nu_i^{(x)}}{d_i} f(t_1, \dots, t_{i-1}, x_i, t_{i+1}, \dots, t_n) + \frac{\nu_i^{(b)}}{d_i} f(t_1, \dots, t_{i-1}, b_i, t_{i+1}, \dots, t_n),$$

where,

$$(3.8) \quad \nu_i^{(a)} = \alpha_i - a_i, \quad \nu_i^{(x)} = \beta_i - \alpha_i, \quad \nu_i^{(b)} = b_i - \beta_i, \quad d_i = b_i - a_i.$$

Then, from (3.1)-(3.2) for $f(\cdot)$ absolutely continuous in the i^{th} variable $t_i \in [a_i, b_i]$ we have

$$(3.9) \quad \mathfrak{N}_i(f) = \frac{1}{d_i} \int_{a_i}^{b_i} r_i(x_i, t_i) \frac{\partial f}{\partial t_i} dt_i = (H_i - \lambda_i)(f), \quad i = 1, 2, \dots, n.$$

where $H_i(f)$ and $\lambda_i(f)$ are as given by (3.7)-(3.8) and (2.2) respectively.

If we now follow the work of the previous section and let $c^{(0)} = (c_1, c_2, \dots, c_n)$, where now $c_i = a_i, x_i$ or b_i in the i^{th} partition for $i = 1, 2, \dots, n$. Then $\sigma_0(c^{(0)})$ which is the set of all such vectors consists of 3^n possibilities. Further, let χ_n be as in (2.17) where now, $C_j = \nu_j^{(a)}$ or $\nu_j^{(x)}$ or $\nu_j^{(b)}$ are as defined by (3.8) with the exception that k of the $C_j = d_j$ and so $\frac{C_j}{d_j} = 1$.

Further, $c^{(k)}$ is a vector taking on fixed values of either a_i, x_i or b_i in the i^{th} position except for k of the positions which are variable, t_\bullet . Let \mathcal{M}_k be k -dimensional integral means for $f(c^{(k)})$, then the following theorem holds.

THEOREM 6. *Let $f : I^n \rightarrow \mathbb{R}$ be absolutely continuous and be such that all partial derivatives of order one in each of the variables exist.*

Then

$$\begin{aligned}
 (3.10) \quad B_n(f) &= \sum_0 \chi_0 f(c^{(0)}) - \sum_1 \chi_1 \mathcal{M}_1 + \sum_2 \chi_2 \mathcal{M}_2 \\
 &\quad - \cdots - (-1)^{n-1} \sum_{n-1} \chi_{n-1} \mathcal{M}_{n-1} + (-1)^n \mathcal{M}_n \\
 &: = \Psi_n \left(\underset{\sim}{a}, \underset{\sim}{\alpha}, \underset{\sim}{x}, \underset{\sim}{\beta}, \underset{\sim}{b} \right),
 \end{aligned}$$

where

$$(3.11) \quad \chi_k = \prod_{j=1}^n \binom{k}{j} \frac{C_j}{d_j}, \quad k = 0, 1, \dots, n,$$

with $C_j = \nu_j^{(a)}, \nu_j^{(x)}$ or $\nu_j^{(b)}$ as defined in (3.8) except for k of the $C_j = d_j$ giving $\frac{C_j}{d_j} = 1$, \mathcal{M}_k is the k -dimensional integral mean of $f(c^{(k)})$ and specifically,

$$\mathcal{M}_n = \frac{1}{D_n} \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} f(c^{(n)}) dt_1 \dots dt_n, \quad c^{(n)} = (t_1, t_2, \dots, t_n).$$

Finally, \sum_k is a sum over all $\binom{n}{k} 3^{n-k}$ terms and $B_n(f)$ is as defined in (3.5).

Proof. Similar to that of Theorem 4 except that for each variable t_i there are now three possible choices for evaluation of either a_i, x_i , or b_i . \square

The following theorem gives bounds for the $\Psi_n \left(\underset{\sim}{a}, \underset{\sim}{\alpha}, \underset{\sim}{x}, \underset{\sim}{\beta}, \underset{\sim}{b} \right)$ as given in either (3.10) or (3.5).

THEOREM 7. *Let the conditions of Theorem 6 continue to hold. Then,*

$$(3.12) \quad D_n \left| \Psi_n \left(\underset{\sim}{a}, \underset{\sim}{\alpha}, \underset{\sim}{x}, \underset{\sim}{\beta}, \underset{\sim}{b} \right) \right| \leq \begin{cases} \prod_{i=1}^n S_i(1) \left\| \frac{\partial^n f}{\partial t_n \cdots \partial t_1} \right\|_{\infty}, & \frac{\partial^n f}{\partial t_n \cdots \partial t_1} \in L_{\infty}[I^n]; \\ \left(\prod_{i=1}^n S_i(q) \right)^{\frac{1}{q}} \left\| \frac{\partial^n f}{\partial t_n \cdots \partial t_1} \right\|_p, & \frac{\partial^n f}{\partial t_n \cdots \partial t_1} \in L_{\infty}[I^n], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1 \\ \prod_{i=1}^n \zeta_i \left\| \frac{\partial^n f}{\partial t_n \cdots \partial t_1} \right\|_1, & \frac{\partial^n f}{\partial t_n \cdots \partial t_1} \in L_1[I^n]; \end{cases}$$

where $\|h\|_p$, $1 \leq p < \infty$ and $\|h\|_\infty$ are defined by (1.11) and (1.12)

$\Psi_n \left(\underset{\sim}{a}, \underset{\sim}{\alpha}, \underset{\sim}{x}, \underset{\sim}{\beta}, \underset{\sim}{b} \right)$ is defined by (3.5) or, explicitly, by (3.10),

$$(3.13) \quad (q+1) S_i(q) = (\alpha_i - a_i)^{q+1} + (x_i - \alpha_i)^{q+1} + (\beta_i - x_i)^{q+1} + (b_i - \beta_i)^{q+1}$$

$$(3.14) \quad \zeta_i = \frac{1}{2} \left\{ \frac{b_i - a_i}{2} + \left| \alpha_i - \frac{a_i + x_i}{2} \right| + \left| \beta_i - \frac{x_i + b_i}{2} \right| + \left| x_i - \frac{a_i + b_i}{2} \right| + \left| \alpha_i - \frac{a_i + x_i}{2} \right| - \left| \beta_i - \frac{x_i + b_i}{2} \right| \right\}.$$

Proof. From (3.10) and (3.5)

$$(3.15) \quad \begin{aligned} & \left| \Psi_n \left(\underset{\sim}{a}, \underset{\sim}{\alpha}, \underset{\sim}{x}, \underset{\sim}{\beta}, \underset{\sim}{b} \right) \right| \\ &= |B_n(f)| \\ &\leq \frac{1}{D_n} \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \left| \prod_{i=1}^n r_i(x_i, t_i) \frac{\partial^n f}{\partial t_n \cdots \partial t_1} \right| dt_1 \cdots dt_n. \end{aligned}$$

Now, for $\frac{\partial^n f}{\partial t_n \cdots \partial t_1} \in L_\infty [I^n]$ then

$$(3.16) \quad D_n |B_n(f)| \leq \left\| \frac{\partial^n f}{\partial t_n \cdots \partial t_1} \right\|_\infty \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \left| \prod_{i=1}^n r_i(x_i, t_i) \right| dt_1 \cdots dt_n,$$

where $\|h\|_\infty$ is defined by (1.12).

Here

$$\int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \left| \prod_{i=1}^n r_i(x_i, t_i) \right| dt_1 \cdots dt_n = \prod_{i=1}^n \int_{a_i}^{b_i} |r_i(x_i, t_i)| dt_i,$$

where

$$\begin{aligned} & \int_{a_i}^{b_i} |r_i(x_i, t_i)| dt_i \\ &= \int_{a_i}^{\alpha_i} (\alpha_i - t_i) dt_i + \int_{\alpha_i}^{x_i} (t_i - \alpha_i) dt_i \\ & \quad + \int_{x_i}^{\beta_i} (\beta_i - t_i) dt_i + \int_{\beta_i}^{b_i} (t_i - \beta_i) dt_i \\ &= \frac{1}{2} \left[(\alpha_i - a_i)^2 + (x_i - \alpha_i)^2 + (\beta_i - x_i)^2 + (b_i - \beta_i)^2 \right]. \end{aligned}$$

Substitution of the above results into (3.16) gives the first inequality in (3.12) with $S_i(1)$ given from (3.13).

Moreover, using the Hölder inequality for multiple integrals we have, from (3.15) for $\frac{\partial^n f}{\partial t_n \dots \partial t_1} \in L_p[I^n]$, $1 < p < \infty$,

$$(3.17) \quad D_n |B_n(f)| \leq \left\| \frac{\partial^n f}{\partial t_n \dots \partial t_1} \right\|_p \left(\int_{a_n}^{b_n} \dots \int_{a_1}^{b_1} \left| \prod_{i=1}^n r_i(x_i, t_i) \right|^q dt_1 \dots dt_n \right)^{\frac{1}{q}},$$

where

$$\begin{aligned} & \int_{a_n}^{b_n} \dots \int_{a_1}^{b_1} \left| \prod_{i=1}^n r_i(x_i, t_i) \right|^q dt_1 \dots dt_n \\ &= \prod_{i=1}^n \int_{a_i}^{b_i} |p_i(x_i, t_i)|^q dt_i \\ &= \prod_{i=1}^n \left\{ \int_{a_i}^{\alpha_i} (\alpha_i - t_i)^q dt_i + \int_{\alpha_i}^{x_i} (t_i - \alpha_i)^q dt_i \right. \\ & \quad \left. + \int_{x_i}^{\beta_i} (\beta_i - t_i)^q dt_i + \int_{\beta_i}^{b_i} (t_i - \beta_i)^q dt_i \right\} \\ &= \frac{1}{(q+1)^n} \prod_{i=1}^n \left[(\alpha_i - a_i)^{q+1} + (x_i - \alpha_i)^{q+1} \right. \\ & \quad \left. + (\beta_i - x_i)^{q+1} + (b_i - \beta_i)^{q+1} \right]. \end{aligned}$$

Substitution of the above calculations into (3.17) gives the second inequality in (3.12) with $S_i(q)$ as given by (3.13).

Finally, for $\frac{\partial^n f}{\partial t_n \dots \partial t_1} \in L_1[I^n]$ we have from (3.15)

$$\begin{aligned} & D_n |B_n(f)| \\ & \leq \sup_{\tilde{t} \in [\tilde{a}, \tilde{b}]} \left| \prod_{i=1}^n r_i(x_i, t_i) \right| \left\| \frac{\partial^n f}{\partial t_n \dots \partial t_1} \right\|_1 \\ & = \prod_{i=1}^n \sup_{t_i \in [a_i, b_i]} |r_i(x_i, t_i)| \left\| \frac{\partial^n f}{\partial t_n \dots \partial t_1} \right\|_1 \\ & = \prod_{i=1}^n \max \{ \alpha_i - a_i, x_i - \alpha_i, \beta_i - x_i, b_i - \beta_i \} \left\| \frac{\partial^n f}{\partial t_n \dots \partial t_1} \right\|_1. \end{aligned}$$

Using the result that

$$\max \{X, Y\} = \frac{X + Y}{2} + \frac{|X - Y|}{2}$$

twice produces the final inequality in (3.12). \square

REMARK 4. The bounds obtained above in Theorem 7 are the product of the bounds for the one dimensional integral results. These were studied extensively in Cerone and Dragomir [8]. It should further be noted that the three point results of the current section recaptures the generalised trapezoidal results of the previous section if we take $\alpha_i = \beta_i = x_i$. In addition, the Ostrowski type results of Cerone [5] are recaptured if we take $\alpha_i = a_i$ and $\beta_i = b_i$.

If we take $n = 2$ then the results of Hanna et al. [18] are recaptured. Chapter 6 by Hanna in [15] treats double integrals while in Chapter 5 of the same book, Barnett et al. produce related but different results for multiple integrals involving only evaluation at an interior point.

Finally, we notice that the best choice of the α_i, β_i and the x_i in (3.12) is at their respective midpoints, providing the sharpest bound. That is, $\alpha_i = \frac{a_i + x_i}{2}$, $\beta_i = \frac{x_i + b_i}{2}$ and $x_i = \frac{a_i + b_i}{2}$, $i = 1, 2, \dots, n$.

4. Concluding remarks

Perturbed rules using the Chebychev functional as determined by Cerone [5] for multiple integrals obtained using an Ostrowski functional as a *generator* can also be produced here for both the trapezoidal and three point developments. This, however, will not be pursued further here. The procedure developed in [5] and extended here may also be used to include higher order formulae involving the behaviour of higher derivatives for its bounds. Multidimensional results based on an m branched Peano kernel producing function evaluations at $m+1$ points are also possible using the methodology demonstrated in Section 3 for $m = 2$. Finally, weighted rules are also possible using a weighted functional as a generator, however, this may be restricted to product form weight functions in the multidimensional result. These will also not be investigated further here.

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