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ORDERED GROUPS IN WHICH ALL CONVEX JUMPS ARE CENTRAL

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Abstract. $(G,<)$ is an ordered group if '$<$' is a total order relation on G in which $f<g$ implies that $xfy<xgy$ for all $f,g,x,y\in G$. We say that $(G,<)$ is centrally ordered if $(G,<)$ is ordered and $[G,D]\subseteq C$ for every convex jump $C\prec D$ in G. Equivalently, if $f^{-1}gf\le g^2$ for all $f,g\in G$ with $g>1$. Every order on a torsion-free locally nilpotent group is central. We prove that if every order on every two-generator subgroup of a locally soluble orderable group G is central, then G is locally nilpotent. We also provide an example of a non-nilpotent two-generator metabelian orderable group in which all orders are central.

1. Introduction

Let G be a group and $<$ be a total order on G. If, further, $f<g$ in $(G,<)$ implies that $xfy<xgy$ for all $x,y\in G$ then we call $(G,<)$ an *ordered group* (or simply write that G is an ordered group). If $<$ is any partial ordering on a group G and $G_+=\{g\in G: g>1\}$, then G is an ordered group if and only if G_+ is a normal subsemigroup of G and $g\in G_+$ or $g^{-1}\in G_+$ for all non-identity elements $g\in G$.

If C is a subset of an ordered group G, then C is said to be *convex* if $c_1\le g\le c_2$ with $c_1,c_2\in C$ and $g\in G$ imply $g\in C$.

Let C and D be convex subgroups of an ordered group (so convex with respect to some specific order on the group). Then $C\subseteq D$ or $D\subseteq C$ ([1], Lemma 3.1.2). Hence if G is an ordered group then for each non-identity element $g\in G$, there is a convex subgroup C_g maximal not containing g; viz: the union of all convex subgroups not containing g.

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The intersection C_g^* of all convex subgroups of G containing g is such a convex subgroup. Moreover $C_g \lhd C_g^*$ and is called a *convex jump*. In the terminology of [5], every ordered group has a *series* of convex jumps; that is, each $g \in G \setminus \{1\}$ belongs to a convex jump (namely, $C_g^* \backslash C_g$) and the intersection of all the C_g (as g ranges over all non-identity elements of G) is just $\{1\}$.

If every convex jump is *central* (i.e., $[C_g^*, G] \subseteq C_g$ for all $g \in G \backslash \{1\}$), then G is said to be a *weakly Abelian ordered group*. Our own preference is to call such ordered groups *centrally ordered* and we will adopt that name here.

Note that if G is centrally ordered, then $C_g \lhd G$ for all $g \in G \backslash \{1\}$. Since every convex subgroup C is the intersection of all C_g for which $g \notin C$, we have that every convex subgroup in a centrally ordered group is normal.

Equivalently, an ordered group G is centrally ordered if $f^{-1}gf \leq g^2$ for all $f, g \in G$ with $g \in G_+$ (see [10], Section 6.2); aliter, $[f, g] \ll |f|, |g|$, where $|h| = \max\{h, h^{-1}\}$, $[f, g] = f^{-1}g^{-1}fg$ and we write $x \ll y$ for $x^n \leq y$ for all integers n (see [1], Lemma 6.4.1 or [12]). Thus we have the crucial property that in any centrally ordered group G, the centre of G contains all elements $g \in G$ with $C_g = \{1\}$.

A subgroup that is convex in some order on an orderable group G is called *relatively convex in* G. The union and intersection of a chain of relatively convex subgroups of an orderable group are always relatively convex ([7], or see [13], Theorem 1.4.5 or [8], Chapter II Section 3 Proposition 1).

The lower central series of an orderable group can often be used to obtain a central order on the group. We illustrate this technique:

If V is a vector space over the rationals, let $\{b_i : i \in I\}$ be a basis for V. Totally order I and define $\sum_{i \in I} q_i b_i > \mathbf{0}$ if $q_{i_0} > 0$ (in $\mathbb{Q}$) where $i_0 = \min\{i \in I : q_i \neq 0\}$. Since every Abelian torsion-free group is a subgroup of a rational vector space, all torsion-free Abelian groups are orderable. If G is a group in which each Abelian group $\gamma_k(G)/\gamma_{k+1}(G)$ is torsion-free and $\bigcap_{k=1}^{\infty} \gamma_k(G) = \{1\}$, then we can order G as follows: Put orders on each $\gamma_k(G)/\gamma_{k+1}(G)$. Let $g \in G$ with $g \neq 1$. Then $g \in \gamma_m(G) \backslash \gamma_{m+1}(G)$ for a unique m. Let $g > 1$ iff $g\gamma_{m+1}(G) > \gamma_{m+1}(G)$. This is a central order on G. Thus, in particular, nilpotent groups in which all the factors $\gamma_k(G)/\gamma_{k+1}(G)$ are torsion-free have central orders.

The factors $\gamma_k(G)/\gamma_{k+1}(G)$ need not be torsion-free for a general torsion-free nilpotent group G. However such groups always possess a torsion-free central series, for example the upper central series $\zeta_k(G)$ which is defined in the next section. Following the above procedure with the new central series replacing the lower central series $\gamma_k(G)$, we see that every torsion-free nilpotent group has a central order. In fact every order on every torsion-free locally nilpotent group is central (see [1], Theorem 6.D or [10], Section 6.2 or [18] or [15]). It is thus natural to ask if the converse is true for some well known large classes of groups such as locally soluble orderable groups or finitely generated orderable groups.

This leads to the following questions:

(1) If every order on a finitely generated orderable group G is central, then is G soluble?

(2) If every order on every subgroup of a finitely generated orderable group G is central, then is G nilpotent?

(3) If every order on every subgroup of a locally soluble orderable group G is central, then is G locally nilpotent?

(4) If every order on a finitely generated soluble orderable group is central, then is G nilpotent?

We will provide the first step towards answering these problems in the following three theorems. We will show that if every order on every two generator subgroup of a locally soluble orderable group G is central, then G is locally nilpotent; but that there is a two-generator non-nilpotent metabelian orderable group in which all orders are central (so the answers to (3) is "yes" and the answer to (4) is "no").

Our main results are as follows:

THEOREM A. *There is a non-nilpotent two-generator metabelian orderable group in which all orders central.*

The example we give has a two-generator subgroup with a non-central order.

THEOREM B. *If every order on every two-generator subgroup of a locally soluble orderable group G is central, then G is locally nilpotent.*

The next result follows as corollary of the above result with the application of Tits' Dichotomy for linear groups.

THEOREM C. *If every order on every two-generator subgroup of an orderable linear group G is central, then G is locally nilpotent.*

The group of formal power series under substitution provides an example of a residually torsion-free nilpotent group G that contains a free group of rank two and Kopytov has shown it has the property that every partial order on G can be extended to a total order on G. It can be shown that G also has the property that every order on G is central. The group is denoted by $G(Q)$ in Johnson [6] and by $F_s(K)$ in Kopytov [9] where details are provided. This group is not finitely generated and does not provide an answer to question (1) nor (2).

2. Definitions and background

As is standard we have written $\gamma_n(G)$ for the n^{th} term in the lower central series of G; i.e., $\gamma_1(G) = G$ and $\gamma_n(G)$ is the normal subgroup generated by all commutators $[g_1, \ldots, g_n]$ where $[g_1, \ldots, g_{k+1}] = [[g_1, \ldots, g_k], g_{k+1}]$. We will write $[a,\ _n\ g]$ for $[a, g_1, \ldots, g_n]$ where $g_1 = \cdots = g_n = g$. If H, K are subgroups of a group G, then we write $[H, K]$ for the subgroup of G generated by all elements $[h, k]$ with $h \in H,\ k \in K$. We use G' to denote the derived subgroup $[G, G]$ and $\zeta(G)$ to denote the centre of G. We inductively define $\zeta_{k+1}(G)$ to be that subgroup of G given by $\zeta_{k+1}(G)/\zeta_k(G) = \zeta(G/\zeta_k(G))$.

If G is a torsion-free nilpotent group, then each $\zeta_{k+1}(G)/\zeta_k(G)$ is a torsion-free Abelian group (see [4]) and so can be ordered. Using this upper central series we see that every torsion-free nilpotent group is orderable.

We will use the following standard shorthand (see, e.g., [3]):

We will write $\mathcal{G}$ for the class of all finitely generated groups,

$\mathcal{G}_2$ for the class of all two-generator groups,

$\mathcal{A}$ for the class of all Abelian groups,

$\mathfrak{F}$ for the class of finite groups,

$\mathfrak{F}_p$ for the class of finite p-groups (each group has size a power of the prime p),

$\mathfrak{N}$ for the class of all nilpotent groups,

$\mathfrak{N}_m$ for the class of all nilpotent class m groups,

P$\mathcal{A}$ for the class of all soluble (polyabelian) groups,

$\mathcal{C}$ for the class of orderable groups in which every order is central, and

$\mathcal{C}_2$ for the class of orderable groups in which every order on every two-generator subgroup is central.

Let $\mathfrak{X}$ and $\mathcal{Y}$ be classes of groups (closed under isomorphisms). We will write $G \in \mathfrak{X}\mathcal{Y}$ if there is a normal subgroup N of G such that $N \in \mathfrak{X}$ and $G/N \in \mathcal{Y}$ and say that G is an $\mathfrak{X}$-by-$\mathcal{Y}$ group. If $\mathfrak{X}$ and $\mathcal{Y}$ denote varieties of groups, then $\mathfrak{X}\mathcal{Y}$ is just the product variety.

We say that a group G is *locally* $\mathfrak{X}$ (L$\mathfrak{X}$) if every finitely generated subgroup of G belongs to $\mathfrak{X}$. We say that a group G is *residually* $\mathfrak{X}$ (R$\mathfrak{X}$) if there is a family $\{N_i : i \in I\}$ of normal subgroups of G such that each $G/N_i \in \mathfrak{X}$ and $\bigcap_{i\in I} N_i = \{1\}$.

Since "orderability" is a local condition ([14] or [1], Lemma 2.2.1 or [13], Theorem 3.1.2), locally torsion-free nilpotent groups are orderable as are residually torsion-free nilpotent groups (and have central orders).

With this notation, our first two theorems can be easily restated:

Theorem A: $\mathcal{A}\mathcal{A} \cap \mathcal{G}_2 \cap \mathcal{C} \not\subseteq \mathfrak{N}$.

Theorem B: $\text{LP}\mathcal{A} \cap \mathcal{C}_2 \subseteq \text{L}\mathfrak{N}$.

By the equivalent condition, every order on every subgroup of an orderable group is central if every order on every two-generator subgroup of an orderable group is; i.e., $\mathcal{C}_2 \subseteq \mathcal{C}$. In fact, we will show that this containment is strict.

For a subgroup H of a group G, the *isolator of* H is the subgroup of G generated by all $g \in G$ for which there is a positive integer n such that $g^n \in H$. If $H \lhd G$, then the isolator of H is also normal in G. If we define $H_0 = H$ and H_{n+1} to be the isolator of H_n, then $\overline{H} = \bigcup_n H_n$ is the smallest isolated subgroup of G that contains H ($x^n \in \overline{H}$ & $n \in \mathbb{Z}_+ \to x \in \overline{H}$). Note that $H \lhd G$ implies that $\overline{H} \lhd G$ and $G/\overline{H}$ is torsion-free. Also observe that relatively convex subgroups are isolated.

LEMMA 2.1. ([1], Corollary 2.1.5, [13], Theorem 2.2.4 and [8], Chapter II Section 4 Theorem 3) *If G is an orderable group and $[x^m, y] = 1$ for some $m \in \mathbb{Z}\backslash\{0\}$, then $[x, y] = 1$. Hence the centre of an orderable group is isolated, and $G/\zeta(G)$ is also an orderable group.*

We will write $\overline{\gamma}_k(G)$ as shorthand for $\overline{\gamma_k(G)}$.

If G is an orderable group and $H \lhd G$, then $\overline{H}$ is relatively convex if $G/\overline{H}$ is orderable. But if G is an ordered group, then $G/\overline{\gamma}_k(G)$ is a torsion-free nilpotent group (of nilpotency class at most k) and so can be ordered. Using this and Kokorin's result we get

COROLLARY 2.2. *If G is an ordered group, then $\overline{\gamma}_k(G)$ is a relatively convex subgroup of G for all positive integers k. Hence $\bigcap_{k=1}^{\infty} \overline{\gamma}_k(G)$ is a relatively convex subgroup of G.*

We will also use

LEMMA 2.3. ([13], Theorem 3.1.7) *Let $\prec$ be a partial order on an orderable metabelian group G such that $\{g \in G : 1 \prec g\}$ is a normal subsemigroup of G. Then $\prec$ can be lifted to a total order on G with respect to which G is an ordered group.*

In contrast to Lemma 2.3, it is unknown for finitely generated Abelian-by-nilpotent groups if every partial order of the type described in Lemma 2.3 can be lifted to a total order (compatible with the group operations).

We will need two lemmata from group theory (see [17] or the references below):

LEMMA 2.4. (i) (see [11]) *Any finitely generated nilpotent group satisfies the maximal condition on normal subgroups. In particular, every normal subgroup is defined by a finite set of relations.*

(ii) (see [3]) *Every finitely generated Abelian-by-nilpotent group satisfies the maximal condition on normal subgroups.*

LEMMA 2.5. (see [2]) *Torsion-free nilpotent groups are residually finite p-groups for all primes p.*

3. Proofs

Proof of Theorem A: Let $M = M(a, b)$ be the free metabelian group on free generators a, b. Then M is residually torsion-free nilpotent by [17], Part 2, Corollary p.146, and M' is generated by c as a normal subgroup of M (where $c = [a, b]$). We adopt the usual shorthand $c^x = x^{-1}cx$, $c^{mx+y} = (c^m)^x c^y$, etc.. Since M is metabelian, we have $c^{ab} = c^{ba}$, $[c, a] = c^{a-1}$, $[c, a, a] = c^{(a-1)^2}$, etc.. So each element of M' has a unique representation:

$$c^{n_0+n_1a^{k_1}b^{m_1}+\cdots+n_sa^{k_s}b^{m_s}}, \quad n_i, k_i, m_i \in \mathbb{Z}. \tag{1}$$

It follows from (1) that for $k \geq 2$, the cosets of the elements $[c,\ _n\, a,\ _m\, b]$ with $n, m \geq 0$ and $n + m = k - 2$ generate $\gamma_k(M)/\gamma_{k+1}(M)$.

Let

$$w = c^{a+a^{-1}+b+b^{-1}-4},$$

and $G = M/N$ where N is the normal subgroup of M generated by w. Note that G is a two-generator metabelian group and that $N \subseteq M'$ since $w \in M'$.

The defining relation for G is equivalent to the relation

$$c^{a^2b+b+ab^2+a-4ab} = 1. \tag{2}$$

We can rewrite this in either of the forms

$$c^{(a-1)^2+(b-1)^2+(a-1)^2(b-1)+(a-1)(b-1)^2} = 1, \tag{3}$$

or

$$[c, a, a] \cdot [c, b, b] \cdot [c, a, a, b] \cdot [c, a, b, b] = 1, \tag{4}$$

the latter being most convenient for calculations in the lower central series of G.

We wish to prove that G is a residually torsion-free nilpotent group (whence G is orderable). To establish this it is enough to show that each $\gamma_k(G)/\gamma_{k+1}(G)$ is a free Abelian group and that $\bigcap_k \gamma_k(G) = \{1\}$.

Now $[c,\ _n\, a,\ _m\, b] = [c,\ _{n+2}\, a,\ _{m-2}\, b]$ modulo $\gamma_{n+m+3}(G)$ for $m > 1$ by (4). It follows by induction that the elements $[c,\ _k\, a]\gamma_{k+3}(G)$ and $[c,\ _{k-1}\, a, b]\gamma_{k+3}(G)$ generate $\gamma_{k+2}(G)/\gamma_{k+3}(G)$.

We claim that these elements are free generators of the Abelian group $\gamma_{k+2}(G)/\gamma_{k+3}(G)$; i.e., each $\gamma_{k+2}(G)/\gamma_{k+3}(G)$ is the free Abelian group on these two generators if $k > 0$, and on c if $k = 0$ (which is obvious). So assume that $k > 0$, but $\gamma_{k+2}(G)/\gamma_{k+3}(G)$ is not freely generated by $[c,\ _k\, a]\gamma_{k+3}(G)$ and $[c,\ _{k-1}\, a, b]\gamma_{k+3}(G)$. Then, in G, we would have

$$c^{n_1(a-1)^k+n_2(a-1)^{k-1}(b-1)} = c^{f_1(a-1,b-1)f_2(a,b,a^{-1},b^{-1})},$$

where $f_1(a-1, b-1)$ is a polynomial in the variables $a-1,\ b-1$ and $f_2(a, b, a^{-1}, b^{-1})$ is a polynomial in the variables a, b, a^{-1}, b^{-1}. Moreover the degree of every monomial of $f_1(a-1, b-1)$ is greater than k. Together with (3), this leads to a relation in M:

$$c^{n_1(a-1)^k+n_2(a-1)^{k-1}(b-1)}$$

$$= c^{f_1(a-1,b-1)f_2(a,b,a^{-1},b^{-1})+((a-1)^2+(b-1)^2+(a-1)^2(b-1)+(a-1)(b-1)^2)f_3(a,b,a^{-1},b^{-1})}$$

for some polynomial $f_3(a,b,a^{-1},b^{-1})$ in the variables a,b,a^{-1},b^{-1}. Conjugate both sides by suitable powers of a and b to eliminate all negative powers of a and b and obtain the equivalent relation

$$c^{(n_1(a-1)^k+n_2(a-1)^{k-1}(b-1))a^{t_1}b^{t_2}}$$
$$= c^{f_1(a-1,b-1)f_4(a,b)+((a-1)^2+(b-1)^2+(a-1)^2(b-1)+(a-1)(b-1)^2)f_5(a,b)}.$$

By (1) we get

$$\begin{aligned}(n_1(a-1)^k &+ n_2(a-1)^{k-1}(b-1))a^{t_1}b^{t_2}\\ &= f_1(a-1,b-1)f_4(a,b) + ((a-1)^2+(b-1)^2\\ &\quad + (a-1)^2(b-1) + (a-1)(b-1)^2)f_5(a,b).\end{aligned}$$

Now write $a^{t_1}b^{t_2}$, $f_4(a,b)$ and $f_5(a,b)$ as polynomials in $(a-1)$ and $(b-1)$. So

$$a^{t_1}b^{t_2} = 1 + f_6(a-1,b-1)$$

where every monomial appearing in f_6 has strictly positive degree;

$$f_4(a,b) = f_7(a-1,b-1);$$

$$f_5(a,b) = f_8(a-1,b-1) + f_9(a-1,b-1) + f_{10}(a-1,b-1)$$

and the monomials of f_8 have degree at most $k-3$, f_9 is a homogeneous polynomial of degree $k-2$ and every monomial of f_{10} has degree exceeding $k-2$. So we have

$$\begin{aligned}&(n_1(a-1)^k + n_2(a-1)^{k-1}(b-1))(1+f_6(a-1,b-1))\\ &= f_1(a-1,b-1)f_7(a-1,b-1) + ((a-1)^2+(b-1)^2\\ &\quad + (a-1)^2(b-1) + (a-1)(b-1)^2)\\ &\quad \times (f_8(a-1,b-1) + f_9(a-1,b-1) + f_{10}(a-1,b-1))\end{aligned}$$

By comparing the homogeneous components of every degree we get that $f_8 = 0$ and

$$n_1(a-1)^k + n_2(a-1)^{k-1}(b-1) - ((a-1)^2+(b-1)^2)f_9 = 0.$$

But the last equality is impossible because the first and the second monomials are divisible by $(a-1)^{k-1}$ and $((a-1)^2+(b-1)^2)f_9$ is not. This is the desired contradiction. Consequently, $\gamma_{k+2}(G)/\gamma_{k+3}(G)$ is the free two-generator Abelian group for each $k > 0$ and has rank 1 for $k = 0$. But $\gamma_1(G)/\gamma_2(G) \cong G/G'$ is the free Abelian group of rank 2. Since every $\gamma_k(G)/\gamma_{k+1}(G)$ is a free Abelian group, all quotients $G/\gamma_k(G)$ $(k \in \mathbb{Z}_+)$ are torsion-free.

It remains to prove that $\bigcap_k \gamma_k(G) = 1$. Let $h \in \gamma_2(G) \setminus \{1\}$. Among the conjugates of h, choose one, say $c^{f(a,b)}$, so that $f(a,b)$ is a polynomial in a, b and has least possible degree in b. Then the degree of b is at most 1, since we can divide $f(a,b)$ or $af(a,b)$ by $a + b - 4ab + a^2b + ab^2$ by (2). Thus $f(a,b) = f_1(a) + f_2(a)b = f_1(a) + f_2(a) + f_2(a)(b-1) = f_3(a-1) + f_4(a-1)(b-1)$ where $f_3(X) = n_0 + n_1X + \cdots + n_sX^s$ and $f_4(X) = m_0 + m_1X + \cdots + m_rX^r$. But $c^{f_3(a-1)+f_4(a-1)(b-1)}$ is a conjugate of $h \neq 1$. So not all n_i, m_j are equal to zero. Let i be the least subscript such that $n_i \neq 0$ (it is undefined if all $n_k = 0$), and j be the least subscript such that $m_j \neq 0$ (it is undefined if all $m_k = 0$). Let $t = \min\{i, j+1\}$, if both i and j are defined; $t = i$ if only i is defined, and $t = j+1$ if only j is defined. Now we can rewrite $c^{f(a,b)}$ in the form $[c,\ _t\ a]^{n_t} + [c,\ _{t-1}\ a, b]^{m_{t-1}}$ modulo $\gamma_{t+1}(G)$. Since at least one of n_t, m_{t-1} is not zero and the elements $[c,\ _t\ a]$, $[c,\ _{t-1}\ a, b]$ are independent modulo $\gamma_{t+1}(G)$, we have $c^{f(a,b)} \neq 1$ modulo $\gamma_{t+1}(G)$. Consequently, $h \notin \gamma_{t+1}(G)$, whence $\bigcap_k \gamma_k(G) = 1$.

This completes the proof that G is a residually torsion-free nilpotent group; so G is orderable.

However, G is not nilpotent since it contains an isomorphic copy of the non-nilpotent group $\mathbb{Z} \wr \mathbb{Z}$; viz: the subgroup $\langle a, c \rangle$.

Since G is metabelian, the relation $c^{4-a-a^{-1}-b-b^{-1}} = 1$ implies that

$$c^{(n_1g_1+\cdots+n_kg_k)(4-a-a^{-1}-b-b^{-1})} = 1$$

for all $n_1, \ldots, n_k \in \mathbb{Z}$ and $g_1, \ldots, g_k \in G$. Since G' is generated by the elements $\{c^g : g \in G\}$, we deduce

$$f^4 = f^{a+a^{-1}+b+b^{-1}}$$

for all $f \in G'$.

Let $<$ be any total order on G and $1 < f \in G'$. Let $C_f \prec C_f^*$ be the convex jump defined by f. If $f \ll f^a$ then $f \ll f^a \cdot f^{a^{-1}+b+b^{-1}} = f^4$ (because $f^{a^{-1}+b+b^{-1}} > 1$), a contradiction. Similarly $f \ll f^{a^{-1}}$, $f \ll f^b$ and $f \ll f^{b^{-1}}$ are all impossible. Therefore f, f^a and f^b are Archimedeanly equivalent, whence $C_f = C_f^a = C_f^b$; thus $C_f \lhd G$.

Consider $\bar{G} = G/C_f$ and $\bar{G}^* = C_f^*/C_f$. Then $\bar{G}^*$ is isomorphic to a subgroup of $\mathbb{R}$ and $\bar{G}$ acts on $\bar{G}^*$ by multiplication by positive real numbers (see [1], section 4.1). So $\bar{f}^{\bar{a}} = \alpha\bar{f}$ and $\bar{f}^{\bar{b}} = \beta\bar{f}$ where $\alpha, \beta \in \mathbb{R}_+$. Hence

$$\bar{f}^{\bar{a}+\bar{a}^{-1}+\bar{b}+\bar{b}^{-1}} = (\alpha + \alpha^{-1} + \beta + \beta^{-1})\bar{f} = 4\bar{f}.$$

But $x + x^{-1} \geq 2$ for any positive real number x, and equality occurs iff $x = 1$. Therefore $\alpha = \beta = 1$. Thus $[C_f^*, G] \subseteq C_f$ for every $f \in G'$.

Suppose now that $f \in G_+ \backslash G'$ and $C_f \prec C_f^*$ is not central. In this case there is an element $g \in G$ such that $[f,g]^{n+1} > f$ for some $n \in \mathbb{N}$. If $f \ll [f,g]$, then $f \in C_{[f,g]}$. By the previous paragraph, $C_{[f,g]} \lhd G$ since $[f,g] \in G'$. Hence, $[f,g] = f^{-1}f^g \in C_{[f,g]}$, a contradiction. Thus f is Archimedeanly equivalent to $[f,g]$ and $C_f = C_{[f,g]} \lhd G$. With our previous notation, we may choose $n \in \mathbb{N}$ so that $[\bar{f},\bar{g}]^{n+1} \geq \bar{f} > [\bar{f},\bar{g}]^n$. As before, $[\bar{f},\bar{g}]^{\bar{g}} = [\bar{f},\bar{g}]$, whence

$$[\bar{f},\bar{g}]^n \cdot \bar{f}^{-1} \cdot \bar{f}^{\bar{g}} = [\bar{f},\bar{g}]^{n+1} = ([\bar{f},\bar{g}]^{n+1})^{\bar{g}} \geq \bar{f}^{\bar{g}}.$$

Therefore $[\bar{f},\bar{g}]^n \geq \bar{f}$, the desired contradiction.

Consequently, every convex jump is central. □

As observed, the subgroup $\langle a, c\rangle$ of our example is isomorphic to $\mathbb{Z} \wr \mathbb{Z}$ and so has a non-central order as we now show:

LEMMA 3.1. *The wreath product $\mathbb{Z} \wr \mathbb{Z}$ has a non-central order.*

Proof: Write an arbitrary element of the (restricted) wreath product uniquely in the form $(\mathbf{f}, n)$ where $\mathbf{f} : \mathbb{Z} \to \mathbb{Z}$ is any function with $\text{supp}(\mathbf{f}) = \{k \in \mathbb{Z} : \mathbf{f}(k) \neq 0\}$ finite. Define such a non-identity element to exceed 1 iff $n > 0$ or $(\mathbf{f}(\ell) > 0 = n)$ where ℓ is the least element of $\text{supp}(\mathbf{f})$. This order is clearly non-central as all powers of any $(\mathbf{f}, 0) > 1$ are less than its conjugate by $(\mathbf{0}, 1)$. □

By Lemma 3.1 and Theorem A we get that $\mathcal{C} \neq \mathcal{C}_2$.

Proof of Theorem B: Let $G \in \mathfrak{N}\mathcal{A} \cap \mathcal{G} \cap \mathcal{C}$. Assume that G is not nilpotent.

(1) We first show that since G is not nilpotent, there is a relatively convex normal subgroup N that is maximal such that the quotient G/N is not nilpotent.

Let $\{K_i : i \in I\}$ be a chain of relatively convex normal subgroups of G such that G/K_i is not nilpotent. Then $K = \bigcup_{i \in I} K_i$ is relatively convex by Kokorin's result. If G/K were nilpotent, then $K \supseteq \gamma_n(G)$ for some n. By Lemma 2.4(i) applied to $G/\gamma_n(G)$, $K = \langle g_1, \ldots, g_m\rangle^G$ for some $g_1, \ldots, g_m \in G$. Since $\{K_i : i \in I\}$ is a chain of relatively convex normal subgroups of G, there is $j \in I$ such that $g_1, \ldots, g_m \in K_j$. Hence $K = K_j$, the desired contradiction (G/K_j is not nilpotent but G/K is nilpotent). Thus, by Zorn's Lemma, there is a relatively convex normal subgroup N that is maximal such that the quotient G/N is not nilpotent.

Now every order on G/N lifts to G in the natural way (and $G/N \in \mathfrak{N}\mathcal{A} \cap \mathcal{G}$). Hence G/N satisfies all the hypotheses of the theorem and is not nilpotent. By replacing G by G/N (if necessary), we may assume that if $M \neq \{1\}$ is any relatively convex normal subgroup of G then G/M is nilpotent.

(2) By Lemma 2.1, the centre $\zeta(G)$ of an orderable group G is relatively convex. By (1), $G/\zeta(G)$ is nilpotent if $\zeta(G) \neq \{1\}$ (whence G is nilpotent). We may therefore assume that $\zeta(G) = \{1\}$.

(3) Let $J = \bigcap_k \overline{\gamma}_k(G)$. We show that for all positive integers m we have $\gamma_m(G) \not\subseteq J$ (whence G/J is not nilpotent). Assume that $\gamma_k(G) \subseteq J$. Since G is not nilpotent, there is a non-identity element $g \in \gamma_k(G)$. Fix an order on G and let C_g be the convex subgroup maximal with respect to not containing g. Then $C_g \lhd G$. Now $C_g \neq \{1\}$ (otherwise $g \in \zeta(G)$, contradicting (2)). By (1), G/C_g is nilpotent. Therefore $\gamma_{k+\ell}(G) \subseteq C_g$ for some positive integer ℓ. Since convex subgroups are isolated, we get $g \in \gamma_k(G) \subseteq J \subseteq \overline{\gamma}_{k+\ell}(G) \subseteq C_g$, a contradiction. Consequently G/J is not nilpotent. But J is relatively convex by Corollary 2.2. By (1) we have $J = \{1\}$. Thus $\bigcap_k \gamma_k(G) = \{1\}$; i.e., G is residually torsion-free nilpotent.

(4) By Lemma 2.5, G is a residually finite p-group for all primes p. For each prime p, let $\{G_{i,p} : i \in I_p\}$ be a family of normal subgroups of G with $G/G_{i,p} \in \mathfrak{F}_p$ and $\bigcap\{G_{i,p} : i \in I_p\} = \{1\}$. By Zorn's Lemma, G has a maximal normal Abelian subgroup A. Moreover, $A \neq \{1\}$ since G is soluble-by-finite. For each prime p, we have $B_p = \bigcap_{i\in I_p} AG_{i,p} \supseteq A$ and $\bigcap_{i\in I_p}(AG_{i,p})' \subseteq \bigcap_{i\in I_p} G_{i,p} = \{1\}$. Hence B_p is an Abelian normal subgroup of G containing A, and so equals A by maximality. Thus $A = \bigcap_{i\in I_p} AG_{i,p}$ whence G/A is a residually finite p-group for each prime p. By [16], G/A is an orderable group. Let $\succ$ be such an order. Consequently, G becomes an ordered group if we define $g > 1$ if $gA \succ A$ or $g \in A_+$ in some original order. Therefore A is a relatively convex subgroup of G. By (1), G/A is nilpotent; hence $G \in \mathcal{A}\mathfrak{N}$ (even as an ordered group) and is not nilpotent.

(5) Let N be a normal nilpotent subgroup of G maximal such that G/N is Abelian. Such a subgroup N exists since G' is nilpotent and G/G' is a finitely generated Abelian group and hence satisfies the maximal condition for subgroups. As in (4), since G is a residually finite p-group, there is a set of normal subgroups $\{G_{i,p} : i \in I_p\}$ of G with $G/G_{i,p} \in \mathfrak{F}_p$ and $\bigcap\{G_{i,p} : i \in I_p\} = \{1\}$. Let $B_p = \bigcap_{i\in I_p} VG_{i,p} \supseteq V$ where $V = \zeta(N)$. Then $\bigcap_{i\in I_p}[VG_{i,p}, N] \subseteq \bigcap_{i\in I_p} G_{i,p} = \{1\}$. Hence $[B_p, N] = 1$. Therefore $[N, N] \subseteq [B_pN, B_pN] = [B_p, B_p][N, N] \subseteq$

$(\bigcap_{i,j \in I_p}[VG_{i,p}, VG_{j,p}])[N,N] \subseteq (\bigcap_{i \in I_p} G_{i,p})[N,N] = [N,N]$. Thus $\gamma_2(B_pN) = \gamma_2(N)$. Since N is nilpotent, so is B_pN. Since $G/(B_pN)$ is Abelian, we have $B_p \subseteq N$ by the choice of N (as a maximal such nilpotent normal subgroup of G). Thus $B_p \subseteq \zeta(N) = V$, whence equality. That is, $V = \bigcap_{i \in I_p} VG_{i,p}$ whence G/V is a residually finite p-group. Since this holds for each prime p, we deduce that G/V is orderable.

Thus V is a normal Abelian relatively convex subgroup of G that is centralized by $N \supseteq G'$. By (1), G/V is nilpotent. Let $C = C_G(V)$ be the centraliser of V in G. Then $C \lhd G$ (since $V \lhd G$) and $G \neq C$ (otherwise $\zeta(G) \supseteq V \neq \{1\}$, contradicting (2)). By Lemma 2.1, C is isolated; and $C \supseteq N$ since $V = \zeta(N)$. Hence $T = G/C$ is a non-trivial finitely generated torsion-free Abelian group, whence T is orderable. The metabelian group $J = V \rtimes T$ (with T acting on V in the same way as G/C) is well defined since conjugation by any element $g \in G$ is the same as by any element of gC. Moreover, J is orderable by taking any G-order on V and extending this by any order on T.

If J had a non-central order, say $\prec$, there are $z \in V$, and $t \in T$ with $1 \prec z \prec [z,t]$. Thus we have a partial order with z and $[z,t]z^{-1}$ both positive. This order can be extended to a partial order on J with positive cone P where $V \subseteq P \cup P^{-1}$, using Lemma 2.3. Use this order on V and any order on G/V to obtain a total order on G. This is possible since any J-order on V is also a G-order on V. Since this order on G is not central, we have a contradiction. Thus we conclude that every order on J is central. Furthermore, J is not nilpotent (otherwise $\{1\} \neq V \cap \zeta(J) \subseteq \zeta(G)$, contradicting (2)). We may therefore assume that G is metabelian.

(6) Now let G be a locally soluble group in which every two-generator subgroup is a $\mathcal{C}$-group. In order to show that G is locally nilpotent, it suffices to assume that it is a finitely generated soluble group and prove that it is nilpotent. We may assume, by induction on solubility length, that $G \in \mathfrak{N}\mathcal{A}$. By (1) — (5), we may further assume that G is metabelian.

As a finitely generated metabelian orderable group, G has a normal Abelian subgroup A such that G/A is torsion-free Abelian. Since G satisfies the maximal condition for normal subgroups, A is finitely generated as a G-subgroup. In other words, $A = \langle X^G \rangle$ for some finite set $X \subset A$. If $\langle x, g \rangle$ were nilpotent for every $x \in X$ and every $g \in G$, then A would be finitely generated since G/A is a finitely generated Abelian group. This would imply that A is contained in the hypercentre of G whence G is nilpotent. Therefore, to complete the proof of Theorem B it suffices to assume that $G = \langle a, t\rangle$, $A = \langle a^G\rangle$ is Abelian, and show that G is nilpotent.

(7) Now $G = \langle a\rangle^G \rtimes \langle t\rangle$ and A, viewed as a $\langle t\rangle$-module has the form $\mathbb{Z}[t, t^{-1}]/I(t)$, where $I(t)$ is the ideal generated by the minimal polynomial (if any) in t that annhilates a and is $\{0\}$ if no such polynomial exists.

If $I(t)$ were non-trivial, then $\sum_{i=0}^{n} c_i t^i \in I(t)$, where $c_0, \ldots, c_n$ are integers and $c_0 \neq 0 \neq c_n$. Then

$$a(c_n t^n) \in R = \langle a\rangle + \cdots + \langle at^{n-1}\rangle$$

and hence $at^n \in \overline{R}$, the isolator of R in G. In the same way we see that $at^m \in \overline{R}$ for all $m \geq n$. Similarly, from

$$\sum_{i=0}^{n} c_i t^{i-n} \in I(t)$$

we get $at^{-m} \in \overline{R}$ for all $m \geq 1$. Hence $A = \overline{R}$ and the rank of A (and therefore the rank of G) is finite. Since G is residually torsion-free nilpotent, the rank of $A \cap \gamma_k(G)$ decreases as k increases. Since $\gamma_2(G) = [A, \langle t\rangle] = A \cap \gamma_2(G)$, it follows that $\gamma_k(G) = 1$ for all k greater than the rank of G. Consequently, G is nilpotent.

Thus we may assume that $I(t)$ is trivial. In this case G is isomorphic to $\langle a\rangle \wr \langle t\rangle$. By Lemma 3.1, G has an order that is not central. This contradiction completes the proof of the theorem. □

The proof we have just given for Theorem B establishes that:

COROLLARY 3.2. *Let G be a finitely generated soluble-by-finite orderable group. If every order on every two-generator subgroup of G is central then G is nilpotent.*

Proof of Theorem C: Let G be an orderable linear group in which every order on every two-generator subgroup of G is central. Let H be a finitely generated subgroup of G. By Tits' Dichotomy [19], H is either soluble-by-finite or has a subgroup that is a free group of rank two. But the free group $F = < x, y >$ on two generators has orders on it that are not central:

Consider $W = \langle a\rangle \wr \langle t\rangle$ with a non-central order on it by Lemma 3.1. Let P be the set of positive elements of such an order on W. The map $x \mapsto a$, $y \mapsto t$ can be extended to a homomorphism ϕ from F onto W. Let $K = ker(\phi)$. Then K is a relatively convex subgroup of F since W is orderable. Thus there is an order on K that is F-invariant. We extend this order to a total order on F by letting an element $g \in F \setminus K$

be positive if $\phi(g) \in P$. This order on F is not central since the order on W is not central.

Hence $H \in (\mathrm{P}\mathcal{A})\mathfrak{F}$, whence by Corollary 3.2, it is nilpotent. □

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