J. Korean Math. Soc. **40** (2003), No. 2, pp. 207–224

A WEIGHTED OSTROWSKI TYPE INEQUALITY FOR FUNCTIONS WITH VALUES IN HILBERT SPACES AND APPLICATIONS

S. S. Dragomir

Abstract. Some weighted Ostrowski type integral inequalities for vector-valued functions in Hilbert spaces are given. Applications for quadrature rules with values in Hilbert spaces are also pointed out.

1. Introduction

In [8], Pečarić and Savić obtained the following Ostrowski type inequality for weighted integrals (see also [5, Theorem 3]):

Theorem 1. *Let $w : [a,b] \to [0,\infty)$ be a weight function on $[a,b]$. Suppose that $f : [a,b] \to \mathbb{R}$ satisfies*

$$|f(t) - f(s)| \leq N\,|t-s|^{\alpha}, \text{ for all } t,s \in [a,b], \tag{1.1}$$

where $N > 0$ and $0 < \alpha \leq 1$ are some constants. Then for any $x \in [a,b]$

$$\left| f(x) - \frac{\int_a^b w(t)\,f(t)\,dt}{\int_a^b w(t)\,dt} \right| \leq N \cdot \frac{\int_a^b |t-x|^{\alpha}\,w(t)\,dt}{\int_a^b w(t)\,dt}. \tag{1.2}$$

Further, if for some constants c and λ

$$0 < c \leq w(t) \leq \lambda c, \text{ for all } t \in [a,b],$$

then for any $x \in [a,b]$, we have

$$\left| f(x) - \frac{\int_a^b w(t)\,f(t)\,dt}{\int_a^b w(t)\,dt} \right| \leq N \cdot \frac{\lambda L(x)\,J(x)}{L(x) - J(x) + \lambda J(x)}, \tag{1.3}$$

Received May 6, 2002.
2000 Mathematics Subject Classification: Primary 26D25, 26D10; Secondary 41A55, 46B10.
Key words and phrases: Ostrowski's inequality, vector-valued functions, Bochner's integral.

where

$$L(x) := \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^{\alpha}$$

and

$$J(x) := \frac{(x-a)^{1+\alpha} + (b-x)^{1+\alpha}}{(1+\alpha)(b-a)}.$$

The inequality (1.2) was rediscovered in [3] where further applications for different weights and in Numerical Analysis were given.

For other results in connection to weighted Ostrowski inequalities, see [2], [6] and [7].

In the present paper we extend the weighted Ostrowski's inequality for vector-valued functions and Bochner integrals in Hilbert spaces.

Let X be a Banach space and $-\infty < a < b < \infty$. We denote by $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators acting on X. The norms of vectors or operators acting on X will be denoted by $\|\cdot\|$.

A function $f : [a,b] \to X$ is called *measurable* if there exists a sequence of simple functions $f_n : [a,b] \to X$ which converges punctually almost everywhere on $[a,b]$ at f. We recall also that a measurable function $f : [a,b] \to X$ is *Bochner integrable* if and only if its norm function (i.e. the function $t \mapsto \|f(t)\| : [a,b] \to \mathbb{R}_+$) is Lebesgue integrable on $[a,b]$.

The following theorem holds [1].

THEOREM 2. *Assume that* $B : [a,b] \to \mathcal{L}(X)$ *is Hölder continuous on* $[a,b]$, *i.e.,*

$$\|B(t) - B(s)\| \le H|t-s|^{\alpha} \quad \text{for all } t,s \in [a,b], \tag{1.4}$$

where $H > 0$ *and* $\alpha \in (0,1]$.

If $f : [a,b] \to X$ is Bochner integrable on $[a,b]$, then we have the inequality:

$$(1.5)\qquad \left\| B(t)\int_a^b f(s)\,ds - \int_a^b B(s) f(s)\,ds \right\|$$

$$\leq H\int_a^b |t-s|^{\alpha}\,\|f(s)\|\,ds$$

$$\leq H \times \begin{cases} \dfrac{(b-t)^{\alpha+1}+(t-a)^{\alpha+1}}{\alpha+1}\,\|\|f\|\|_{[a,b],\infty} & \text{if } f \in L_{\infty}([a,b];X); \\ \left[\dfrac{(b-t)^{q\alpha+1}+(t-a)^{q\alpha+1}}{q\alpha+1}\right]^{\frac{1}{q}} \|\|f\|\|_{[a,b],p} & \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ & \text{and } f \in L_p([a,b];X); \\ \left[\dfrac{1}{2}(b-a)+\left|t-\dfrac{a+b}{2}\right|\right]^{\alpha} \|\|f\|\|_{[a,b],1} \end{cases}$$

for any $t \in [a,b]$.

The following corollary holds.

COROLLARY 1. *Assume that $B : [a,b] \to \mathcal{L}(X)$ is Lipschitzian with the constant $L > 0$. Then we have the inequality*

$$(1.6)\qquad \left\| B(t)\int_a^b f(s)\,ds - \int_a^b B(s) f(s)\,ds \right\| \leq L\int_a^b |t-s|\,\|f(s)\|\,ds$$

$$\leq L \times \begin{cases} \left[\frac{1}{4}(b-a)^2 + \left(t - \frac{a+b}{2}\right)^2\right] |||f|||_{[a,b],\infty} & \\ \qquad \text{if } f \in L_\infty([a,b];X); & \\ \left[\frac{(b-t)^{q+1} + (t-a)^{q+1}}{q+1}\right]^{\frac{1}{q}} |||f|||_{[a,b],p} & \\ \qquad \text{if } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1 & \\ \qquad \text{and } f \in L_p([a,b];X); & \\ \left[\frac{1}{2}(b-a) + \left|t - \frac{a+b}{2}\right|\right] |||f|||_{[a,b],1} & \end{cases}$$

for any $t \in [a,b]$.

REMARK 1. If we choose $t = \frac{a+b}{2}$ in (1.5) and (1.6), then we get the following midpoint inequalities:

$$\left\| B\left(\frac{a+b}{2}\right) \int_a^b f(s)\,ds - \int_a^b B(s) f(s)\,ds \right\| \tag{1.7}$$

$$\leq H \int_a^b \left| s - \frac{a+b}{2} \right|^\alpha \|f(s)\|\,ds$$

$$\leq H \times \begin{cases} \frac{1}{2^\alpha(\alpha+1)}(b-a)^{\alpha+1} |||f|||_{[a,b],\infty} & \\ \qquad \text{if } f \in L_\infty([a,b];X); & \\ \frac{1}{2^\alpha(q\alpha+1)^{\frac{1}{q}}}(b-a)^{\alpha+\frac{1}{q}} |||f|||_{[a,b],p} & \\ \qquad \text{if } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1 & \\ \qquad \text{and } f \in L_p([a,b];X); & \\ \frac{1}{2^\alpha}(b-a)^\alpha |||f|||_{[a,b],1} & \end{cases}$$

and

$$\left\| B\left(\frac{a+b}{2}\right)\int_a^b f(s)\,ds - \int_a^b B(s) f(s)\,ds\right\| \tag{1.8}$$

$$\begin{aligned} &\leq L\int_a^b \left|s-\frac{a+b}{2}\right| \|f(s)\|\,ds \\ &\leq L\times \begin{cases} \frac{1}{4}(b-a)^2 \||f|\|_{[a,b],\infty} & \text{if } f\in L_\infty([a,b];X); \\ \frac{1}{2(q+1)^{\frac{1}{q}}}(b-a)^{1+\frac{1}{q}} \||f|\|_{[a,b],p} & \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ & \text{and } f\in L_p([a,b];X); \\ \frac{1}{2}(b-a)\||f|\|_{[a,b],1} \end{cases} \end{aligned}$$

respectively.

REMARK 2. Consider the function $\Psi_\alpha : [a,b] \to \mathbb{R}$, $\Psi_\alpha(t) := \int_a^b |t-s|^\alpha \|f(s)\|\,ds$, $\alpha \in (0,1)$. If f is continuous on $[a,b]$, then Ψ_α is differentiable and

$$\begin{aligned} \frac{d\Psi_\alpha(t)}{dt} &= \frac{d}{dt}\left[\int_a^t (t-s)^\alpha \|f(s)\|\,ds + \int_t^b (s-t)^\alpha \|f(s)\|\,ds\right] \\ &= \alpha\left[\int_a^t \frac{\|f(s)\|}{(t-s)^{1-\alpha}}ds - \int_t^b \frac{\|f(s)\|}{(s-t)^{1-\alpha}}ds\right]. \end{aligned}$$

If $t_0 \in (a,b)$ is such that

$$\int_a^{t_0} \frac{\|f(s)\|}{(t_0-s)^{1-\alpha}}ds = \int_{t_0}^b \frac{\|f(s)\|}{(s-t_0)^{1-\alpha}}ds$$

and $\Psi_s'(\cdot)$ is negative on (a,t_0) and positive on (t_0,b), then the best inequality we can get in the first part of (1.5) is the following one

(1.9)

$$\left\|B(t_0)\int_a^b f(s)\,ds - \int_a^b B(s) f(s)\,ds\right\| \leq H\int_a^b |t_0-s|^\alpha \|f(s)\|\,ds.$$

If $\alpha = 1$, then, for

$$\Psi(t) := \int_a^b |t-s|\,\|f(s)\|\,ds,$$

we have

$$\frac{d\Psi(t)}{dt} = \int_a^t \|f(s)\|\,ds - \int_t^b \|f(s)\|\,ds, \quad t \in (a,b),$$
$$\frac{d^2\Psi(t)}{dt^2} = 2\|f(t)\| \geq 0, \quad t \in (a,b),$$

which shows that Ψ is convex on (a,b).

If $t_m \in (a,b)$ is such that

$$\int_a^{t_m} \|f(s)\|\,ds = \int_{t_m}^b \|f(s)\|\,ds,$$

then the best inequality we can get from the first part of (1.6) is

$$\left\| B(t_m)\int_a^b f(s)\,ds - \int_a^b B(s)\,f(s)\,ds \right\| \leq L\int_a^b \operatorname{sgn}(s-t_m)\,s\,\|f(s)\|\,ds. \tag{1.10}$$

We recall that a function $F : [a,b] \to \mathcal{L}(X)$ is said to be *strongly continuous* if for all $x \in X$, the maps $s \mapsto F(s)\,x : [a,b] \to X$ are continuous on $[a,b]$. In this case the function $s \mapsto \|B(s)\| : [a,b] \to \mathbb{R}_+$ is (Lebesgue) measurable and bounded ([4]). The linear operator $L = \int_a^b F(s)\,ds$ (defined by $Lx := \int_a^b F(s)\,xds$ for all $x \in X$) is bounded, because

$$\|Lx\| \leq \left(\int_a^b \|F(s)\|\,ds\right) \cdot \|x\| \quad \text{for all } x \in X.$$

The following theorem also holds [1].

THEOREM 3. *Assume that* $f : [a,b] \to X$ *is Hölder continuous, i.e.,*

$$\|f(t) - f(s)\| \leq K\,|t-s|^{\beta} \quad \text{for all } t,s \in [a,b], \tag{1.11}$$

where $K > 0$ *and* $\beta \in (0,1]$.

If $B : [a,b] \to \mathcal{L}(X)$ *is strongly continuous on* $[a,b]$, *then we have the inequality:*

$$\left\| \left(\int_a^b B(s)\,ds\right) f(t) - \int_a^b B(s)\,f(s)\,ds \right\| \leq K\int_a^b |t-s|^{\beta}\,\|B(s)\|\,ds \tag{1.12}$$

$$\leq K \times \begin{cases} \frac{(b-t)^{\beta+1}+(t-a)^{\beta+1}}{\beta+1} \left\| |B| \right\|_{[a,b],\infty} \\ \qquad\qquad \text{if } \|B(\cdot)\| \in L_\infty([a,b];\mathbb{R}_+); \\ \left[\frac{(b-t)^{q\beta+1}+(t-a)^{q\beta+1}}{q\beta+1}\right]^{\frac{1}{q}} \left\| |B| \right\|_{[a,b],p} \\ \qquad\qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ \qquad\qquad \text{and } \|B(\cdot)\| \in L_p([a,b];\mathbb{R}_+); \\ \left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]^{\beta} \left\| |B| \right\|_{[a,b],1} \end{cases}$$

for any $t \in [a,b]$.

The following corollary holds.

COROLLARY 2. *Assume that* f *and* B *are as in Theorem 3. If, in addition,* $\int_a^b B(s)\,ds$ *is invertible in* $\mathcal{L}(X)$, *then we have the inequality:*

$$\left\| f(t) - \left(\int_a^b B(s)\,ds\right)^{-1} \int_a^b B(s) f(s)\,ds \right\| \leq K \left\| \left(\int_a^b B(s)\,ds\right)^{-1} \right\| \int_a^b |t-s|^{\beta} \|B(s)\|\,ds \tag{1.13}$$

for any $t \in [a,b]$.

REMARK 3. It is obvious that the inequality (1.13) contains as a particular case what is the so called Ostrowski's inequality for weighted integrals (see (1.2)).

2. The results

The following lemma concerning an integral identity that will be used in the following holds.

LEMMA 1. *Let* $f : [a,b] \to H$ *be an absolutely continuous function and* $g : [a,b] \to H$ *be Bochner integrable on* $[a,b]$, *where* $(H, \langle \cdot,\cdot \rangle)$ *is a Hilbert space on* $\mathbb{K}$ $(\mathbb{K} = \mathbb{R}, \mathbb{C})$. *Then for any* $x \in [a,b]$, *we have the identity*

$$\left\langle f(x), (B)\int_a^b g(t)\,dt \right\rangle = \int_a^b \langle f(t), g(t) \rangle\,dt + \int_a^b \langle f'(\tau), K(\tau,x) \rangle\,dx, \tag{2.1}$$

where

$$K(\tau, x) := \begin{cases} (B)\int_a^{\tau} g(s)\,ds & \text{if} \quad a \leq \tau \leq x \leq b; \\ -(B)\int_{\tau}^{b} g(s)\,ds & \text{if} \quad a \leq x < \tau \leq b \end{cases}$$

and $(B)\int_a^b g(t)\,dt$ *denotes the Bochner's integral of* g *on* $[a,b]$.

Proof. Integrating by parts, we may write:

$$\begin{aligned} &\int_a^x \left\langle f'(\tau), (B)\int_a^{\tau} g(s)\,ds \right\rangle d\tau \\ &= \left\langle f(\tau), (B)\int_a^{\tau} g(s)\,ds \right\rangle \Big|_a^x - \int_a^x \langle f(\tau), g(\tau)\rangle\, d\tau \\ &= \left\langle f(x), (B)\int_a^x g(s)\,ds \right\rangle - \int_a^x \langle f(\tau), g(\tau)\rangle\, d\tau \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} &\int_x^b \left\langle f'(\tau), -(B)\int_{\tau}^{b} g(s)\,ds \right\rangle d\tau \\ &= -\left\langle f(\tau), (B)\int_{\tau}^{b} g(s)\,ds \right\rangle \Big|_x^b - \int_x^b \langle f(\tau), g(\tau)\rangle\, d\tau \\ &= \left\langle f(x), (B)\int_x^b g(s)\,ds \right\rangle - \int_x^b \langle f(\tau), g(\tau)\rangle\, d\tau. \end{aligned} \tag{2.3}$$

If we add (2.2) and (2.3) we deduce the desired identity (2.1). □

For g as above, we now define the following associated functions

$$\begin{aligned} \gamma(\tau) &: = \left\| (B)\int_a^{\tau} g(s)\,ds \right\|, \quad \tau \in [a,b] \\ \tilde{\gamma}(\tau) &: = \left\| (B)\int_{\tau}^{b} g(s)\,ds \right\|, \quad \tau \in [a,b] \end{aligned}$$

and

$$\Gamma_1(x) := \int_a^x \gamma(\tau)\,d\tau + \int_x^b \tilde{\gamma}(\tau)\,d\tau, \quad x \in [a,b] \quad \text{provided } \gamma, \tilde{\gamma} \in L_1[a,b],$$

$$\begin{aligned} \Gamma_q(x) &: = \left(\int_a^x \gamma^q(\tau)\,d\tau + \int_x^b \tilde{\gamma}^q(\tau)\,d\tau \right)^{\frac{1}{q}}, \quad x \in [a,b] \\ &\quad \text{provided } \gamma, \tilde{\gamma} \in L_q[a,b], \; q \in (1,\infty), \end{aligned}$$

$$\Gamma_\infty(x) \quad : \quad = \max\left(\sup_{\tau\in[a,x]}\gamma(\tau),\ \sup_{\tau\in[x,b]}\tilde{\gamma}(\tau)\,d\tau\right),\quad x\in[a,b]$$
provided $\gamma,\tilde{\gamma}\in L_\infty[a,b]$.

We may state the following estimation result:

THEOREM 4. *Let f and g be as in Lemma 1. Then for any $x\in[a,b]$ we have the inequality*:

$$\left|\left\langle f(x),(B)\int_a^b g(t)\,dt\right\rangle-\int_a^b\langle f(t),g(t)\rangle\,dt\right| \le \int_a^x \|f'(\tau)\|\,\gamma(\tau)\,d\tau+\int_x^b\|f'(\tau)\|\,\tilde{\gamma}(\tau)\,d\tau=:M(x). \tag{2.4}$$

If we denote (for $a\le\alpha<\beta\le b$)

$$|||f'|||_{[\alpha,\beta],\infty} = ess\sup_{\tau\in[\alpha,\beta]}\|f'(\tau)\|,$$
$$|||f'|||_{[\alpha,\beta],p} = \left(\int_\alpha^\beta\|f'(\tau)\|^p\,d\tau\right)^{\frac{1}{p}},\quad p\ge1$$

then we have the inequalities for $M(x)$ as follows

$$M(x) \le \begin{cases} |||f'|||_{[a,x],\infty}\cdot\int_a^x\gamma(\tau)\,d\tau+|||f'|||_{[x,b],\infty}\cdot\int_x^b\tilde{\gamma}(\tau)\,d\tau \\ \qquad\qquad \textit{if } f'\in L_\infty[a,b],\ \gamma,\tilde{\gamma}\in L_\infty[a,b], \\ |||f'|||_{[a,x],p}\cdot\left(\int_a^x\gamma^q(\tau)\,d\tau\right)^{\frac{1}{q}}+|||f'|||_{[x,b],p}\cdot\left(\int_x^b\tilde{\gamma}^q(\tau)\,d\tau\right)^{\frac{1}{q}} \\ \qquad \textit{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ f'\in L_p[a,b],\ \gamma,\tilde{\gamma}\in L_q[a,b], \\ |||f'|||_{[a,x],1}\cdot\sup_{\tau\in[a,x]}\gamma(\tau)+|||f'|||_{[x,b],1}\cdot\sup_{\tau\in[x,b]}\tilde{\gamma}(\tau)\,d\tau \end{cases}$$
$$\le \begin{cases} \Gamma_1(x)\cdot|||f'|||_{[a,b],\infty}\ \textit{if } f'\in L_\infty[a,b],\ \gamma,\tilde{\gamma}\in L_\infty[a,b], \\ \Gamma_p(x)\cdot|||f'|||_{[a,b],p} \\ \quad \textit{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ f'\in L_p[a,b],\ \gamma,\tilde{\gamma}\in L_q[a,b], \\ \Gamma_\infty(x)\cdot|||f'|||_{[a,b],1}\quad \textit{if } \gamma,\tilde{\gamma}\in L_\infty[a,b]. \end{cases} \tag{2.5}$$

Proof. Taking the modulus in (2.1), we may write that

$$\begin{aligned}
& \left|\left\langle f(x), (B)\int_a^b g(t)\,dt\right\rangle - \int_a^b \langle f(t), g(t)\rangle\,dt\right| \\
\leq & \int_a^x \left|\left\langle f'(\tau), (B)\int_a^\tau g(s)\,ds\right\rangle\right| d\tau \\
& + \int_x^b \left|\left\langle f'(\tau), (B)\int_\tau^b g(s)\,ds\right\rangle\right| d\tau \\
& \text{(by Schwartz's inequality in } H, \langle\cdot,\cdot\rangle\text{)} \\
\leq & \int_a^x \|f'(\tau)\| \left\|(B)\int_a^\tau g(s)\,ds\right\| d\tau \\
& + \int_x^b \|f'(\tau)\| \left\|(B)\int_\tau^b g(s)\,ds\right\| d\tau \\
= & M(x)
\end{aligned}$$

and the inequality (2.4) is proved.

Now, observe that

$$\int_a^x \|f'(\tau)\|\,\gamma(\tau)\,d\tau \leq \left|\left\|f'\right\|\right|_{[a,x],\infty} \cdot \int_a^x \gamma(\tau)\,d\tau$$

and

$$\int_x^b \|f'(\tau)\|\,\tilde{\gamma}(\tau)\,d\tau \leq \left|\left\|f'\right\|\right|_{[x,b],\infty} \cdot \int_x^b \tilde{\gamma}(\tau)\,d\tau$$

giving the first part of the first inequality in (2.5).

Using Hölder's integral inequality, we may write for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ that

$$\begin{aligned}
\int_a^x \|f'(\tau)\|\,\gamma(\tau)\,d\tau & \leq \left(\int_a^x \|f'(\tau)\|^p\,d\tau\right)^{\frac{1}{p}} \cdot \left(\int_a^x \gamma^q(\tau)\,d\tau\right)^{\frac{1}{q}} \\
& = \left|\left\|f'\right\|\right|_{[a,x],p} \left(\int_a^x \gamma^q(\tau)\,d\tau\right)^{\frac{1}{q}}
\end{aligned}$$

and

$$\int_x^b \|f'(\tau)\|\,\tilde{\gamma}(\tau)\,d\tau \leq \left|\left\|f'\right\|\right|_{[x,b],p} \cdot \left(\int_x^b \tilde{\gamma}^q(\tau)\,d\tau\right)^{\frac{1}{q}}$$

giving the second part of the first inequality in (2.5).

The last part is obvious.

For the second inequality in (2.5) we observe that

$$\begin{aligned}
& \left|\left|\left|f'\right|\right|\right|_{[a,x],\infty} \cdot \int_a^x \gamma(\tau)\,d\tau + \left|\left|\left|f'\right|\right|\right|_{[x,b],\infty} \cdot \int_x^b \tilde{\gamma}(\tau)\,d\tau \\
\leq\ & \max\left\{\left|\left|\left|f'\right|\right|\right|_{[a,x],\infty}, \left|\left|\left|f'\right|\right|\right|_{[x,b],\infty}\right\} \cdot \left\{\int_a^x \gamma(\tau)\,d\tau + \int_x^b \tilde{\gamma}(\tau)\,d\tau\right\} \\
=\ & \left|\left|\left|f'\right|\right|\right|_{[a,b],\infty} \cdot \Gamma_1(x),
\end{aligned}$$

$$\begin{aligned}
& \left|\left|\left|f'\right|\right|\right|_{[a,x],p} \cdot \left(\int_a^x \gamma^q(\tau)\,d\tau\right)^{\frac{1}{q}} + \left|\left|\left|f'\right|\right|\right|_{[x,b],p} \cdot \left(\int_x^b \tilde{\gamma}^q(\tau)\,d\tau\right)^{\frac{1}{q}} \\
\leq\ & \left(\left|\left|\left|f'\right|\right|\right|_{[a,x],p}^p + \left|\left|\left|f'\right|\right|\right|_{[x,b],p}^p\right)^{\frac{1}{p}} \\
& \times \left(\left[\left(\int_a^x \gamma^q(\tau)\,d\tau\right)^{\frac{1}{q}}\right]^q + \left[\left(\int_x^b \tilde{\gamma}^q(\tau)\,d\tau\right)^{\frac{1}{q}}\right]^q\right)^{\frac{1}{q}} \\
=\ & \left|\left|\left|f'\right|\right|\right|_{[a,b],p} \cdot \Gamma_q(x),
\end{aligned}$$

and

$$\begin{aligned}
& \left|\left|\left|f'\right|\right|\right|_{[a,x],1} \cdot \sup_{\tau\in[a,x]} \gamma(\tau) + \left|\left|\left|f'\right|\right|\right|_{[x,b],1} \cdot \sup_{\tau\in[x,b]} \tilde{\gamma}(\tau) \\
\leq\ & \max\left\{\sup_{\tau\in[a,x]} \gamma(\tau), \sup_{\tau\in[x,b]} \tilde{\gamma}(\tau)\,d\tau\right\} \left[\left|\left|\left|f'\right|\right|\right|_{[a,x],1} + \left|\left|\left|f'\right|\right|\right|_{[x,b],1}\right] \\
=\ & \left|\left|\left|f'\right|\right|\right|_{[a,b],1} \cdot \Gamma_\infty(x),
\end{aligned}$$

and the last part of the inequality (2.5) is thus proved. □

REMARK 4. If $g \in L_\infty[a,b]$, then, for any $\tau \in [a,b]$

$$\gamma(\tau) \leq (\tau - a)\left|\left|\left|g\right|\right|\right|_{[a,\tau],\infty} \quad \text{and} \quad \tilde{\gamma}(\tau) \leq (b-\tau)\left|\left|\left|g\right|\right|\right|_{[\tau,b],\infty},$$

if $g \in L_q[a,b]$ $\left(q > 1, \frac{1}{p} + \frac{1}{q} = 1\right)$, then

$$\gamma(\tau) \leq (\tau - a)^{\frac{1}{p}}\left|\left|\left|g\right|\right|\right|_{[a,\tau],q} \quad \text{and} \quad \tilde{\gamma}(\tau) \leq (b-\tau)^{\frac{1}{p}}\left|\left|\left|g\right|\right|\right|_{[\tau,b],q},$$

and finally, if $g \in L_1[a,b]$, then

$$\gamma(\tau) \leq \left|\left|\left|g\right|\right|\right|_{[a,\tau],1} \quad \text{and} \quad \tilde{\gamma}(\tau) \leq \left|\left|\left|g\right|\right|\right|_{[\tau,b],1}.$$

Consequently, we may bound $M(x)$ (defined in Theorem 4) in an other manner than those in Theorem 4, as follows:

$$
(2.6)\quad M(x) \le \begin{cases}
\int_a^x \|f'(\tau)\|(\tau-a)\,|||g|||_{[a,\tau],\infty}\,d\tau \\
\qquad + \int_x^b \|f'(\tau)\|(b-\tau)\,|||g|||_{[\tau,b],\infty}\,d\tau \\
\qquad \text{if } g\in L_\infty[a,b], \\
\int_a^x \|f'(\tau)\|(\tau-a)^{\frac{1}{p}}\,|||g|||_{[a,\tau],q}\,d\tau \\
\qquad + \int_x^b \|f'(\tau)\|(b-\tau)^{\frac{1}{p}}\,|||g|||_{[\tau,b],q}\,d\tau \\
\qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ g\in L_p[a,b], \\
\int_a^x \|f'(\tau)\|\,|||g|||_{[a,\tau],1}\,d\tau \\
\qquad + \int_x^b \|f'(\tau)\|\,|||g|||_{[\tau,b],1}\,d\tau \\
\qquad \text{if } g\in L_1[a,b],
\end{cases}
$$

$$
\le \begin{cases}
|||g|||_{[a,x],\infty}\int_a^x \|f'(\tau)\|(\tau-a)\,d\tau \\
\qquad + |||g|||_{[x,b],\infty}\int_x^b \|f'(\tau)\|(b-\tau)\,d\tau =: M_\infty(x) \\
\qquad \text{if } g\in L_\infty[a,b], \\
|||g|||_{[a,x],q}\int_a^x \|f'(\tau)\|(\tau-a)^{\frac{1}{p}}\,d\tau \\
\qquad + |||g|||_{[x,b],q}\int_x^b \|f'(\tau)\|(b-\tau)^{\frac{1}{p}}\,d\tau =: M_q(x) \\
\qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ g\in L_p[a,b], \\
|||g|||_{[a,x],1}\,|||f'|||_{[a,x],1} + |||g|||_{[x,b],1}\,|||f'|||_{[x,b],1} =: M_1(x) \\
\qquad \text{if } g\in L_1[a,b].
\end{cases}
$$

Since

$$\text{(2.7)} \qquad \int_a^x \left\| f'(\tau) \right\| (\tau - a)\, d\tau$$

$$\leq \begin{cases} |||f'|||_{[a,x],\infty} \int_a^x (\tau - a)\, d\tau \\ |||f'|||_{[a,x],\alpha} \left(\int_a^x (\tau - a)\, d\tau \right)^{\frac{1}{\beta}}, \ \alpha > 1, \ \frac{1}{\alpha} + \frac{1}{\beta} = 1 \\ (x-a)\, |||f'|||_{[a,x],1} \end{cases}$$

$$= \begin{cases} \frac{1}{2} (x-a)^2 \, |||f'|||_{[a,x],\infty} & \text{if } \ f' \in L_\infty [a,b] \\ \frac{1}{(\beta+1)^{\frac{1}{\beta}}} (x-a)^{1+\frac{1}{\beta}} \, |||f'|||_{[a,x],\alpha} & \\ \qquad \text{if } \ f' \in L_\alpha [a,b], \ \alpha > 1, \ \frac{1}{\alpha} + \frac{1}{\beta} = 1 \\ (x-a)\, |||f'|||_{[a,x],1} \end{cases}$$

and, similarly,

$$\text{(2.8)} \qquad \int_x^b \left\| f'(\tau) \right\| (b - \tau)\, d\tau$$

$$\leq \begin{cases} \frac{1}{2} (b-x)^2 \, |||f'|||_{[x,b],\infty} & \text{if } \ f' \in L_\infty [a,b] \\ \frac{1}{(\beta+1)^{\frac{1}{\beta}}} (b-x)^{1+\frac{1}{\beta}} \, |||f'|||_{[x,b],\alpha} & \\ \qquad \text{if } \ f' \in L_\alpha [a,b], \ \alpha > 1, \ \frac{1}{\alpha} + \frac{1}{\beta} = 1 \\ (b-x)\, |||f'|||_{[x,b],1} \end{cases}$$

then we get the following bound for $M_\infty(x)$ (defined in (2.6))

$$(2.9)\quad M_\infty(x) \le \begin{cases} \frac{1}{2}\Big[(x-a)^2 |||g|||_{[a,x],\infty} |||f'|||_{[a,x],\infty} \\ \qquad + (b-x)^2 |||g|||_{[x,b],\infty} |||f'|||_{[x,b],\infty}\Big] \\ \qquad \text{if } g \in L_\infty[a,b] \text{ and } f' \in L_\infty[a,b]; \\ \frac{1}{(\beta+1)^{\frac{1}{\beta}}}\Big[(x-a)^{1+\frac{1}{\beta}} |||g|||_{[a,x],\infty} |||f'|||_{[a,x],\alpha} \\ \qquad + (b-x)^{1+\frac{1}{\beta}} |||g|||_{[x,b],\infty} |||f'|||_{[x,b],\alpha}\Big] \\ \qquad \text{if } g \in L_\infty[a,b] \\ \qquad \text{and } f' \in L_\alpha[a,b],\ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ (x-a) |||g|||_{[a,x],\infty} |||f'|||_{[a,x],1} + (b-x) |||g|||_{[x,b],\infty} |||f'|||_{[x,b],1} \\ \qquad \text{if } g \in L_\infty[a,b] \text{ and } f' \in L_1[a,b]. \end{cases}$$

We also have

$$(2.10)\quad \int_a^x \|f'(\tau)\| (\tau-a)^{\frac{1}{p}} d\tau \le \begin{cases} |||f'|||_{[a,x],\infty} \int_a^x (\tau-a)^{\frac{1}{p}} d\tau \\ |||f'|||_{[a,x],\alpha} \left(\int_a^x (\tau-a)^{\frac{\beta}{p}} d\tau\right)^{\frac{1}{\beta}},\ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1 \\ (x-a)^{\frac{1}{\beta}} |||f'|||_{[a,x],1} \end{cases}$$

$$= \begin{cases} \frac{p}{p+1}(x-a)^{\frac{1}{p}+1} |||f'|||_{[a,x],\infty} & \text{if } f' \in L_\infty[a,b] \\ \left(\frac{p}{p+\beta}\right)^{\frac{1}{\beta}} (x-a)^{\frac{1}{p}+\frac{1}{\beta}} |||f'|||_{[a,x],\alpha}, & \text{if } f' \in L_\alpha[a,b] \\ & \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1 \\ (x-a)^{\frac{1}{p}} |||f'|||_{[a,x],1} \end{cases}$$

and, similarly,

$$\text{(2.11)}\quad \int_x^b \left\| f'(\tau) \right\| (b-\tau)^{\frac{1}{p}} d\tau$$

$$\leq \begin{cases} \frac{p}{p+1}(b-x)^{\frac{1}{p}+1} \left|\left|\left| f' \right|\right|\right|_{[x,b],\infty} & \text{if } f' \in L_\infty[a,b] \\ \left(\frac{p}{p+\beta}\right)^{\frac{1}{\beta}} (b-x)^{\frac{1}{p}+\frac{1}{\beta}} \left|\left|\left| f' \right|\right|\right|_{[x,b],\alpha}, & \text{if } f' \in L_\alpha[a,b] \\ & \alpha > 1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1 \\ (b-x)^{\frac{1}{p}} \left|\left|\left| f' \right|\right|\right|_{[x,b],1} \end{cases}$$

and thus we may point out the following bounds for the $M_q(x)$ (defined in (2.6))

$$\text{(2.12)}\ M_q(x)$$

$$\leq \begin{cases} \frac{p}{p+1}\Big[(x-a)^{\frac{1}{p}+1} \left|\left|\left| g \right|\right|\right|_{[a,x],q} \left|\left|\left| f' \right|\right|\right|_{[a,x],\infty} \\ \quad + (b-x)^{\frac{1}{p}+1} \left|\left|\left| g \right|\right|\right|_{[x,b],q} \left|\left|\left| f' \right|\right|\right|_{[x,b],\infty}\Big] \\ \quad \text{if } g \in L_q[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1,\quad f' \in L_\infty[a,b]; \\ \left(\frac{p}{p+\beta}\right)^{\frac{1}{\beta}} \Big[(x-a)^{\frac{1}{p}+\frac{1}{\beta}} \left|\left|\left| g \right|\right|\right|_{[a,x],q} \left|\left|\left| f' \right|\right|\right|_{[a,x],\alpha} \\ \quad + (b-x)^{\frac{1}{p}+\frac{1}{\beta}} \left|\left|\left| g \right|\right|\right|_{[x,b],q} \left|\left|\left| f' \right|\right|\right|_{[x,b],\alpha}\Big] \\ \quad \text{if } g \in L_q[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \quad f' \in L_\alpha[a,b],\ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ (x-a)^{\frac{1}{p}+1} \left|\left|\left| g \right|\right|\right|_{[a,x],q} \left|\left|\left| f' \right|\right|\right|_{[a,x],1} \\ \quad + (b-x)^{\frac{1}{p}+1} \left|\left|\left| g \right|\right|\right|_{[x,b],q} \left|\left|\left| f' \right|\right|\right|_{[x,b],1} \\ \quad \text{if } g \in L_q[a,b],\quad p>1,\ \frac{1}{p}+\frac{1}{q}=1. \end{cases}$$

In the following section we will show by an example how the above inequalities can be used in approximating the integral $\int_a^b \langle f(t), g(t) \rangle dt$, where f, g are as in Lemma 1.

3. A quadrature formula

In this section we will show, by example, how the integral $\int_a^b \langle f(t), g(t)\rangle dt$, where $f : [a,b] \to H$ is an absolutely continuous function with values in the Hilbert space H and $g : [a,b] \to H$ is a Bochner integrable function on $[a,b]$, may be approximated with order one accuracy in terms of simpler quantities as described below.

For this purpose, by the use of inequalities (2.4), (2.6) and (2.11), we may consider the inequality

$$\begin{aligned}
&\left| \int_a^b \langle f(t), g(t)\rangle dt - \left\langle f(x), (B)\int_a^b g(t)\,dt \right\rangle \right| \\
&\leq \int_a^x \|f'(\tau)\|\,\gamma(\tau)\,d\tau + \int_x^b \|f'(\tau)\|\,\tilde{\gamma}(\tau)\,d\tau \\
&\leq \frac{1}{2}\Big[(x-a)^2 \,|||g|||_{[a,x],\infty}\, |||f'|||_{[a,x],\infty} \\
&\quad + (b-x)^2\, |||g|||_{[x,b],\infty}\, |||f'|||_{[x,b],\infty}\Big] \\
&\leq \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)^2\, |||g|||_{[a,b],\infty}\, |||f'|||_{[a,b],\infty},
\end{aligned} \tag{3.1}$$

provided $g \in L_\infty([a,b];H)$ and $f' \in L_\infty([a,b];H)$.

For the sake of simplicity, we will use only the last inequality in (3.1) and, in particular, the "mid-point" inequality

$$\begin{aligned}
&\left| \int_a^b \langle f(t), g(t)\rangle dt - \left\langle f\left(\frac{a+b}{2}\right), (B)\int_a^b g(t)\,dt \right\rangle \right| \\
&\qquad\qquad \leq \frac{1}{4}(b-a)^2\, |||g|||_{[a,b],\infty}\, |||f'|||_{[a,b],\infty}.
\end{aligned} \tag{3.2}$$

Suppose that the interval $[a,b]$ is partitioned by the division $I_n : a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ into n smaller intervals with the step size $h_i := x_{i+1} - x_i$, $i = 0, \ldots, n-1$. Denote $\nu(I_n) := \max_{0\leq i\leq n-1} h_i$.

We may consider a sequence of intermediate points $\xi = (\xi_0, \ldots, \xi_{n-1})$ with the property that $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, \ldots, n-1)$ and the sum

$$\sigma_n(f, g, I_n, \xi) := \sum_{i=0}^{n-1} \left\langle f(\xi_i), (B)\int_{x_i}^{x_{i+1}} g(\tau)\,d\tau \right\rangle. \tag{3.3}$$

We may state the following result.

THEOREM 5. *With the assumptions in Lemma 1 and if* $g \in L_{\infty}([a,b];H)$, $f' \in L_{\infty}([a,b];H)$, I_n *and* ξ *are as above, then we have*

$$\int_a^b \langle f(t), g(t)\rangle\, dt = \sigma_n(f,g,I_n,\xi) + R_n(f,g,I_n,\xi), \tag{3.4}$$

where $\sigma_n(f,g,I_n,\xi)$ *is given by* (3.3) *and the remainder* $R_n(f,g,I_n,\xi)$ *satisfies the bound*

$$\begin{aligned} &|R_n(f,g,I_n,\xi)| \\ &\le |||f'|||_{[a,b],\infty}\, |||g|||_{[a,b],\infty} \sum_{i=0}^{n-1}\left[\frac{1}{4}h_i^2 + \left(\xi_i - \frac{x_i+x_{i+1}}{2}\right)^2\right] \\ &\le |||f'|||_{[a,b],\infty}\, |||g|||_{[a,b],\infty}\, \frac{1}{2}\sum_{i=0}^{n-1} h_i^2 \\ &\le \frac{1}{2}(b-a)\, |||f'|||_{[a,b],\infty}\, |||g|||_{[a,b],\infty}\, \nu(I_n). \end{aligned} \tag{3.5}$$

Proof. It is obvious from the inequality (3.1) applied in the interval $[x_i, x_{i+1}]$ and at the point ξ_i, that

$$\begin{aligned} &\left|\int_{x_i}^{x_{i+1}} \langle f(t), g(t)\rangle\, dt - \left\langle f(\xi_i), (B)\int_{x_i}^{x_{i+1}} g(\tau)\, d\tau\right\rangle\right| \\ &\qquad \le \left[\frac{1}{4}h_i^2 + \left(\xi_i - \frac{x_i+x_{i+1}}{2}\right)^2\right] |||g|||_{[a,b],\infty}\, |||f'|||_{[a,b],\infty} \end{aligned} \tag{3.6}$$

for each $i \in \{0,\dots,n-1\}$.

Summing in (3.6) over i from 0 to $n-1$ and using the generalized triangle inequality we get (3.5). □

The best quadrature in the class is obviously the mid-point one. This may be stated in the following corollary.

COROLLARY 3. *With the assumptions in Theorem 5, we have*

$$\int_a^b \langle f(t), g(t)\rangle\, dt = T_n(f,g,I_n) + R_n(f,g,I_n), \tag{3.7}$$

where

$$T_n(f,g,I_n) := \sum_{i=0}^{n-1}\left\langle f\left(\frac{x_i+x_{i+1}}{2}\right), (B)\int_{x_i}^{x_{i+1}} g(\tau)\, d\tau\right\rangle. \tag{3.8}$$

The remainder $R_n(f,g,I_n)$ *satisfies the estimate*

$$|R_n(f,g,I_n)| \leq \left\|\left|f'\right|\right\|_{[a,b],\infty} \left\||g|\right\|_{[a,b],\infty} \frac{1}{4}\sum_{i=0}^{n-1} h_i^2 \leq \frac{1}{4}(b-a)\left\|\left|f'\right|\right\|_{[a,b],\infty} \left\||g|\right\|_{[a,b],\infty} \nu(I_n). \tag{3.9}$$

References

[1] N. S. Barnett, C. Buşe, P. Cerone, and S. S. Dragomir, *On weighted Ostrowski type inequalities for operators and vector-valued functions*, J. Ineq. Pure & Appl. Math. **3** (1) (2002), Article 12.

[2] S. S. Dragomir, N. S. Barnett, and P. Cerone, *An n−dimensional version of Ostrowski's inequality for mappings of Hölder type*, Kyungpook Math. J. **40** (2000), no. 1, 65–75.

[3] S. S. Dragomir, P. Cerone, J. Roumeliotis, and S. Wang, *A weighted version of Ostrowski inequality for mappings of Hölder type and applications in numerical analysis*, Bull. Math. Soc. Sci. Math. Roumanie **42**(90)(4) (1992), 301–304.

[4] E. Hille and R. S. Philips, *Functional Analysis and Semi-Groups,* Amer. Math. Soc. Colloq. Publ. Vol. **31**, Providence, R.I., (1957).

[5] M. Matić, J. E. Pečarić and N. Ujević, *Generalisation of weighted version of Ostrowski's inequality and some related results*, J. Inequal. Appl. **5** (2000), 639–666.

[6] G. V. Milovanović, *On some integral inequalities*, Univ. Beograd Publ. Elektrotehn Fak. Ser. Mat. Fiz., No. 498-541 (1975), 119–124.

[7] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Inequalities Involving Functions and Their Integrals and Derivatives,* Kluwer Academic Publishers, Dordrecht, 1991.

[8] J. E. Pečarić and B. Savić, *O novom postupku razvijanja funkcija u red i nekim primjenama*, Zb. Rad. V A KoV (Beograd) **9** (1983), 171–202.

School of Communications and Informatics
Victoria University of Technology
PO Box 14428
Melbourne City MC
8001, Victoria, Australia
E-mail: sever@matilda.vu.edu.au